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Finite-dimensional flexible algebras associated with directed and weighted CW complexes

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Abstract

In this paper, we study a link between directed and weighted CW complexes (also called configurations) and flexible algebras determining which configurations are associated with those algebras. Some important elements that can be read from the (pseudo)digraph that is associated with a flexible algebra are studied. Moreover, the isomorphism classes of each 2-dimensional configuration associated with these algebras is analyzed, providing a new method to classify them. In order to complement the theoretical study, two algorithmic methods are implemented: the first one checks if a given directed and weighted CW complex is associated or not with a flexible algebra, while the second one constructs and draws the (pseudo)digraph associated with a given flexible algebra.

1 Introduction

Nowadays, one of the most important and stimulating research in Mathematics is finding and studying new links between different fields. Alternative techniques and procedures allow researchers to solve many unsolved problems, improve known theories and achieve new results. This paper deals with the relation between Graph Theory and flexible algebras. More concretely, the main goal is to make progress on the research line started in [2, 5], where a

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mapping between Lie algebras and directed CW complexes was introduced in order to translate properties of Lie algebras into the language of Graph Theory and vice versa. Now, the main goal is to obtain an analogous mapping for flexible algebras.

Non-associative algebras have been deeply studied due to their own theoretical importance and their many applications to different fields like Physics, Engineering or Applied Mathematics [11, 16]. A particular type of these algebras is formed by flexible algebras. There exists a close relation between flexible algebras and other types of algebras. In this sense, every Lie, Jordan, associative or alternative algebra is flexible. The first papers dealing with flexible algebras were written by Oehmke and Schafer in 1954 and 1958. In [22], Schafer studied the algebras generated by the Cayley-Dickson process over a field and proved that they satisfy the flexible identity. In [19], Oehmke studied several properties on flexible algebras. More recently, Pumpln [21] has analyzed algebraic constructions that yield to flexible quadratic algebras and Behn Correa and Hentzel in [3] have studied flexible algebras satisfying the polynomial identity $x(yz) = y(zx)$. Flexible algebras have also been investigated in terms of degree of algebras [14, 17]. A very important example of flexible algebras is formed by the octonions, which are a normed division algebra over the real number field. They are non-commutative, non-associative and flexible. Octonions have applications in many different fields such as string theory, special relativity and quantum logic (see [1]). Other characterizations and applications of these algebras can be found in [13, 15] and references therein.

Currently, Graph Theory has become a very useful tool to deal with a wide range of problems in many research fields. This theory may be used to study non-associative algebras in general and flexible algebras in particular. Concerning flexible algebras, there is no reference in the literature about the study of the link among graph theory and these algebras. However, this theory has been essential in order to study other non-associative algebras such as Lie, Leibniz, Malcev, Zinbiel and evolution algebras. For instance, in case of Lie algebras, trees perform an important role to determine the Dynkin diagrams associated to such algebras [23] and graphs are used to represent Lie algebras [20]. Leibniz algebras have also been studied and classified starting from their associated graphs [6]. A similar study was done for Malcev and Zinbiel algebras [7, 9]. Another example is the use of finite connected bipartite graph to construct finite-dimensional indecomposable semisimple Leibniz algebras [25]. One of the most important types of non-associative algebras are evolution algebras. There are several papers dealing with the link between these algebras and graphs [8, 10, 24, 4]. In those papers graphs are used in order to study several properties of their associated evolution algebra. Finally, in [18], the authors considered to solve some open problems related to graphicable algebras.

This paper is organized as follows: in Section 2, some well-known concepts on Graph Theory and flexible algebras are recalled. An algorithmic procedure to associate directed and weighted CW complexes with flexible algebras and vice versa is developed in Section 3. Next, Section 4 shows some properties flexible algebras that can be read from its associated directed and weighted CW complex such that the center and derived algebra. Section 5 studies the structure of (pseudo)digraphs associated with flexible algebras and some of their properties. For each configuration, the type of flexible algebra considering solvability and nilpotency is analyzed. Section 6 is devoted to determining the isomorphism classes of the 2-dimensional algebras obtained in the previous section. Section 7 shows the implementation of the two algorithmic methods used in the previous sections. The first one is designed to check if a given directed and weighted CW complex is associated or not with a flexible algebra and the second one draws the (pseudo)digraph, if possible, associated with a given finite-dimensional flexible algebra. Moreover, a brief computational study, showing the complexity order and computing time of the routines of the algorithm is given. At the end of the paper, there is a conclusion section, acknowledgements and references. Finally, and in order to make the paper more legible, an appendix section in included, which contains several lists of restrictions from some results of Sections 4 and 6.

2 Preliminaries

This section recalls some preliminary concepts, results and notations about flexible algebras, Graph Theory and CW complexes. Concerning the former, the reader can consult [19]. Regarding the latters, [12] is an introductory reference to Graph Theory and CW complexes were introduced by J.H.C. Whitehead [27].

2.1 Flexible algebras

Let $\mathbb{K}$ be a field. A *flexible algebra* $\mathcal{F}$ is a vector space over $\mathbb{K}$ with a second bilinear inner composition law ($[\cdot,\cdot]$), called the *bracket product* or *commutator*, which satisfies

$$[X,[Y,X]] = [[X,Y],X]$$

This is known as the flexible identity and we will use the following notation: $F(X,Y) = [X,[Y,X]] - [[X,Y],X]$. Given a basis $\{e_i\}_{i=1}^n$ of $\mathcal{F}$, its *structure (or Maurer-Cartan) constants* are defined by the coefficients $c_{i,j}^h$ which determines the law of the algebra: $[e_i,e_j] = \sum_{h=1}^n c_{i,j}^h e_h$, for $1 \le i < j \le n$.

Given a flexible algebra $\mathcal{F}$, its *center* is defined as

$$Z(\mathcal{F})=\{X\in\mathcal{F}\mid[X,Y]=0,\,\forall\,Y\in\mathcal{F}\}.$$

The flexible algebra $\mathcal{F}$ is *abelian* if $Z(\mathcal{F})$ and $\mathcal{F}$ are isomorphic. In that case, $\mathcal{F}$ is called zero or trivial algebra.

The *derived series* of a given finite-dimensional flexible algebra $\mathcal{F}$ is

$$\mathcal{C}_1(\mathcal{F})=\mathcal{F},\ \mathcal{C}_2(\mathcal{F})=[\mathcal{F},\mathcal{F}],\ \ldots,\ \mathcal{C}_k(\mathcal{F})=[\mathcal{C}_{k-1}(\mathcal{F}),\mathcal{C}_{k-1}(\mathcal{F})],\ \ldots$$

We say that $\mathcal{F}$ is *solvable* if there exists $m\in\mathbb{N}$, $m>1$ such that $\mathcal{C}_m(\mathcal{F})=\{0\}$. In addition, if $\mathcal{C}_{m-1}(\mathcal{F})\neq\{0\}$ also holds, then m is known as the *solvability index* or *solvindex* of $\mathcal{F}$ and it is said that $\mathcal{F}$ is $(m-1)$-step solvable.

The *central series* of a given finite-dimensional flexible algebra $\mathcal{F}$ is

$$\mathcal{C}^1(\mathcal{F})=\mathcal{F},\ \mathcal{C}^2(\mathcal{F})=[\mathcal{F},\mathcal{F}],\ \ldots,\ \mathcal{C}^k(\mathcal{F})=[\mathcal{C}^{k-1}(\mathcal{F}),\mathcal{F}],\ \ldots$$

We say that $\mathcal{F}$ is *nilpotent* if there exists $m\in\mathbb{N}$, $m>1$ such that $\mathcal{C}^m(\mathcal{F})=\{0\}$. In addition, if $\mathcal{C}^{m-1}(\mathcal{F})\neq\{0\}$ also holds, then m is known as the *nilpotency index* or *nilindex* of $\mathcal{F}$ and it is said that $\mathcal{F}$ is $(m-1)$-step nilpotent.

Notice that every nilpotent algebra is trivially solvable because $\mathcal{C}_i(\mathcal{F})\subseteq\mathcal{C}^i(\mathcal{F})$ for all $i\in\mathbb{N}$.

The derived algebra of a flexible algebra $\mathcal{F}$ will be denoted by $D\mathcal{F}=\mathcal{C}_2(\mathcal{F})=\mathcal{C}^2(\mathcal{F})$. A flexible algebra $\mathcal{F}$ is perfect if $\mathcal{F}$ and $D\mathcal{F}$ are isomorphic.

2.2 Graph Theory and CW complexes

A *digraph* consists of an ordered pair $G=(V,E)$, where V is a non-empty set called vertex-set and E is a set of ordered pairs (edges) of two vertices, called edge-set. It is possible to associate a *weight* to each edge. In that case G will be a weighted digraph.

