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On generalized morphic modules

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Abstract

Aim of the present article is to extend generalized morphic ring to modules. Let R be a commutative ring with a unity and M an R-module. M is said to be a generalized morphic module if for each $m \in M$, there exists $a \in R$ such that $ann_R(m) = (a) + ann_R(M)$, where (a) is the principal ideal generated by an element $a \in R$. Many examples and characterizations of generalized morphic modules are given. Moreover, as an application of generalized morphic modules, we use them to characterize Baer modules and principal ideal rings.

1 Introduction

Throughout this article, we focus only on commutative rings with a unity and nonzero unital modules. Let R will always denote such a ring and M will denote such an R-module. In commutative algebra, the concept of von Neumann regular ring (for short, vn-regular ring) and its generalizations have a significiant place. A ring R is called a *vn-regular ring* if for each $a \in R$, there exists $x \in R$ such that $a = a^2x$ [22]. Note that a ring R is a vn-regular ring if and only if for each $a \in R$, the principal ideal (a) is generated by an idempotent element $e \in R$, namely, $(a) = (e)$. R is called a *Baer* (sometimes called *PP* or *complemented*) *ring* if each annihilator $ann(a)$ of an element $a \in R$ is generated by an idempotent element $e \in R$ [8]. It is easy to see

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that every vn-regular ring is also a Baer ring but the converse is not true in general (just consider an integral domain which is not a field). Let R be a ring and $T(R)$ its total quotient ring. Then R is called a *quasi regular ring* if its total quotient ring $T(R)$ is a vn-regular ring [8]. In [8, Theorem 2.2], the author showed that a ring R is quasi regular if and only if it is a reduced ring satisfying the following property: for each $a \in R$, there exists $b \in R$ such that $ann(ann(a)) = ann(b)$. Also, R is called a *generalized morphic* (briefly, *g-morphic*) *ring* if each annihilator $ann(a)$ of an element $a \in R$ is a principal ideal, namely, $ann(a) = (b)$ for some $b \in R$ [23]. The notion of vn-regular ring and its above generalizations have been studied in many papers. See, for example, [1], [6], [7], [8], [9], [10], [11], [12], [13] and [14]. This paper aims to extend the notion of g-morphic ring to modules and to characterize some class of rings and modules in terms of g-morphic modules.

Now for the sake of completeness, we give some definitions and notations which will be followed in the sequel. Let M be an R-module, N, K be two submodules of M, and J be an ideal of R. The residual of N by K and J is defined as follows:

$$(N :_R K) = \{a \in R : aK \subseteq N\}$$
$$(N :_M J) = \{m \in M : Jm \subseteq N\}.$$

Particularly, we use $ann_R(K)$ and $ann_M(J)$ to denote $(0 :_R K)$ and $(0 :_M J)$, respectively. Also, for each $m \in M$, we use $ann_R(m)$ instead of $ann_R(Rm)$, where Rm is the cyclic submodule of M. Jayaram and Tekir, in their recent paper [10], extended the notion of idempotent element to modules and also they introduced and studied vn-regular modules. Let M be an R-module. An element $e \in R$ is called a *weak idempotent* element if $e - e^2 \in ann(M)$, or equivalently $em = e^2m$ for each $m \in M$. It is clear that all idempotents in R are weak idempotents and the converse holds provided that M is a *faithful module*, i.e., $ann(M) = 0$. An R-module M is said to be a *vn-regular* module if for each $m \in M$, there exists $a \in R$ such that $Rm = aM = a^2M$ [10]. By [10, Lemma 5], a finitely generated (briefly, f.g.) R-module M is vn-regular if and only if for each $m \in M$, the cyclic submodule $Rm = eM$ for some weak idempotent element $e \in R$. Afterwards, In [11], the authors introduced the notion of Baer modules in terms of weak idempotent elements: an R-module M is called a *Baer module* if for each $m \in M$, there exists a weak idempotent element $e \in R$ such that $ann_R(m)M = eM$. In [11], the authors gave many properties and characterizations of Baer modules. Also, the authors in [12], extended the property "$ann(ann(a)) = ann(b)$" in rings to modules as follows: an R-module M is called a *weak quasi regular module* if for each $m \in M$, there exists $a \in R$ such that $ann_M(ann_R(m)) = ann_M(a)$ [12]. In [11] and [12], they gave the relations between aforementioned class of modules as follows:

f.g. vn-regular module $\Rightarrow$ f.g Baer module $\Rightarrow$ weak quasi regular module