A *loop* in the digraph $G=(V,E)$ is an edge that connects a vertex with itself. If the digraph G contains loops, then G is called a *pseudodigraph*. A vertex $v\in V$ is called *simple* if there is no loop on this vertex.

Throughout the paper, weighted (pseudo)digraphs will be considered.

Given a (pseudo)digraph $G=(V,E)$, a *sub(pseudo)digraph* $G'=(V',E')$ of G is a (pseudo)digraph verifying $V'\subseteq V$ and $E'\subseteq E$. A sub(pseudo)digraph H is said to be *induced* by a vertex-subset $V(H)$ in G if the edge-set of H consists of all the edges of G between two vertices in $V(H)$.

Two vertices $u,v\in V$ are adjacent if there is an edge from vertex u to v or viceversa. In that case, we will say that those vertices are incident with that edge. A vertex $v\in V$ is a *sink* (resp. a *source*) if each edge incident with v is oriented towards v (resp. from v). See Figure 1.

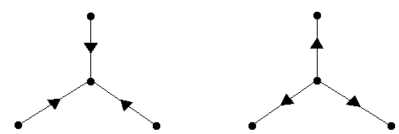

Figure 1: Example of sink and source, respectively.

A sequence of consecutive vertices and edges in a digraph is known as a *walk*. A (pseudo)digraph is *connected* if there is a walk between any pair of vertices. Otherwise, we will say that the digraph is non-connected.

A *CW complex* is a topologycal space built out of smaller spaces iteratively by a process called attaching cells, where a k-cell is a k-dimensional disc

$$D^k = \{x \in \mathbb{R}^k \, : \, |x| \leq 1\}$$

A CW complex is directed when we establish a direction in the elements that formed the complex. In case that we have some weight over the edges of that complex, we call it a directed and weighted CW complex. This class of spaces generalizes simplicial complexes and retains a combinatorial structure nature. Discrete points, digraphs and directed full triangles (in the sense that will be seen in Section 3) are examples of 0,1 and 2-dimensional directed CW complexes, respectively.

3 Associating flexible algebras with directed and weighted CW complexes

Let $\mathcal{F}$ be an n-dimensional flexible algebra with basis $\mathcal{B} = \{e_i\}_{i=1}^n$. The structure constants are given by $[e_i, e_j] = \sum_{h=1}^n c_{i,j}^h e_h$ and, hence, the pair $(\mathcal{F}, \mathcal{B})$ is associated with an directed and weighted CW complex by using the following procedure.

a) Draw a vertex i for each vector $e_i \in \mathcal{B}$.

b) For every vertex i verifying $[e_i, e_i] \neq 0$, draw a loop such that its weight is the n-tuple $(c_{i,i}^1, c_{i,i}^2, \ldots, c_{i,i}^n)$. See Figure 2.

c) Given two positive integers $i < j \leq n$ verifying $(c_{i,j}^j, c_{j,i}^j) \neq (0,0)$, draw a directed edge from vertex i to j whose weight is given by the pair $(c_{i,j}^j, c_{j,i}^j)$. Draw another directed edge with weight $(c_{i,j}^i, c_{j,i}^i)$, but now from j to i, in case that $(c_{i,j}^i, c_{j,i}^i) \neq (0,0)$. See Figure 3.

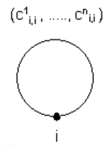

Figure 2: Loop over vertex i.

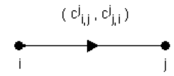

Figure 3: Directed edge.

d) Given three positive integers $i < j < k \leq n$ such that $(c^k_{i,j}, c^k_{j,i}, c^i_{j,k}, c^i_{k,j}, c^j_{i,k}, c^j_{k,i}) \neq (0,0,0,0,0,0)$, draw a full triangle ijk such that the edges ij, jk and ik have weights $(c^k_{i,j}, c^k_{j,i})$, $(c^i_{j,k}, c^i_{k,j})$ and $(c^j_{i,k}, c^j_{k,i})$, respectively; see Figure 4. Moreover,

  d1) a discontinuous line (named *ghost edge*) will be used for edges with weight $(0,0)$.

  d2) If two triangles ijk and ijl satisfy that $(c^k_{i,j}, c^k_{j,i}) = (c^l_{i,j}, c^l_{j,i})$, only one edge between vertices i and j shared by both triangles will be drawn. See Figure 5.

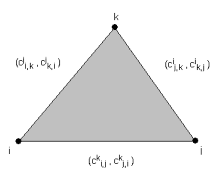

Figure 4: Full triangle.

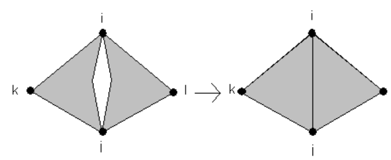

Figure 5: Two triangles sharing an edge.

Therefore, every flexible algebra with a given basis can be associated with a directed and weighted CW complex.

Example 1. *The* 3*-dimensional flexible algebra with non-zero brackets* $[e_1, e_2] = e_1 = [e_2, e_1]$*,* $[e_2, e_2] = e_2$*,* $[e_1, e_3] = [e_3, e_1] = e_2$*,* $[e_2, e_3] = [e_3, e_2] = 2e_2$ *and* $[e_3, e_3] = e_3$ *is associated with the directed and weighted CW complex shown in Figure 6.*

Now, we see how to define the flexible algebra associated with a fixed pseudodigraph (directed and weighted 1-dimensional CW complex). Let $G = (V, E)$ be a pseudodigraph with $V = \{1, \ldots, n\}$. Then G can be associated with a flexible algebra $\mathcal{F}$ with basis B as follows:

a) Define the vector space $W = \{e_1, \ldots, e_n\}$ from the set of vertices V.

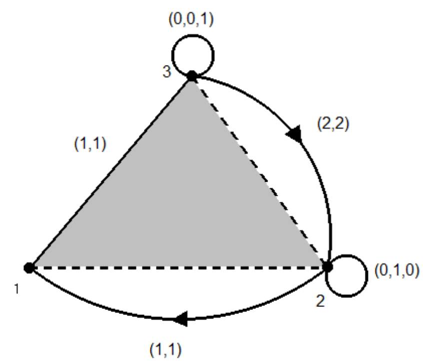

Figure 6: Directed and weighted CW complex associated with a 3-dimensional flexible algebra.

b) In case that $i \quad (1 \leq i \leq n)$ is an isolated vertex, we define $[e_i, e_i] = 0$.

c) If two vertices i and $j \quad (1 \leq i < j \leq n)$ are not adjacent, then $[e_i, e_j] = [e_j, e_i] = 0$

d) In case that $i \quad (1 \leq i \leq n)$ is a non-simple vertex, then we define $[e_i, e_i] = \sum_{h=1}^{n} c_{i,i}^{h} e_i$.

e) Given two vertices i and j with $1 \leq i < j \leq n$, if there is a directed edge from i to j and there is no directed edge from j to i, then we define $[e_i, e_j] = c_{i,j}^{j} e_j$, $[e_j, e_i] = c_{j,i}^{j} e_j$ with $(c_{i,j}^{j}, c_{j,i}^{j}) \neq (0, 0)$.

f) Given two vertices i and j with $1 \leq i < j \leq n$, in case that there is a directed edge between i and j and also a directed edge between j and i, then we define $[e_i, e_j] = c_{i,j}^{i} e_i + c_{i,j}^{j} e_j$, $[e_j, e_i] = c_{j,i}^{i} e_i + c_{j,i}^{j} e_j$ with $(c_{i,j}^{i}, c_{j,i}^{i}), (c_{i,j}^{j}, c_{j,i}^{j}) \neq (0, 0)$.

The above definitions and linear extension provide a product on V. Finally, we have to impose the flexible identity in order to obtain a flexible algebra.

4 Reading properties from the directed and weighted CW complexes

In this section, we analyze the properties that can be read from the directed and weighted CW complex associated with a flexible algebra: being a Lie algebra, the center and the derived algebra.

The method shown in Section 3 for flexible algebras provides a generalization of the one described in [2] for Lie algebras, as it is proved in the following

Proposition 1. *Given a flexible algebra which is also a Lie algebra, its associated directed and weighted CW complex satisfies the following conditions*

1. *There are no loops.*
2. *The weight for the edge from vertex i to vertex j is given by $(c_{i,j}^j, -c_{i,j}^j)$.*
3. *The weight for the edges in a full triangle ijk is given by $(c_{i,j}^k, -c_{i,j}^k)$, $(c_{j,k}^i, -c_{j,k}^i)$ and $(c_{i,k}^j, -c_{i,k}^j)$.*

Proof. Trivial from the self-annihilation and the skew-symmetry of the commutator. □

Remark 1. *Given an edge in a directed and weighted CW complex associated with an flexible algebra, both coordinates in Conditons 2 and 3 from Proposition 1 are opposite each other and, then, only one coordinate is required for saving the information of the structure constants as happened in [2].*

Lemma 1. *Let us denote by G the directed and weighted CW complex associated with a flexible algebra $\mathcal{F}$. Then,*

$$Z(\mathcal{F}) \supseteq \text{span}\{e_i \mid i \text{ is an isolated vertex}\}$$

Proof. It follows from the fact that isolated vertices without loops on this structure correspond to basis vectors in the center of the flexible algebra. □

Remark 2. *If a directed and weighted CW complex is formed only by isolated vertices (trivial graph), then it will be associated with an abelian flexible algebra. From here on, only non-abelian flexible algebras will be considered.*

Regarding Lemma 1, let us note that the center of a flexible algebra may contain basis vectors which do not correspond to isolated vertex.

Example 2. *Let $\mathcal{F}$ be the 3-dimensional flexible algebra with basis $B = \{e_1, e_2, e_3\}$ and law $[e_1, e_2] = e_2$, $[e_3, e_2] = -e_2$. This algebra is associated with digraph a) of Figure 10 and $Z(\mathcal{F}) = \text{span}(e_1 + e_3)$.*

Lemma 2. *Let G the connected (pseudo)digraph associated with a flexible algebra $\mathcal{F}$. Then it is verified that*

$$D\mathcal{F} = \text{span}\left(\sum_{h=1}^{n} c_{i,j}^h \, e_h \mid h \text{ is not a simple source vertex}\right).$$

Proof. First, it is trivial that the derived algebra of $\mathcal{F}$ is given by $D\mathcal{F} = span(\{[e_i, e_j] \mid 1 \leq i, j \leq n\}) = span\left(\sum_{h=1}^{n} c_{i,j}^h \, e_h\right)$. Let us note there is no edge directed to a simple source vertex. Therefore, we conclude that $D\mathcal{F} = \text{span}\left(\sum_{h=1}^{n} c_{i,j}^h \, e_h \mid h \text{ is not a simple source vertex}\right)$.

□

Corollary 1. *Let G the connected (pseudo)digraph associated with a flexible algebra $\mathcal{F}$. If G contains a simple source vertex, then $\mathcal{F}$ is not perfect.*

Proof. If G contains a source vertex i, then there is no edge directed to i. Therefore, we can affirm that $e_i \notin D\mathcal{F}$ and, hence, $\mathcal{F}$ is not perfect. □

5 Flexible algebras associated to pseudodigraphs

This section studies the structure of (pseudo)digraphs associated with flexible algebras. For each case, the condition on the structure constants and the type of flexible algebra is analyzed according to its solvability. Let $\mathcal{F}$ be a non-trivial or non-zero flexible algebra with basis $\mathcal{B}$ whose directed and weighted CW complex G consists of a (pseudo)digraph; that is, there are no triangles in G. This assertion is equivalent to affirm that the law of $\mathcal{F}$ with respect to the basis $\mathcal{B} = \{e_i\}_{i=1}^n$ is given by

$$[e_i, e_j] = c_{i,j}^i e_i + c_{i,j}^j e_j, \; 1 \leq i \neq j \leq n; \quad [e_k, e_k] = \sum_{h=1}^{n} c_{k,k}^h e_h \tag{1}$$

and the rest of brackets are null.

Proposition 2. *Let $\mathcal{F}$ be a 2-dimensional non-abelian flexible algebra associated to a non-connected pseudodigraph G. Then, G is isomorphic to one configuration in Figure 7.*

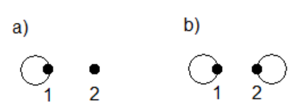

Figure 7: Non-connected pseudodigraphs with two vertices.

Proof. Figure 7 includes all the possible 2-vertices non-connected pseudodigraphs. Following the procedure of Section 3, we can construct the algebra associated with each configuration. For each of them, flexible identity is imposed. Therefore, every configuration in Figure 7 is associated with a flexible algebra if and only if the restrictions indicated in Section 9 hold for each of them. □

Proposition 3. *Let $\mathcal{F}$ be a 2-dimensional non-abelian flexible algebra associated to a connected (pseudo)digraph G. Then, G is isomorphic to one configuration in Figure 8.*

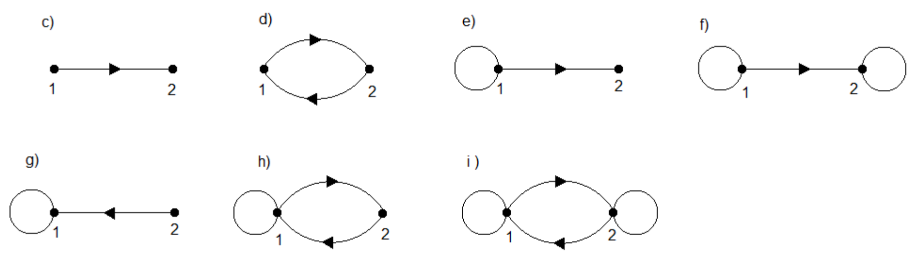

Figure 8: Connected (pseudo)digraphs with two vertices.

Proof. Figure 8 includes all the possible 2-vertices and connected pseudodigraphs. Following the procedure of Section 3, we consider the algebra associated with each configuration. Imposing the flexible identity, we obtain that every configuration in Figure 8 is associated with a flexible algebra if and only if the restrictions indicated in Section 9 hold for each of them. □

Proposition 4. *Under the assumptions in Propositions 2 and 3,*

- *Configuration a) is associated with a 2-dimensional 2-step solvable flexible algebra if $c^1_{1,1} = 0$. Otherwise, we obtain a non-solvable algebra.*
- *Configurations b) is associated with a 2-dimensional 2-step solvable flexible algebra if $c^1_{2,2} = -\frac{(c^1_{1,1})^3}{(c^2_{1,1})^2}, c^2_{2,2} = -\frac{(c^1_{1,1})^2}{c^2_{1,1}}$. Otherwise, we obtain a non-solvable algebra.*
- *Configurations c) is associated with a 2-dimensional 2-step solvable flexible algebra.*

- *Configurations d) is associated with a* 2*-dimensional* 2*-step solvable flexible algebra if* $c_{1,2}^2(c_{1,2}^1 + c_{2,1}^1) = 0$*. Otherwise, we obtain a non-solvable algebra.*

- *Configuration e) is associated with a* 2*-dimensional* 2*-step solvable flexible algebra if* $c_{1,1}^1 = 0$*. Otherwise, we obtain a perfect flexible algebra.*

- *Configuration f) is associated with a* 2*-dimensional non-solvable flexible algebra if* $c_{1,1}^2 = 0$*. Otherwise, we obtain a perfect flexible algebra.*

- *Configuration g) is associated with a* 2*-dimensional non-solvable flexible algebra if* $c_{1,1}^1 = c_{2,2}^1 = 0$*. Otherwise, we obtain a perfect flexible algebra.*

- *Configurations h) and i) are associated with a* 2*-dimensional* 2*-step solvable flexible algebra if* $c_{1,1}^2 \neq 0$ *and* $c_{1,2}^1 = -\frac{(c_{1,1}^1)^2}{2c_{1,1}^2}, c_{1,2}^2 = -\frac{c_{1,1}^1}{2}$*. Otherwise, we obtain a non-solvable algebra.*