Now, we introduce a new class of modules which is an extension of g-morphic rings to modules. Let M be an R-module. Then M is called a *g-morphic module* if for each $m \in M$, there exists $a \in R$ such that $ann_R(m) = (a) + ann(M)$. Among other results in this paper, we show that the class of g-morphic modules is an intermediate class between f.g. Baer modules and weak quasi regular modules (See Proposition 2.1). We characterize g-morphic modules in terms of the factor ring $R/ann(M)$ (See Proposition 2.2, Proposition 2.3 and Theorem 2.1). Also, we investigate the behaviour of g-morhic modules under homomorphism, under localization, under idealization of a module, in direct product of modules, in direct summands of modules (See Proposition 2.4, Proposition 2.5, Proposition 2.7 and Proposition 2.6). We give a characterization of principal ideal rings in terms of g-morphic modules (See Theorem 2.2). Furthermore, we use the g-moprhic modules to characterize Baer modules (See Proposition 2.8, Proposition 2.9 and Theorem 2.3). Finally, in Section 3, we investigate the extension of g-morphic modules to polynomial modules and formal power series modules (See Theorem 3.2).

2 Characterization of generalized morphic modules

Definition 2.1. *Let M be an R-module. Then M is said to be a g-morphic module if for each $m \in M$, there exists $a \in R$ such that $ann_R(m) = (a) + ann(M)$.*

Example 2.1. *A ring R is a g-morphic ring if and only if R is a g-morphic R-module.*

Example 2.2. *Every torsion free module is a g-morphic module. Let M be a torsion free module and m a nonzero element of M. Then clearly $ann_R(m) = (0) + ann(M)$. Hence, M is a g-morphic module.*

Example 2.3. *Suppose that M is an R-module in which $ann(M) \in Max(R)$, where $Max(R)$ denotes the set of maximal ideals of R. Take an element $m \in M$. Since $ann(M) \subseteq ann_R(m)$ and $ann(M) \in Max(R)$, we can conclude either $ann_R(m) = (0) + ann(M)$ or $ann_R(m) = R = (1) + ann(M)$. Therefore M is a g-morphic module.*

Example 2.4. *Every simple module is a g-morphic module. Let M be a simple module and $0 \neq m \in M$. Then $Rm = M$ and thus $ann_R(m) = ann(M) = (0) + ann(M)$ which is needed.*

Example 2.5. *Let M be an R-module such that $R/ann(M)$ is a principal ideal ring. Take an element $m \in M$. Put $I = ann_R(m)/ann(M)$. Then I is a principal ideal so that $I = ann_R(m)/ann(M) = (a + ann(M))$ for some $a \in R$. Then we can conlude that $ann_R(m) = (a) + ann(M)$, that is, M is a g-morphic module. In particular, every module over a principal ideal ring is a g-morphic.*

Example 2.6. *Let $n \geq 2$ be an integer. Then $\mathbb{Z}$-module $\mathbb{Z}_n$ is a g-morphic module.*

Proposition 2.1. *(i) Every finitely generated Baer module is a g-morphic module.*
(ii) Every g-morphic module is a weak quasi regular module.

Proof. (i): Suppose that M is a finitely generated Baer module and take an element $m \in M$. Since M is a Baer module, there exists a weak idempotent $e \in R$ such that $ann_R(m)M = eM$. Then we can conclude that

$$M = eM + (1 - e)M = [ann_R(m) + (1 - e)]M.$$

By [2, Corollary 2.5], $ann_R(m) + (1 - e) = R$ and so $1 = r + s(1 - e)$ for some $r \in ann(m)$ and $s \in R$. Then $e = re + se(1 - e) \in ann_R(m)$ so that $(e) + ann(M) \subseteq ann_R(m)$. Now, let $x \in ann_R(m)$. Then $xM \subseteq ann_R(m)M = eM$ and thus $(1 - e)xM = 0$ and this yields that $(1 - e)x \in ann(M)$. Then we have $x = ex + (1 - e)x \in (e) + ann(M)$ and hence $ann_R(m) = (e) + ann(M)$. Therefore, M is a g-morphic module.

(ii) Let $m \in M$. By definition of g-morphic module, there exists $a \in R$ such that $ann_R(m) = (a) + ann(M)$ and thus

$$\begin{aligned} ann_M(ann_R(m)) &= ann_M((a) + ann(M)) \\ &= ann_M(a). \end{aligned}$$

Hence, M is a weak quasi regular module. □

The converse of previous proposition (i) is not always true. See the following example.