Proof. Let us denote by $\mathcal{F}_x$ the flexible algebra associated with Configuration x and we will consider restrictions indicated in the appendix section for every configuration. In case of $\mathcal{F}_a$, if $c_{1,1}^1 = 0$, then $c_{1,1}^2$ must be different from zero and $\mathcal{C}_2(\mathcal{F}_a) = \text{span}(c_{1,1}^2\, e_2)$, $\mathcal{C}_3(\mathcal{F}_a) = \{0\}$. Therefore, $\mathcal{F}_a$ is 2-step solvable. In case that $c_{1,1}^1 \neq 0$, then $\mathcal{C}_2(\mathcal{F}_a) = \text{span}(c_{1,1}^1 e_1 + c_{1,1}^2 e_2) = \mathcal{C}_3(\mathcal{F}_a)$. For $\mathcal{F}_b$, in case that $c_{2,2}^1 = -\frac{(c_{1,1}^1)^3}{(c_{1,1}^2)^2}, c_{2,2}^2 = -\frac{(c_{1,1}^1)^2}{c_{1,1}^2}$, then $[e_2, e_2] = -\left(\frac{c_{1,1}^1}{c_{1,1}^2}\right)^2 [e_1, e_1]$, $\mathcal{C}_2(\mathcal{F}_b) = \text{span}([e_1, e_1])$ and $\mathcal{C}_3(\mathcal{F}_b) = \{0\}$. Otherwise, $\mathcal{F}_b$ and $\mathcal{C}_2(\mathcal{F}_b)$ are isomorphic. Next, $\mathcal{F}_c$ is 2-step solvable since $\mathcal{C}_2(\mathcal{F}_c) = \text{span}(e_2)$, $\mathcal{C}_3(\mathcal{F}_c) = \{0\}$. In case of $\mathcal{F}_d$, if $c_{1,2}^2(c_{1,2}^1 + c_{2,1}^1) = 0$, then $[e_2, e_1] = \frac{c_{2,1}^1}{c_{1,2}^1}[e_1, e_2]$ and $[[e_2, e_1], [e_2, e_1]] = 0$. Therefore, $\mathcal{C}_2(\mathcal{F}_d) = \text{span}([e_2, e_1])$ and $\mathcal{C}_3(\mathcal{F}_d) = \{0\}$. Otherwise, $\mathcal{F}_d$ and $\mathcal{C}_2(\mathcal{F}_d)$ are isomorphic and, consequently, $\mathcal{F}_d$ is non-solvable. Now, we consider the algebras $\mathcal{F}_e$. If $c_{1,1}^1 = 0$, then $c_{1,2}^2 = c_{2,1}^2 \neq 0$ and $c_{1,1}^2 \neq 0$. Clearly, $\mathcal{C}_2(\mathcal{F}_e) = \text{span}(e_2)$ and $\mathcal{C}_3(\mathcal{F}_e) = \{0\}$. The cases of $\mathcal{F}_f$ and $\mathcal{F}_g$ are analogous. Finally, we consider $\mathcal{F}_h$ with law $[e_1, e_1] = c_{1,1}^1 e_1 + c_{1,1}^2 e_2$, $[e_1, e_2] = c_{1,2}^1 e_1 + c_{1,2}^2 e_2$, $[e_2, e_1] = c_{2,1}^1 e_1 + c_{2,1}^2 e_2$. According to the restrictions shown in Section 9, $[e_2, e_1] = \frac{c_{2,1}^1}{c_{1,2}^1}[e_1, e_2]$. If $c_{1,1}^2 = 0$, then $c_{1,1}^1 \neq 0$ and $[e_1, e_2] = [e_2, e_1] = c_{1,2}^1 e_1 + c_{1,2}^2 e_2$. In this case, $\mathcal{F}_h$ is perfect since $\mathcal{F}_h$ and $\mathcal{C}_2(\mathcal{F}_h)$ are isomorphic. We suppose that $c_{1,1}^2 \neq 0$. If $c_{1,2}^1 = -\frac{(c_{1,1}^1)^2}{2c_{1,1}^2}, c_{1,2}^2 = -\frac{c_{1,1}^1}{2}$, then $[e_1, e_2] = [e_2, e_1] = -\frac{c_{1,1}^1}{2c_{1,1}^2}[e_1, e_1]$. Consequently, $\mathcal{C}_2(\mathcal{F}_h) = \text{span}([e_1, e_1])$ and $\mathcal{C}_3(\mathcal{F}_h) = \{0\}$. Otherwise, we obtain a non-solvable flexible algebra. The case of algebra $\mathcal{F}_i$ is similar to $\mathcal{F}_h$. □

Proposition 5. *If G is a non-connected (pseudo)digraph of 3 vertices, then G is associated with a 3-dimensional flexible algebra $\mathcal{F}$ if and only if G is isomorphic to one of the configurations of Figure 9.*

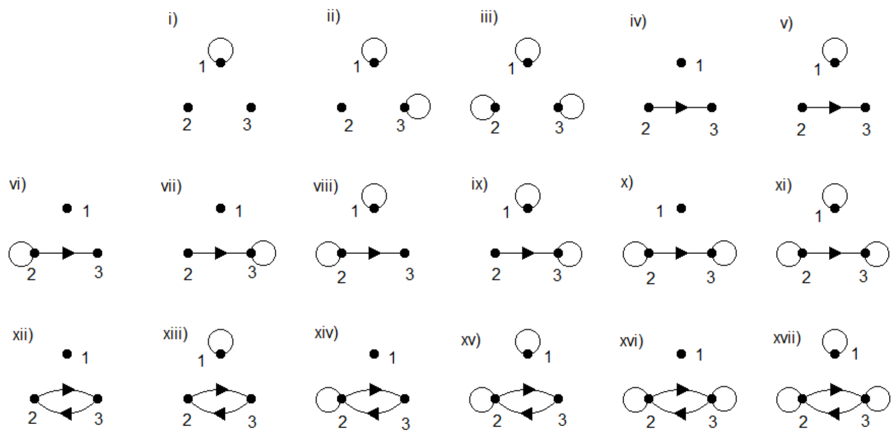

Figure 9: Disconnected (pseudo)digraphs with 3 vertices.

Proof. First, Figure 9 includes all the possible disconnected (pseudo)digraphs of three vertices. Every configuration is associated with a flexible algebra if and only if the restrictions included in Section 9 hold for each of them

□

Proposition 6. *Under the assumptions in Proposition 5,*

- *Configuration i) is associated with a 2-step nilpotent flexible algebra if* $c^1_{1,1} = 0$. *Otherwise, it is non-solvable and non-perfect.*
- *Configuration ii) is associated with a 2-step solvable flexible algebra if* $c^1_{1,1} = -\frac{(c^3_{3,3})^2}{c^1_{3,3}}$, $c^2_{1,1} = -\frac{(c^3_{3,3})^2 c^2_{3,3}}{(c^1_{3,3})^2}$. *Otherwise, it is non-solvable and non-perfect. In case that* $c^1_{1,1} = c^3_{1,1} = c^1_{3,3} = c^3_{3,3} = 0$, *then the associated flexible algebra is 2-step nilpotent.*
- *Configuration iii) is associated with a 2-step solvable flexible algebra if* $c^1_{1,1} = c^1_{2,2} = c^1_{3,3} = 0, c^2_{1,1} = \frac{c^3_{1,1} c^2_{2,2}}{c^3_{2,2}}, c^2_{3,3} = -\frac{(c^2_{2,2})^3}{(c^3_{2,2})^2}, c^3_{3,3} = -\frac{(c^2_{2,2})^2}{c^3_{2,2}}$ *or* $c^2_{1,1} = c^2_{2,2} = c^2_{3,3} = 0, c^1_{1,1} = -\frac{(c^3_{3,3})^2}{c^1_{3,3}}$, $c^3_{1,1} = -\frac{(c^3_{3,3})^2 c^2_{3,3}}{(c^1_{3,3})^2}$, $c^1_{2,2} = \frac{c^3_{2,2} c^1_{3,3}}{c^3_{3,3}}$. *Otherwise, it is non-solvable and non-perfect.*

- *Configuration iv) is associated with a 2-step solvable non-nilpotent flexible algebra.*
- *Configuration v) is associated with a non-solvable flexible algebra if* $c_{1,1}^1 \neq 0$. *In case that* $c_{1,1}^1 = c_{1,1}^2 = c_{3,2}^3 = 0$, *then the algebra is 2-step solvable. If* $c_{1,1}^1 = c_{1,1}^3 = c_{2,3}^3 = 0$ *then it is 3-step solvable.*
- *Configuration vi) is associated with a non-solvable flexible algebra if* $c_{2,2}^2 \neq 0$. *Otherwise, it is 2-step solvable.*
- *Configuration vii) is associated with a 3-step solvable non-nilpotent flexible algebra if* $c_{3,3}^3 = c_{3,3}^2 = 0$. *Otherwise, it is non-solvable and non-perfect.*
- *Configuration viii) is associated with a 3-step solvable non-nilpotent flexible algebra if* $c_{1,1}^3 = c_{2,2}^3 = 0$, $c_{2,2}^1 = -\frac{(c_{1,1}^1)^3}{(c_{1,1}^2)^2}$, $c_{2,2}^2 = -\frac{(c_{1,1}^1)^2}{c_{1,1}^2}$. *In case that* $c_{1,1}^1 = c_{1,1}^2 = c_{2,2}^1 = c_{2,2}^2 = 0$ *and* $c_{2,3}^3 = c_{3,2}^3$, *then it is associated with a 2-step solvable flexible algebra. Otherwise, it is non-solvable.*
- *Configurations* $ix), x)$ *and xi) are associated with a 2-step solvable non-nilpotent flexible algebra if* $c_{1,1}^1 = c_{1,1}^2 = c_{2,2}^1 = c_{2,2}^2 = 0$ *and* $c_{2,3}^3 = c_{3,2}^3$. *Otherwise, they are not solvable.*
- *Configuration xii) is associated with a 2-step solvable non-nilpotent flexible algebra if* $c_{2,3}^2 = -c_{3,2}^2$ *and* $c_{2,3}^3 = -c_{3,2}^3$. *Otherwise, it is non-solvable and non-perfect.*
- *Configuration xiii) is associated with a 2-step solvable non-nilpotent flexible algebra if* $c_{1,1}^1 = 0$ *and* $c_{1,1}^2 c_{3,2}^3 = c_{1,1}^3 c_{3,2}^2$. *Otherwise, it is non-solvable and non-perfect.*
- *Configuration xiv) is associated with a 2-step solvable non-nilpotent flexible algebra if* $c_{2,2}^2 = c_{2,3}^2 = c_{3,2}^2 = 0$ *or* $c_{2,2}^1 = 0$, $c_{2,2}^2 = -2c_{2,3}^3$, $c_{2,2}^3 = -2\frac{(c_{2,3}^3)^2}{c_{2,3}^2}$, $c_{2,3}^2 = c_{3,2}^2$, $c_{2,3}^3 = c_{3,2}^3$. *Otherwise, it is non-solvable and non-perfect.*
- *Configuration xv) is associated with a 2-step solvable non-nilpotent flexible algebra if* $c_{2,3}^2 = c_{3,2}^2$, $c_{2,3}^3 = c_{3,2}^3$ *and either* $c_{1,1}^1 = c_{1,1}^2 = c_{2,2}^1 = c_{2,2}^2 = c_{2,3}^3 = 0$ *or* $c_{1,1}^1 = c_{2,2}^1 = 0$, $c_{1,1}^3 = \frac{c_{1,1}^2 c_{2,3}^3}{c_{2,3}^2}$, $c_{2,2}^2 = -2c_{2,3}^3$, $c_{2,2}^3 = -\frac{2(c_{3,3}^3)^2}{c_{2,3}^2}$. *Otherwise, it is non-solvable.*
- *Configurations xvi) and xvii) are associated with a solvable flexible algebra if* $c_{2,3}^2 = c_{3,2}^2$, $c_{2,3}^3 = c_{3,2}^3$ *and* $c_{2,2}^1 = -\frac{(c_{2,2}^2)^2 c_{3,3}^1}{4(c_{2,3}^2)^2}$, $c_{2,2}^3 = -\frac{(c_{2,2}^2)^2}{2c_{2,3}^2}$, $c_{2,3}^3 =$

$-\frac{(c_{2,2}^2)}{2}$, $c_{3,3}^2 = c_{3,3}^3 = 0$ *or* $c_{2,2}^1 = -\frac{(c_{3,3}^3)^2 c_{3,3}^1}{(c_{3,3}^2)^2}$, $c_{2,2}^2 = \frac{-2c_{2,3}^3 c_{3,3}^2 + (c_{3,3}^3)^2}{c_{3,3}^2}$, $c_{2,2}^3 = -\frac{c_{3,3}^3(2c_{2,3}^3 c_{3,3}^2 + (c_{3,3}^3)^2)}{(c_{3,3}^2)^2}$, $c_{2,3}^2 = \frac{c_{3,3}^2 c_{2,3}^3}{c_{3,3}^3}$. *Otherwise, it is non-solvable.*

Proof. Due to reasons of length, we only show the proof for Configuration $xv)$ and the reasoning is similar for the other configurations. According to the proof of Proposition 5, the flexible algebra associated with Configuration $xv)$ has basis $\{e_1, e_2, e_3\}$ and brackets

$$[e_1, e_1] = c_{1,1}^1 e_1 + c_{1,1}^2 e_2 + c_{1,1}^3 e_3, \quad [e_2, e_2] = c_{2,2}^1 e_1 + c_{2,2}^2 e_2 + c_{2,2}^3 e_3,$$

$$[e_2, e_3] = c_{2,3}^2 e_2 + c_{2,3}^3 e_3, \quad [e_3, e_2] = c_{3,2}^2 e_2 + c_{3,2}^3 e_3$$

verifying $c_{2,2}^3(c_{2,3}^2 - c_{3,2}^2) = c_{2,2}^3(c_{2,3}^3 - c_{3,2}^3) = c_{2,2}^j(c_{2,3}^2 - c_{3,2}^2) = c_{3,2}^3 c_{2,3}^2 - c_{2,3}^3 c_{3,2}^2 = 0$, $(c_{2,3}^2, c_{3,2}^2), (c_{2,3}^3, c_{3,2}^3) \neq (0,0)$, $(c_{j,j}^1, c_{j,j}^2, c_{j,j}^3) \neq (0,0,0)$, for $j = 1,2$.

From the previous restrictions, we obtain that $c_{2,3}^2 = c_{3,2}^2, c_{2,3}^3 = c_{3,2}^3$. Consequently, $[e_2, e_3] = [e_3, e_2]$ and $\mathcal{C}_2(\mathcal{F}) = \text{span}([e_1, e_1], [e_2, e_2], [e_2, e_3])$. If, in addition, $c_{1,1}^1 = c_{1,1}^2 = c_{2,2}^1 = c_{2,2}^2 = c_{2,3}^3 = 0$, then $\mathcal{C}_2(\mathcal{F}) = \text{span}(e_3)$ is an abelian ideal and, therefore, $\mathcal{C}_3(\mathcal{F}) = \{0\}$ and $\mathcal{F}$ is 2-step solvable and non-nilpotent since $\mathcal{C}^k(\mathcal{F}) = \text{span}(e_3)$, for $k \geq 2$. In case that $c_{1,1}^1 = c_{2,2}^1 = 0$, $c_{1,1}^3 = \frac{c_{1,1}^2 c_{2,3}^3}{c_{2,3}^2}$, $c_{2,2}^2 = -2c_{2,3}^3$, $c_{2,2}^3 = -\frac{2(c_{3,3}^3)^2}{c_{2,3}^2}$, then $[e_1, e_1] = \frac{c_{1,1}^2}{c_{2,3}^2}[e_2, e_3]$, $[e_2, e_2] = \frac{-2c_{2,3}^3}{c_{2,3}^2}[e_2, e_3]$ and $[[e_2, e_3], [e_2, e_3]] = 0$. Consequently, $\mathcal{C}_2(\mathcal{F}) = \text{span}([e_2, e_3]) = \mathcal{C}^k(\mathcal{F})$, $\forall k \geq 2$, $\mathcal{C}_3(\mathcal{F}) = \{0\}$ and we conclude that $\mathcal{F}$ is 2-step solvable and non-nilpotent.

Notice that if one of the previous cases is not satisfied, $\mathcal{C}_2(\mathcal{F}) = \mathcal{C}_k(\mathcal{F})$, for $k \geq 3$ and $\mathcal{F}$ is non-solvable.

□

Proposition 7. *Let $\mathcal{F}$ be a 3-dimensional non-abelian flexible algebra associated to a connected digraph G. Then, G is isomorphic to one configuration in Figure 10.*

Proof. Figure 10 includes all the possible 3-vertices connected digraphs. Following the procedure of Section 3, we can construct the algebra associated with each configuration. For each of them, flexible identity is imposed. Therefore, every configuration in Figure 10 is associated with a flexible algebra if and only if the restrictions indicated in Section 9 hold for each of them. □

Proposition 8. *Under the assumptions in Proposition 7,*

- *Configurations a) and c) are associated with a 2-step solvable non-nilpotent flexible algebra.*

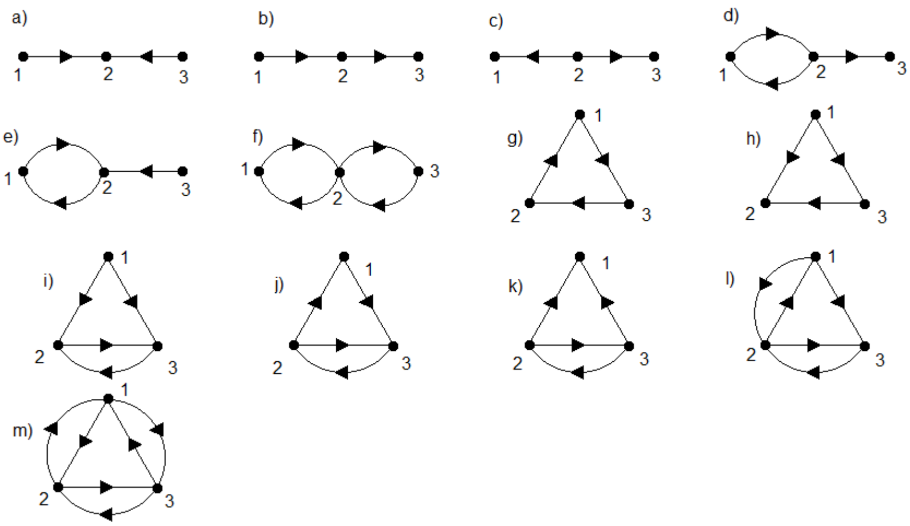

Figure 10: Connected digraphs with three vertices.