Example 2.7. *Consider $\mathbb{Z}$-rmodule $\mathbb{Z}_4$. Then by Example 2.6, it is a g-morphic module but not a Baer module.*

Lemma 2.1. *Let M be an R-module. Then M is a g-morphic module if and only if for each $m_1, m_2, \ldots, m_n \in M$, there exists $a \in R$ such that*

$$\bigcap_{i=1}^{n} ann_R(m_i) = (a) + ann(M).$$

Proof. The "if" part clearly shows that M is a g-morphic module. Assume that M is a g-morphic module. We use induction on n to show that for each $m_1, m_2, \ldots, m_n \in M$,

$$\bigcap_{i=1}^{n} ann_R(m_i) = (a) + ann(M)$$

for some $a \in R$. If $n = 1$, the claim follows from the fact that M is a g-morphic module. Let $n = 2$. Take two elements $m_1, m_2 \in M$. Since M is a g-morphic module, there exists $a_1, a_2, a \in R$ such that

$$\begin{aligned} ann_R(m_1) &= (a_1) + ann(M) \\ ann_R(m_2) &= (a_2) + ann(M) \\ ann_R(a_1 m_2) &= (a) + ann(M). \end{aligned}$$

Now, let $x \in ann_R(m_1) \cap ann_R(m_2)$. Then $x = a_1 y + z$ for some $y \in R$ and $z \in ann(M)$. Since $x \in ann_R(m_2)$, we obtain that $xm_2 = (a_1 y + z)m_2 = ya_1 m_2 = 0$ and so $y \in ann_R(a_1 m_2) = (a) + ann(M)$. This implies that $y = ra + s$ for some $r \in R$ and $s \in ann(M)$. Then we conclude that $x = ra_1 a + a_1 s + z \in (a_1 a) + ann(M)$. Also note that $(a_1 a) + ann(M) \subseteq (a_1) + ann(M) \subseteq ann_R(m_1)$. Since $(a) \subseteq ann_R(a_1 m_2)$, we have $(a_1 a) + ann(M) \subseteq ann_R(m_2)$. Then we have $ann_R(m_1) \cap ann_R(m_2) = (a_1 a) + ann(M)$ which shows the claim is true for $n = 2$. Now assume that the claim is true for all $k < n$. Take arbitrary elements $m_1, m_2, \ldots, m_n \in M$. By induction hypothesis

$$\begin{aligned} ann_R(Rm_1 + Rm_2 + \cdots + Rm_{n-1}) = \bigcap_{i=1}^{n-1} ann_R(m_i) &= (a'_1) + ann(M) \\ ann_R(m_n) &= (a'_2) + ann(M) \\ ann_R(a'_1 m_n) &= (a') + ann(M). \end{aligned}$$

Similar argument in the case $n = 2$ shows that $\bigcap_{i=1}^{n} ann_R(m_i) = (a'_1 a') + ann(M)$ which completes the proof. □

Proposition 2.2. *Let M be a finitely generated g-morphic module. Then $R/ann(M)$ is a g-morphic ring.*

Proof. Let M be a finitely generated g-morphic module. Put $R' = R/ann(M)$ and $\overline{a} = a + ann(M) \in R'$ for some $a \in R$. Then we can easily see that $ann_{R'}(\overline{a}) = ann(aM)/ann(M)$. Since M is a finitely generated module, we can write $M = \sum_{i=1}^{n} Rm_i$ for some $m_1, m_2, \ldots, m_n \in M$. This yields that

$ann(aM) = \bigcap_{i=1}^{n} ann_R(am_i)$. Since M is a g-morphic module, by Lemma 2.1, $ann(aM) = \bigcap_{i=1}^{n} ann_R(am_i) = (b) + ann(M)$ for some $b \in R$. Then we conclude that $ann_{R'}(\overline{a}) = [(b) + ann(M)]/ann(M) = (b + ann(M))$. Therefore, $R/ann(M)$ is a g-morphic ring. □

Recall from [11] that an R-module M is said to be a *weak multiplication* module if for each $m \in M$, $ann_R(m) = ann_R(IM)$ for some finitely generated ideal I of R. Note that every multiplication module and every torsion free module are a weak multiplication module so that the class of weak multiplication modules properly contain the class of multiplication modules and torsion free modules.

Proposition 2.3. *Let M be a weak multiplication module and $R/ann(M)$ be a g-morphic ring. Then M is a g-morphic module.*

Proof. Put $R' = R/ann(M)$ and take an element $m \in M$. Since M is a weak multiplication module, there exists a finitely generated ideal I of R such that $ann_R(m) = ann(IM)$. Then we get $I = \sum_{i=1}^{n} Ra_i$ for some $a_1, a_2, \ldots, a_n \in R$. This implies that $ann(IM) = \bigcap_{i=1}^{n} ann(a_iM)$. Also, note that for each $a_i \in R$, $ann_{R'}(\overline{a_i}) = ann(a_iM)/ann(M)$, where $\overline{a_i} = a_i + ann(M)$. As R' is a g-morphic ring, by Lemma 2.1,

$$\bigcap_{i=1}^{n} ann_{R'}(\overline{a_i}) = \left[\bigcap_{i=1}^{n} ann(a_iM)\right]/ann(M) = (\overline{b})$$

for some $\overline{b} = b + ann(M)$. This gives $\bigcap_{i=1}^{n} ann(a_iM) = ann_R(m) = (b) + ann(M)$, namely, M is a g-morphic module. □

Theorem 2.1. *Let M be a f.g. weak multiplication module. The following statements are equivalent.*

(i) M is a g-morphic module.

(ii) $R/ann(M)$ is a g-morphic ring.

Proof. $(i) \Leftrightarrow (ii)$: Follows from Proposition 2.2 and Proposition 2.3. □

Proposition 2.4. *(i) Let $f : M_1 \to M_2$ be a monomorphism and M_2 a g-morphic R-module. Then M_1 is a g-morphic R-module.*

(ii) Every submodule of a g-morphic module is a g-morphic.

(iii) Let M be a g-morphic module and $S \subseteq R$ a multiplicatively closed subset of R. Then $S^{-1}M$ is a g-morphic $S^{-1}R$-module.

Proof. (i) Suppose that $m_1 \in M_1$. Since M_2 is a g-morphic R-module and $f(m_1) \in M_2$, there exists $x \in R$ such that $ann_R(f(m_1)) = (x) + ann(M_2)$. Since f is monomorphism, $ann(M_2) \subseteq ann(M_1)$ and $ann_R(f(m_1)) = ann_R(m_1)$ and this yields that $ann_R(m_1) = (x) + ann(M_2) \subseteq (x) + ann(M_1)$. Also note that $xf(m_1) = f(xm_1) = 0$ and thus $xm_1 = 0$ so that $(x) + ann(M_1) \subseteq ann_R(m_1)$. Then we can conclude that $ann_R(m_1) = (x)+ann(M_1)$. Therefore M_1 is a g-morphic R-module.

(ii) Follows from (i).

(iii) Let $\frac{m}{s} \in S^{-1}M$. Then it is clear that $ann_{S^{-1}R}(\frac{m}{s}) = S^{-1}\left(ann_R(m)\right)$. Since M is a g-morphic module, there exists $a \in R$ such that $ann_R(m) = (a) + ann(M)$ and so

$$\begin{aligned} ann_{S^{-1}R}(\frac{m}{s}) &= S^{-1}\left((a) + ann(M)\right) \\ &= S^{-1}((a)) + S^{-1}(ann(M)) \\ &\subseteq (\frac{a}{1}) + ann_{S^{-1}R}(S^{-1}M). \end{aligned}$$

On the other hand $(a) \subseteq ann_R(m)$ and so $(\frac{a}{1}) \subseteq S^{-1}\left(ann_R(m)\right) = ann_{S^{-1}R}(\frac{m}{s})$ and this yields that $(\frac{a}{1}) + ann_{S^{-1}R}(S^{-1}M) \subseteq ann_{S^{-1}R}(\frac{m}{s})$, whence $ann_{S^{-1}R}(\frac{m}{s}) = (\frac{a}{1}) + ann_{S^{-1}R}(S^{-1}M)$. Therefore, $S^{-1}M$ is a g-morphic $S^{-1}R$-module. □

Proposition 2.5. *Let M_i be an R_i-module for each $i \in \Delta$. Suppose that $R = \prod_{i\in\Delta} R_i$ and $M = \prod_{i\in\Delta} M_i$. Then the following statements are equivalent.*

(i) M is a g-morphic R-module.

(ii) M_i is a g-morphic R_i-module for each $i \in \Delta$.

Proof. $(i) \Rightarrow (ii)$: Suppose that M is a g-morphic R-module and take $i_0 \in \Delta$. Let $m_{i_0} \in M_{i_0}$. Now put

$$m_i = \begin{cases} m_{i_0}\ ;\ i = i_0 \\ 0\ ;\ i \neq i_0 \end{cases}$$

and $m = (m_i)_{i\in\Delta} \in M$. Since M is a g-morphic R-module, we can conclude that $ann_R(m) = (x)+ann(M)$ for some $x \in R$. Assume that $x = (x_i)_{i\in\Delta}$ and note that $ann_R(m) = \prod_{i\in\Delta} ann_{R_i}(m_i)$, $ann(M) = \prod_{i\in\Delta} ann_{R_i}(M_i)$ and also

$(x) = \prod_{i\in\Delta} R_i x_i$. Then we have

$$\begin{aligned} ann_R(m) &= \prod_{i\in\Delta} ann_{R_i}(m_i) \\ &= \prod_{i\in\Delta} R_i x_i + \prod_{i\in\Delta} ann_{R_i}(M_i) \\ &= \prod_{i\in\Delta} [R_i x_i + ann_{R_i}(M_i)] \end{aligned}$$

and this yields $ann_{R_{i_0}}(m_{i_0}) = R_{i_0}x_{i_0} + ann_{R_{i_0}}(M_{i_0})$. Therefore, M_{i_0} is a g-morphic R_{i_0}-module.