- *Configurations* $b)$ *and* $h)$ *are associated with a* 3*-step solvable non-nilpotent flexible algebra.*
- *Configurations* $d)$ *and* $e)$ *are associated with a* 3*-step solvable non-nilpotent flexible algebra if* $c_{1,2}^1 = -c_{2,1}^1$, $c_{1,2}^2 = -c_{2,1}^2$. *Otherwise, it is non-solvable and non-perfect. The same happens for Configurations* $i)$ *and* $k)$ *with conditions* $c_{3,2}^2 = -c_{2,3}^2$, $c_{3,2}^3 = -c_{2,3}^3$.
- *Configuration* $f)$ *is associated with a non-solvable flexible algebra.*
- *Configurations* $g)$ *and* $\ell)$ *are associated with a perfect flexible algebra.*
- *Configuration* $j)$ *is associated with a* 3*-step solvable flexible algebra if* $c_{2,3}^2 = 0$. *Otherwise, it is a perfect flexible algebra.*
- *Configuration* $m)$ *is associated with a* 2*-step solvable flexible algebra if* $c_{1,2}^1 = -c_{2,3}^3$, $c_{1,2}^2 = c_{1,3}^3$, $c_{1,3}^1 = -c_{3,2}^2$, $c_{2,1}^1 = c_{2,3}^3$, $c_{2,1}^2 = -c_{1,3}^3$, $c_{2,3}^2 = -c_{3,2}^2$, $c_{3,1}^1 = c_{3,2}^2$, $c_{3,1}^3 = -c_{1,3}^3$, $c_{3,2}^3 = -c_{2,3}^3$. *Otherwise, it is non-solvable.*

Proof. Let us denote by $\mathcal{F}_x$ the flexible algebra associated with Configuration $x)$ and we will consider restrictions indicated in the appendix section for every configuration. In case of Configurations *a*) and *c*), it is satisfied that $\mathcal{C}_2(\mathcal{F}_a) = \mathrm{span}(e_2)$ and $\mathcal{C}_2(\mathcal{F}_c) = \mathrm{span}(e_1, e_3)$ are abelian ideals. For Configurations *b*) and *h*), we have that $\mathcal{C}_2(\mathcal{F}_b) = \mathrm{span}(e_2, e_3)$, $\mathcal{C}_3(\mathcal{F}_b) = \mathrm{span}(e_3)$, $\mathcal{C}_2(\mathcal{F}_h) = \mathrm{span}(e_2, e_3)$, $\mathcal{C}_3(\mathcal{F}_h) = \mathrm{span}(e_2)$ and $\mathcal{C}_k(\mathcal{F}_b) = \mathcal{C}_k(\mathcal{F}_h) = \{0\}$, for $k \geq 4$. In case of $\mathcal{F}_d$, if $c^1_{1,2} = -c^1_{2,1}$ and $c^2_{1,2} = -c^2_{2,1}$, then $[e_2, e_1] = -[e_1, e_2]$, $\mathcal{C}_2(\mathcal{F}_d) = \mathrm{span}([e_1, e_2], e_3)$, $\mathcal{C}_3(\mathcal{F}_d) = \mathrm{span}(e_3)$ and $\mathcal{C}_p(\mathcal{F}_d) = \{0\}$, for $p \geq 4$. Otherwise, $\mathcal{C}_q(\mathcal{F}_d) = \mathrm{span}([e_1, e_2], e_3)$, for $q \geq 2$. The proof is similar for Configurations *e*), *i*) and *k*). Now, we consider $\mathcal{F}_f$. This algebra is not perfect and could be solvable if $c^1_{2,1} = -c^1_{1,2}$, $c^2_{2,1} = -c^2_{1,2}$, $c^2_{3,2} = -c^2_{2,3}$, $c^3_{3,2} = -c^3_{2,3}$. In such a case, $[e_2, e_1] = -[e_1, e_2]$ and $[e_3, e_2] = -[e_2, e_3]$. Consequently, $\mathcal{C}_2(\mathcal{F}_f) = \mathrm{span}([e_1, e_2], [e_2, e_3])$ and $[[e_1, e_2], [e_1, e_2]] = [[e_2, e_3], [e_2, e_3]] = 0$. However, $\mathcal{C}_k(\mathcal{F}_f) = \mathrm{span}([[e_1, e_2], [e_2, e_3]])$, for $k \geq 3$. Therefore, $\mathcal{F}_f$ is non-solvable. Next, $\mathcal{F}_g$ and $\mathcal{C}_2(\mathcal{F}_g)$ are isomorphic, so $\mathcal{F}_g$ is a perfect flexible algebra. The same happens for $\mathcal{F}_\ell$. If we consider $\mathcal{F}_j$, then $\mathcal{C}_2(\mathcal{F}_j) = \mathrm{span}(e_1, e_3, c^2_{2,3}e_2 + c^3_{2,3}e_3)$. In case that $c^2_{2,3} = 0$, then $\mathcal{C}_2(\mathcal{F}_j) = \mathrm{span}(e_1, e_3)$ and $\mathcal{C}_3(\mathcal{F}_j) = \mathrm{span}(e_3)$ is an abelian ideal. Consequently, $\mathcal{F}_j$ is 3-step solvable. Otherwise, $\mathcal{C}_2(\mathcal{F}_j) = \mathrm{span}(e_1, e_2, e_3) = \mathcal{F}_j$. Finally, we consider the algebra $\mathcal{F}_m$. According to the restrictions in appendix, we have that $[e_2, e_1] = \frac{c^1_{2,1}}{c^1_{1,2}}[e_1, e_2]$, $[e_3, e_2] = \frac{c^2_{3,2}}{c^2_{2,3}}[e_2, e_3]$ and $[e_3, e_1] = \frac{c^1_{3,1}}{c^1_{1,3}}[e_1, e_3]$. In case that $c^1_{1,2} = -c^3_{2,3}$, $c^2_{1,2} = c^3_{1,3}$, $c^1_{1,3} = -c^2_{3,2}$, $c^1_{2,1} = c^3_{2,3}$, $c^2_{2,1} = -c^3_{1,3}$, $c^2_{2,3} = -c^2_{3,2}$, $c^1_{3,1} = c^2_{3,2}$, $c^3_{3,1} = -c^3_{1,3}$, $c^3_{3,2} = -c^3_{2,3}$, then $[e_2, e_1] = -[e_1, e_2]$, $[e_3, e_2] = -[e_2, e_3]$ and $[e_3, e_1] = -[e_1, e_3]$. Therefore $\mathcal{C}_2(\mathcal{F}_m) = \mathrm{span}([e_1, e_2], [e_2, e_3], [e_1, e_3])$, where

$$[e_1, e_2] = -c^3_{2,3}e_1 + c^3_{1,3}e_2, \quad [e_1, e_3] = -c^2_{3,2}e_1 + c^3_{1,3}e_3, \quad [e_2, e_3] = -c^2_{3,2}e_2 + c^3_{2,3}e_3$$

Moreover,

$$\begin{vmatrix} c_{i,i} & c_{i,j} & c_{i,k} \\ c_{j,i} & 0 & c_{j,k} \\ c_{k,i} & c_{k,j} & c_{k,k} \end{vmatrix} = 0$$

and $\mathcal{C}_3(\mathcal{F}_m) = \{0\}$, so $\mathcal{F}_m$ is 2-step solvable.