$(ii) \Rightarrow (i)$: Let M_i be a g-morphic R_i-module for each $i \in \Delta$. Take an element $m = (m_i)_{i\in\Delta} \in M$. Since M_i is a g-morphic R_i-module, there exists $x_i \in R_i$ such that $ann_{R_i}(m_i) = (x_i) + ann_{R_i}(M_i)$. This implies that

$$\begin{aligned} ann_R(m) &= \prod_{i\in\Delta} ann_{R_i}(m_i) \\ &= \prod_{i\in\Delta} [(x_i) + ann_{R_i}(M_i)] \\ &= \prod_{i\in\Delta} (x_i) + \prod_{i\in\Delta} ann_{R_i}(M_i) \\ &= R(x_i)_{i\in\Delta} + ann(M). \end{aligned}$$

Hence, M is a g-morphic R-module. □

Proposition 2.6. *Let $M = \bigoplus_{i\in\Delta} N_i$ be a direct summand of a family of faithful R-modules. Then the following statements are equivalent.*

(i) M is a g-morphic R-module.

(ii) N_i is a g-morphic R-module for each $i \in \Delta$.

Proof. $(i) \Rightarrow (ii)$: Suppose that M is a g-morphic R-module. Then by Proposition 2.4 (ii), N_i is a g-morphic R-module for each $i \in \Delta$.

$(ii) \Rightarrow (i)$: Suppose that N_i is a g-morphic R-module for each $i \in \Delta$. Since N_i is a faithful module, M is a faithful module. Take an element $m \in M$. Then by direct summand, $m = m_{i_1} + m_{i_2} + \cdots + m_{i_n}$ for some $m_{i_k} \in N_{i_k}$. Take an element $r \in ann_R(m)$. Then $rm = r(m_{i_1} + m_{i_2} + \cdots + m_{i_n}) = 0$ and so $rm_{i_1} = -(rm_{i_2} + \cdots + rm_{i_n}) \in N_{i_1} \cap (N_{i_2} + \ldots + N_{i_n}) = 0$. This yields that $r \in ann_R(m_{i_1})$. Similar argument shows that $r \in ann_R(m_{i_k})$ for each $k = 1, 2, \ldots, n$. Then we can conclude that $ann_R(m) = \bigcap_{k=1}^{n} ann_R(m_{i_k})$. As N_i is a g-morphic faithful R-module, there exists $x_k \in R$ such that $ann_R(m_{i_k}) = (x_k)$. Similar argument in the proof of Lemma 2.1, $ann_R(m) = (x)$ for some $x \in R$. Hence, M is a g-morphic R-module. □

Corollary 2.1. *Let $\{M_i\}_{i\in\Delta}$ be a family of faithful R-modules and $M = \prod_{i\in\Delta} M_i$, where Δ is a finite index set. Then the following statements are equivalent.*

(i) M is a g-morphic R-module.

(ii) M_i is a g-morphic R-module for each $i \in \Delta$.

Proof. $(i) \Leftrightarrow (ii)$: Follows from the fact that a direct product of R-modules is isomorphic to a direct summand of R-modules and Proposition 2.6. □

Now, we characterize the Principal ideal rings in terms of g-morphic modules.

Theorem 2.2. *The following statements are equivalent for any commutative ring R.*

(i) R is a principal ideal ring.

(ii) Every R-module is a g-morphic module.

Proof. $(i) \Rightarrow (ii)$: It follows from Example 2.5.

$(ii) \Rightarrow (i)$: Suppose that every module over R is a g-morphic module. Let Q be an ideal of R. Now, we will show that $Q = (x)$ for some $x \in Q$. Put $R' = R \times (R/Q)$. Then by assumption R' is g-morphic R-module. First note that $ann_R(R') = 0$. Let $m = (0, \overline{1}) \in R'$. Since R' is a g-morphic R-module, there exists $x \in R$ such that $ann_R(m) = Q = (x) + ann_R(R') = (x)$. Hence, R is a principal ideal ring. □

Let M be an R-module. The idealization or trivial extension $R \propto M = \{(r, m) : r \in R, m \in M\}$ of M is a commutative ring with componentwise addition and the multiplication $(a, m)(b, m') = (ab, am' + bm)$ for each $a, b \in R$; $m, m' \in M$ [21]. In [9, Theorem 3.1], the authors showed that if $R \propto M$ is a g-morphic ring, then R is a g-morphic ring. Now, we say a lot more than [9, Theorem 3.1] in the next proposition.

Proposition 2.7. *Suppose that $R \propto M$ is a g-morphic ring. Then R is a g-morphic ring and M is a g-morphic R-module.*

Proof. Assume that $R \propto M$ is a g-morphic ring. Then by [9, Theorem 3.1], R is a g-morphic ring. Now we will show that M is a g-morphic R-module. Let $m \in M$. Put $m^* = (0, m) \in R \propto M$. Then we have $ann_{R\propto M}(m^*) = (R \propto M)(r, m')$ for some $(r, m') \in R \propto M$. Then we conclude that $(r, m')(0, m) = (0, rm) = (0, 0)$ and also $(r) + ann(M) \subseteq ann_R(m)$. Take an element $t \in ann_R(m)$. Then $(t, 0)(0, m) = (0, 0)$ and thus $(t, 0) \in ann_{R\propto M}(m^*) = (R \propto M)(r, m')$. This gives $(t, 0) = (r, m')(x, m'')$ for some $(x, m'') \in R \propto M$. Then we have $t = rx \in (r)$ and so $ann_R(m) = (r) + ann(M)$. Hence, M is a g-morphic R-module. □

Let M be an R-module. Then the polynomial module over the polynomial ring $R[X]$ in indeterminate X is denoted by $M[X]$. Recall that an R-module M is said to be a *reduced module* if for each $a \in R$, $m \in M$ and whenever $a^2m = 0$ then $am = 0$ [16].

Proposition 2.8. *Let M be a f.g. Baer R-module, $R_n = R[X]/(X^{n+1})$ and $M_n = M[X]/(X^{n+1})$. Then M_n is a g-morphic R_n-module for each $n \in \mathbb{N}$.*

Proof. Suppose that M is a f.g. Baer module. Take an element $m^*(x) \in M_n$. Then $m^*(x) = m_0 + m_1X + m_2X^2 + \ldots + m_nX^n + (X^{n+1})$ for some $m_i \in M$. By [11, Proposition 1], we know that M is a reduced module. Take an element $r^*(x) = r_0+r_1X+r_2X^2+\ldots+r_nX^n+(X^{n+1}) \in ann_{R_n}(m^*(x))$. Then we conculde that $(r_0+r_1X+r_2X^2+\cdots+r_nX^n+(X^{n+1}))(m_0+m_1X+m_2X^2+\cdots+m_nX^n+(X^{n+1})) = 0_{M_n}$. This yields that

$$\begin{aligned} r_0m_0 &= 0 \\ r_0m_1 + r_1m_0 &= 0 \\ &\cdots \\ r_0m_n + r_1m_{n-1} + \cdots + r_nm_0 &= 0 \end{aligned}$$

Then we have $r_0(r_0m_1 + r_1m_0) = r_0^2m_1 + r_1r_0m_0 = 0$ and so $r_0^2m_1 = 0$. Since M is a reduced module, we conclude that $r_0m_1 = 0$. Similar argument shows that $r_j \in \bigcap_{i=1}^{n} ann_R(m_i)$ for all $j = 0, 1, \ldots, n$. As M is a f.g. Baer module, similar arguing in the proof of Proposition 2.1, $ann_R(m_i) = (e_i) + ann(M)$ for some weak idemptent element $e_i \in R$. Then note that $\bigcap_{i=1}^{n} ann_R(m_i) = (e) + ann(M)$ where $e = e_0e_1\cdots e_n$. Also, one can observe that $ann_{R_n}(m^*(x)) = [\bigcap_{i=1}^{n} ann_R(m_i)][X]/(X^{n+1}) = [(e) + ann(M)][X]/(X^{n+1})$. Now, put $e^*(x) = e \in R$. Then we can conclude that $ann_{R_n}(m^*(x)) = e^*(x)R[X]/(X^{n+1}) + ann_{R_n}(M_n)$. Hence, M_n is a g-morphic R_n-module. $\square$

Proposition 2.9. *Let M be an R-module. Suppose that $M_n = M[X]/(X^{n+1})$ is a g-morphic $R_n = R[X]/(X^{n+1})$-module for each $n \in \mathbb{N}$. Then M is a Baer module.*

Proof. Suppose that $M_n = M[X]/(X^{n+1})$ is a g-morphic $R_n = R[X]/(X^{n+1})$-module. In particular, $M_1 = M[X]/(X^2)$ is a g-morphic $R_1 = R[X]/(X^2)$-module. Take an element $m \in M$. Put $m^*(x) = mX + (X^2) \in M_1$. Then by assumption, there exists $r^*(x) = r_0+r_1X+(X^2) \in R_1$ such that $ann_{R_1}(m^*(x)) = (r^*(x)) + ann_{R_1}(M_1)$. Also note that $ann_{R_1}(M_1) = [Ann(M)][X]/(X^2)$. Since $r^*(x) \in ann_{R_1}(m^*(x))$, we have $(r_0 + r_1X + (X^2))(mX + (X^2)) = r_0mX +$