□

Proposition 9. *Let $\mathcal{F}$ be a 3-dimensional non-abelian flexible algebra associated to a connected pseudodigraph G. Then, G is isomorphic to one configuration in Figure 11.*

Proof. Figure 11 includes all the possible 3-vertices connected pseudodigraphs. Following the procedure of Section 3, we can construct the algebra associated with each configuration. For each of them, flexible identity is imposed. Therefore, every configuration in Figure 11 is associated with a flexible algebra if and only if the restrictions indicated in Section 9 hold for each of them. □

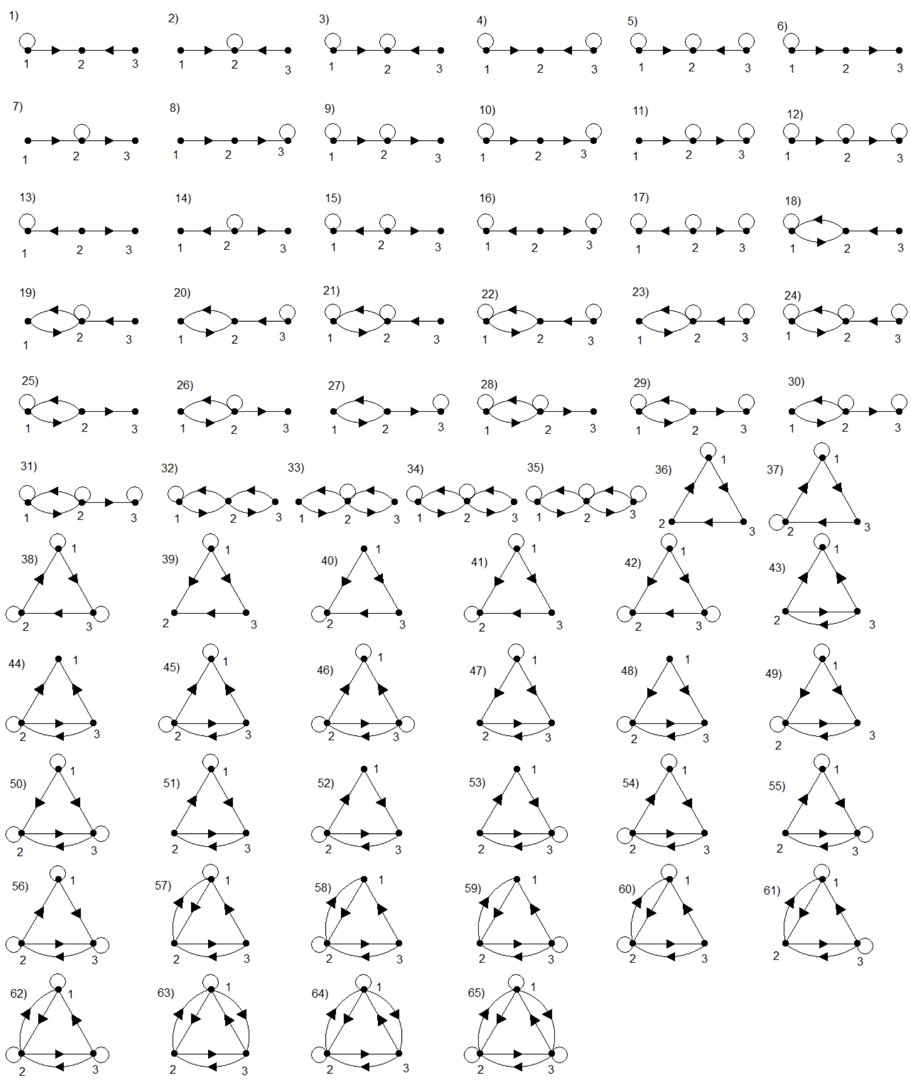

Figure 11: Connected pseudodigraphs with three vertices.

6 Classification of flexible algebras

In this section, the isomorphism class for each configuration with two vertices from Section 5 associated with non-abelian flexible algebras is analyzed. In this way, we provide a new method to classify these algebras is provided.

Proposition 10. *Flexible algebras associated with Configuration a) from Figure 2 belong to the isomorphism class* $\mathcal{F}^2_{0)} = \mathrm{span}(e_1, e_2)$ *defined by the law* $[e_1, e_1] = e_1$ *or* $\mathcal{F}^2_{i)} = \mathrm{span}(e_1, e_2)$ *given by* $[e_1, e_1] = e_2$.

Proof. Let $\mathcal{F}$ be the flexible algebra associated with Configuration $a)$ from Figure 7. Let $\{v_1, v_2\}$ be the basis of $\mathcal{F}$. If $c^1_{1,1} \neq 0$, then we consider the basis change $\phi : \mathcal{F} \to \mathcal{F}$ given by $e_1 = \phi(v_1) = \frac{1}{c^1_{1,1}} v_1 + \frac{c^2_{1,1}}{(c^1_{1,1})^2} v_2$; $e_2 = \phi(v_2) = v_2$ and the law $[e_1, e_1] = e_1$ is obtained. In case that $c^1_{1,1} = 0$, then $c^2_{1,1} \neq 0$ and the basis change $\phi : \mathcal{F} \to \mathcal{F}$ given by $e_1 = \phi(v_1) = \frac{1}{c^2_{1,1}} v_1$; $e_2 = \phi(v_2) = \frac{1}{c^2_{1,1}} v_2$ leads to the law $[e_1, e_1] = e_2$.

□

Proposition 11. *Flexible algebras associated with Configuration b) from Figure 2 belong to the isomorphism class* $\mathcal{F}^2_{ii)} = \mathrm{span}(e_1, e_2)$ *defined by the law* $[e_1, e_1] = e_1$, $[e_2, e_2] = e_2$.

Proof. If $\mathcal{F}$ is the flexible algebra associated with Configuration $b)$ from Figure 7, then its law is given by $[v_1, v_1] = c^1_{1,1} v_1$, $[v_2, v_2] = c^2_{2,2} v_2$. Now, with the basis change $\phi : \mathcal{F} \to \mathcal{F}$ given by $e_i = \phi(v_i) = \frac{1}{c^i_{i,i}} v_i$ for $i = 1, 2$, the law $[e_1, e_1] = e_1$, $[e_2, e_2] = e_2$ is obtained. □

Proposition 12. *Flexible algebras associated with Configuration c) from Figure 8 belong to the isomorphism class* $\mathcal{F}^2_{iii)} = \mathrm{span}(e_1, e_2)$ *defined by the law* $[e_1, e_2] = e_2$, $[e_2, e_1] = \alpha e_2$, *with* $\alpha \in \mathbb{C}$.

Proof. Let $\mathcal{F}$ be the flexible algebra associated with Configuration $c)$ from Figure 8, then its law is given by $[v_1, v_2] = c^2_{1,2} v_2$, $[v_2, v_1] = c^2_{2,1} v_2$, where $(c^2_{1,2}, c^2_{2,1}) \neq (0, 0)$. If $c^2_{1,2} \neq 0$ (the other case is similar) then the basis change $\phi : \mathcal{F} \to \mathcal{F}$ given by $e_1 = \phi(v_1) = \frac{1}{c^2_{1,2}} v_1$, $e_2 = v_2$ leads to the law $[e_1, e_2] = e_2$, $[e_2, e_1] = \frac{c^2_{2,1}}{c^2_{1,2}} e_2$. □

Proposition 13. *Flexible algebras associated with Configuration d) from Figure 8 belong to the isomorphism class* $\mathcal{F}^2_{iv)} = \mathrm{span}(e_1, e_2)$ *defined by the law* $[e_1, e_2] = e_1 + e_2$, $[e_2, e_1] = \alpha(e_1 + e_2)$, *with* $\alpha \in \mathbb{C}$.

Proof. Let $\mathcal{F}$ be the flexible algebra associated with Configuration d) from Figure 8, then its law is given by $[v_1, v_2] = c^1_{1,2}v_1 + c^2_{1,2}v_2$, $[v_2, v_1] = c^1_{2,1}v_1 + c^2_{2,1}v_2$, where $c^1_{1,2}c^2_{2,1} = c^2_{1,2}c^1_{2,1}$ and $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1}) \neq (0, 0)$. Now, with the basis change $\phi : \mathcal{F} \to \mathcal{F}$ given by $e_1 = \phi(v_1) = \frac{1}{c^2_{1,2}}v_1$, $e_2 = \frac{1}{c^1_{1,2}}v_2$ leads to the law $[e_1, e_2] = e_1 + e_2$, $[e_2, e_1] = \frac{c^2_{2,1}}{c^2_{1,2}}(e_1 + e_2)$. □

Proposition 14. *Flexible algebras associated with Configuration e) from Figure 8 belong to the isomorphism class $\mathcal{F}^2_{v)} = \mathrm{span}(e_1, e_2)$ defined by the law $[e_1, e_1] = \alpha e_1 + \beta e_2$, $[e_1, e_2] = [e_2, e_1] = e_2$, where $\alpha \in \mathbb{C}$ and $\beta \in \{0, 1\}$.*