$(X^2) = 0_{M_1}$. Then we can conclude that $(r_0) + ann(M) \subseteq ann_R(m)$. Let $t \in ann_R(m)$. Put $t^*(x) = t + (X^2)$. Then we have $t^*(x) \in ann_{R_1}(m^*(x))$ and so $t^*(x) = (r_0 + r_1X + (X^2))(c_0 + c_1X + (X^2)) + (y_0 + y_1X + (X^2))$ for some $c_0, c_1 \in R$ and $y_0, y_1 \in ann(M)$. This implies that $t = r_0c_0 + y_0 \in (r_0)+ann(M)$ and thus we conclude that $ann_R(m) = (r_0)+ann(M)$. Also, as $X + (X^2) \in ann_{R_1}(m^*(x))$, we can write $X + (X^2) = (r_0 + r_1X + (X^2))(c'_0 + c'_1X+(X^2))+(y'_0+y'_1X+(X^2))$ for some $c'_0, c'_1 \in R$ and $y'_0, y'_1 \in ann(M)$. This yields that

$$r_0c'_0 + y'_0 = 0$$
$$r_0c'_1 + r_1c'_0 + y'_1 = 1$$

Multplying the second equality by r_0, we have $r_0 - r_0^2c'_1 \in ann(M)$ and so $r_0c'_1 - (r_0c'_1)^2 \in ann(M)$, that is, $r_0c'_1 = e'$ is a weak idempotent element of R. Also one can see that $(r_0) + ann(M) = (e') + ann(M)$. Then we have $ann_R(m) = (e')+ann(M)$ and this yields $ann_R(m)M = e'M$. Therefore, M is a Baer module. □

Next , we give a characterization of Baer modules in terms of g-morphic modules.

Theorem 2.3. *Let M be a finitely generated R-module, $R_n = R[X]/(X^{n+1})$ and $M_n = M[X]/(X^{n+1})$. Then the following statements are equivalent.*
(i) M is a Baer module.
(ii) M_n is a g-morphic R_n-module for each $n \in \mathbb{N}$.

Proof. $(i) \Leftrightarrow (ii)$: Follows from Proposition 2.8 and Proposition 2.9. □

3 Extension of generalized morphic modules

Recall that an R-module M is an *Armendariz module* if for each $f(x) = a_0 + a_1X + \cdots + a_nX^n \in R[X]$ and $m(x) = m_0 + m_1X + \cdots + m_kX^k \in M[X]$ such that $f(x)m(x) = 0$, then $a_i \in ann_R(m_j)$ for each $0 \leq i \leq n$ and $0 \leq j \leq k$. Note that all reduced modules are Armendariz [4].

Proposition 3.1. *Let M be an R-module. Then the following statements are satisfied.*
(i) If $M[X]$ is a g-morphic $R[X]$-module, then M is a g-morphic R-module.
(ii) If M is an Armendariz g-morphic module, then $M[X]$ is a g-morphic $R[X]$-module.

Proof. (i): Suppose that $M[X]$ is a g-morphic $R[X]$-module and take an element $m \in M$. Put $m^*(x) = m \in M[X]$. Then note that $ann_{R[X]}(m^*(x)) = [ann_R(m)][X]$ and $ann_{R[X]}(M[X]) = [ann(M)][X]$. As $M[X]$ is a g-morphic $R[X]$-module, there exists $f(x) = a_0 + a_1X + \cdots + a_nX^n \in R[X]$ such that $ann_{R[X]}(m^*(x)) = (f(x)) + ann_{R[X]}(M[X])$. Then we obtain $f(x)m^*(x) = 0$ and so $a_0m = 0$. This yields that $(a_0) + ann(M) \subseteq ann_R(m)$. Now, let $t \in ann_R(m)$. Now put $t^*(x) = t \in R[X]$ and note that $t^*(x) \in ann_{R[X]}(m^*(x))$. Then there exists $g(x) = b_0 + b_1X + \cdots + b_mX^m \in R[X]$ and $h(x) = c_0 + c_1X + \cdots + c_kX^k \in [ann(M)][X]$ such that $t^*(x) = f(x)g(x) + h(x)$ and so $t = a_0b_0 + c_0 \in (a_0) + ann(M)$. Therefore, we get $ann_R(m) = (a_0) + ann(M)$, whence M is a g-morphic R-module.

(ii) Suppose that M is an Armendariz g-morphic module and take an element $m(x) = m_0 + m_1X + \cdots + m_kX^k \in M[X]$. Since M is an Armendariz module, it is easy to see that

$$ann_{R[X]}(m(x)) = [\bigcap_{i=0}^{k} ann_R(m_i)][X].$$

As M is a g-morphic module, by Lemma 2.1, $\bigcap_{i=0}^{k} ann_R(m_i) = (a) + ann(M)$ for some $a \in R$. This implies that

$$ann_{R[X]}(m(x)) = [(a) + ann(M)][X]$$

Now put $\alpha(x) = a \in R[X]$. Then note that $ann_{R[X]}(m(x)) = (\alpha(x)) + ann_{R[X]}(M[X])$.
Therefore, $M[X]$ is a g-morphic $R[X]$-module. □

Theorem 3.1. *Let M be an Armendariz R-module. Then M is a g-morphic R-module if and only if $M[X]$ is a g-morphic $R[X]$-module.*

Proof. Follows from Proposition 3.1. □

Definition 3.1. *Let M be an R-module. M is said to satisfy $(*)$-condition if for each countable subset $\{m_i\}_{i\in\mathbb{N}}$, there exists a finite subset $\{m'_1, m'_2, \ldots, m'_n\} \subseteq M$ such that*

$$\bigcap_{i\in\mathbb{N}} ann_R(m_i) = \bigcap_{i=1}^{n} ann_R(m'_i).$$

Note that every module over artinian ring satisfies $(*)$-property. Let M be an R-module. $M[[X]]$ denotes the formal power series module over formal power series ring $R[[X]]$. Then an R-module M is called a *ps-Armendariz module* if for each $f(x) = \sum_{i=0}^{\infty} a_iX^i \in R[[X]]$ and $m(x) = \sum_{i=0}^{\infty} m_iX^i \in M[[X]]$ such that $f(x)m(x) = 0$, then $a_j \in \bigcap_{i=0}^{\infty} ann_R(m_i)$ for all $j \geq 0$. Note that every ps-Armendariz module is an Armendariz module.

Proposition 3.2. *(i) Let $M[[X]]$ be a g-morphic $R[[X]]$-module. Then M is a g-morphic R-module.*

(ii) Let M be a ps-Armendariz module satisfying $(*)$-property. If M is a g-morphic R-module, then $M[[X]]$ is a g-morphic $R[[X]]$-module.

Proof. (i) Assume that $M[[X]]$ is a g-morphic $R[[X]]$-module and take an element $m \in M$. Put $m^*(x) = m \in M[[X]]$. Then by assumption, there exists an $f(x) = \sum_{i=0}^{\infty} a_i X^i \in R[[X]]$ such that $ann_{R[[X]]}(m^*(x)) = (f(x)) + ann_{R[[X]]}(M[[X]])$. Note that $ann_{R[[X]]}(M[[X]]) = [ann(M)][[X]]$ and $ann_{R[[X]]}(m^*(x)) = [ann_R(m)][[X]]$. Since $f(x) \in ann_{R[[X]]}(m^*(x))$, we have $a_0 \in ann_R(m)$ and thus $(a_0)+ann(M) \subseteq ann_R(m)$. Now, let $t \in ann_R(m)$. Put $t^*(x) = t \in R[[X]]$. Then it is clear that $t^*(x) \in ann_{R[[X]]}(m^*(x))$ and so $t^*(x) = f(x)g(x) + h(x)$ for some $g(x) = \sum_{i=0}^{\infty} b_i X^i \in R[[X]]$ and $h(x) = \sum_{i=0}^{\infty} c_i X^i \in [ann(M)][[X]]$. This yields that $t = a_0 b_0 + c_0 \in (a_0) + ann(M)$ and therefore $ann_R(m) = (a_0) + ann(M)$. Hence, M is a g-morphic R-module.

(ii) Suppose that M is a g-morphic ps-Armendariz module satisfying $(*)$-property. Take an element $m(x) = \sum_{i=0}^{\infty} m_i X^i \in M[[X]]$. Since M is a ps-Armendariz module, it is clear that

$$ann_{R[[X]]}(m(x)) = [\bigcap_{i=0}^{\infty} ann_R(m_i)][[X]]$$

Since M satisfies $(*)$-property, there exists a finite subset $\{m'_1, m'_2, \ldots, m'_n\} \subseteq M$ such that

$$\bigcap_{i=0}^{\infty} ann_R(m_i) = \bigcap_{i=1}^{n} ann_R(m'_i).$$

Then we can conclude that $ann_{R[[X]]}(m(x)) = [\bigcap_{i=1}^{n} ann_R(m'_i)][[X]]$. As M is a g-morphic R-module, by Lemma 2.1, $\bigcap_{i=1}^{n} ann_R(m'_i) = (a) + ann(M)$ for some $a \in R$. Now put $\alpha(x) = a \in R[[X]]$. Then we can conclude that

$$\begin{aligned} ann_{R[[X]]}(m(x)) &= [(a) + ann(M)][[X]] \\ &= (\alpha(x)) + ann_{R[[X]]}(M[[X]]). \end{aligned}$$

Therefore, $M[[X]]$ is a g-morphic $R[[X]]$-module. □

Theorem 3.2. *Let M be a ps-Armendariz module satisfying the $(*)$-condition. Then the following statements are equivalent.*

(i) M is a g-morphic R-module.

(ii) $M[X]$ is a g-morphic $R[X]$-module.

(iii) $M[[X]]$ is a g-morphic $R[[X]]$-module.

Proof. $(i) \Leftrightarrow (ii)$: Follows from Proposition 3.1.

$(i) \Leftrightarrow (iii)$: Follows from Proposition 3.2. □

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