Proof. According to the proof of Proposition 4 and the restrictions indicated in Appendix, there exist two different families of flexible algebras associated with Configuration e) from Figure 8. Let us denote by $\mathcal{F}_1$ and $\mathcal{F}_2$ those algebras and let $\{v_1, v_2\}$ be a basis for them. The law of $\mathcal{F}_1$ is given by $[v_1, v_1] = c^1_{1,1}v_1 + c^2_{1,1}v_2$, $[v_1, v_2] = [v_2, v_1] = c^2_{1,2}v_2$, while the non-zero brackets of $\mathcal{F}_2$ are $[v_1, v_1] = c^2_{1,1}v_2$, $[v_1, v_2] = c^2_{1,2}v_2$, $[v_2, v_1] = c^2_{2,1}v_2$. The basis changes $\phi : \mathcal{F}_1 \to \mathcal{F}_1$ given by $e_1 = \phi(v_1) = \frac{1}{c^2_{1,2}}v_1$; $e_2 = \phi(v_2) = \frac{c^2_{1,1}}{(c^2_{1,2})^2}v_2$ and $\phi' : \mathcal{F}_2 \to \mathcal{F}_2$ given by $e_1 = \phi'(v_1) = \frac{1}{c^2_{1,2}}v_1$; $e_2 = \phi'(v_2) = v_2$ lead to the law $[e_1, e_1] = \frac{c^1_{1,1}}{c^2_{1,2}}e_1 + \beta e_2$, $[e_1, e_2] = e_2$, where $\beta = 0$ for $\mathcal{F}_2$ and $\beta = 1$ for $\mathcal{F}_1$.
□

Proposition 15. *Flexible algebras associated with Configuration f) from Figure 8 belong to the isomorphism class $\mathcal{F}^2_{vi)} = \mathrm{span}(e_1, e_2)$ defined by the law $[e_1, e_1] = \alpha e_1 + e_2$, $[e_1, e_2] = [e_2, e_1] = e_2$, $[e_2, e_2] = \beta e_1 + e_2$, where $\alpha, \beta \in \mathbb{C}$.*

Proof. Let us denote by $\mathcal{F}$ the flexible algebra associated with Configuration f). If $\{v_1, v_2\}$ is a basis of $\mathcal{F}$, according to the proof of Proposition 4 and the restrictions indicated in Appendix, the law of $\mathcal{F}$ is given by $[v_1, v_1] = c^1_{1,1}v_1 + c^2_{1,1}v_2$, $[v_1, v_2] = [v_2, v_1] = c^2_{1,2}v_2$, $[v_2, v_2] = c^1_{2,2}v_1 + c^2_{2,2}v_2$. The basis change $\varphi : \mathcal{F} \to \mathcal{F}$ given by $e_1 = \varphi(v_1) = \frac{1}{c^2_{1,2}}v_1 + \frac{\lambda}{c^2_{1,2}}v_2$; $e_2 = \varphi(v_2) = \mu v_2$, where a and b are the solution of the system

$$\left.\begin{array}{r} c^2_{2,2}\lambda^2 + c^2_{1,2}\lambda + c^2_{1,1} = 0, \\ \mu^2 c^2_{2,2}c^2_{1,2} = \lambda \end{array}\right\},$$

leads to the law $[e_1, e_1] = \frac{c^1_{1,1} + \lambda^2 c^1_{2,2}}{(c^2_{1,2})^2}e_1 + e_2$, $[e_1, e_2] = [e_2, e_1] = e_2$, $[e_2, e_2] = \mu^2 c^1_{2,2}c^2_{1,2}e_1 + e_2$. □

Proposition 16. *Flexible algebras associated with Configuration g) from Figure 8 belong to the isomorphism class* $\mathcal{F}^2_{vii)} = \mathrm{span}(e_1, e_2)$ *defined by the law* $[e_1, e_1] = e_1 + \alpha e_2$, $[e_1, e_2] = [e_2, e_1] = e_1$, *where* $\alpha \in \mathbb{C}$.

Proof. Let us denote by $\mathcal{F}$ the flexible algebra associated with Configuration g). If $\{v_1, v_2\}$ is a basis of $\mathcal{F}$, according to the proof of Proposition 4 and the restrictions indicated in Appendix, the law of $\mathcal{F}$ is given by $[v_1, v_1] = c^1_{1,1}v_1 + c^2_{1,1}v_2, [v_1, v_2] = [v_2, v_1] = c^1_{1,2}v_1$. Now, we consider the basis changes $\phi : \mathcal{F} \to \mathcal{F}$ given by $e_1 = \phi(v_1) = \frac{1}{c^1_{1,1}}v_1 = e_1$, $e_2 = \phi(v_2) = \frac{1}{c^1_{1,2}}v_2 = e_2$. Applying ϕ, we obtain the law $[e_1, e_1] = e_1 + \alpha e_2$, $[e_1, e_2] = [e_2, e_1] = e_1$, where $\alpha = \frac{c^2_{1,1}c^1_{1,2}}{(c^1_{1,1})^2}$. □

Proposition 17. *Flexible algebras associated with Configuration h) from Figure 8 belong to the isomorphism class* $\mathcal{F}^2_{viii)} = \mathrm{span}(e_1, e_2)$ *defined by the law* $[e_1, e_1] = e_1 + \alpha e_2$, $[e_1, e_2] = [e_2, e_1] = e_1$, *where* $\alpha \in \mathbb{C}$.

Proof. We denote by $\mathcal{F}$ the flexible algebra associated with Configuration h). Let $\{v_1, v_2\}$ be a basis of $\mathcal{F}$. According to the proof of Proposition 4 and the restrictions indicated in Appendix, the law of $\mathcal{F}$ is given by $[v_1, v_1] = c^1_{1,1}v_1 + c^2_{1,1}v_2, [v_1, v_2] = [v_2, v_1] = c^1_{1,2}v_1 + c^2_{1,2}v_2$. Now, we consider the basis changes $\phi : \mathcal{F} \to \mathcal{F}$ given by $e_1 = \phi(v_1) = \frac{1}{2c^2_{1,2}+c^1_{1,1}}v_1 + \frac{c^2_{1,2}}{(2c^2_{1,2}+c^1_{1,1})c^1_{1,2}}v_2 = e_1$, $e_2 = \phi(v_2) = \frac{1}{c^1_{1,2}}v_2 = e_2$. Applying ϕ, we obtain the law $[e_1, e_1] = e_1 + \alpha e_2$, $[e_1, e_2] = [e_2, e_1] = e_1$, where $\alpha = \frac{c^1_{1,2}c^2_{1,1} - c^1_{1,1}c^2_{1,2}}{(2c^2_{1,2}+c^1_{1,1})^2}$. □

Proposition 18. *Flexible algebras associated with Configuration i) from Figure 8 belong to the isomorphism class* $\mathcal{F}^2_{ix)} = \mathrm{span}(e_1, e_2)$ *defined by the law* $[e_1, e_1] = e_1 + \alpha e_2$, $[e_2, e_2] = \beta e_1 + e_2$, $[e_1, e_2] = [e_2, e_1] = e_1$, *where* $\alpha, \beta \in \mathbb{C}$ *or* $\mathcal{F}^2_{x)} = \mathrm{span}(e_1, e_2)$ *defined by the law* $[e_1, e_1] = e_1$, $[e_2, e_2] = e_2$, $[e_1, e_2] = (1 - \lambda)e_1 + \lambda e_2$, $[e_2, e_1] = (1 - \mu)e_1 + \mu e_2$, *where* $\lambda, \mu \in \mathbb{C}$

Proof. According to the proof of Proposition 4 and the restrictions indicated in Appendix, there exist three different families of flexible algebras associated with Configuration i) from Figure 8. Let us denote by $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ those algebras and let $\{v_1, v_2\}$ be a basis for them. The law of $\mathcal{F}_1$ is given by $[v_1, v_1] = c^1_{1,1}v_1 + c^2_{1,1}v_2$, $[v_1, v_2] = [v_2, v_1] = c^1_{1,2}v_1 + c^2_{1,2}v_2$, $[v_2, v_2] = c^1_{2,2}v_1 + c^2_{2,2}v_2$, while the non-zero brackets of $\mathcal{F}_2$ are $[v_1, v_1] = c^1_{1,1}v_1 + c^2_{1,1}v_2$, $[v_1, v_2] = [v_2, v_1] = c^1_{1,2}v_1 + c^2_{1,2}v_2$, $[v_2, v_2] = c^1_{2,2}v_1 + c^2_{2,2}v_2$ and the law of $\mathcal{F}_3$ is $[v_1, v_1] = c^1_{1,1}v_1 + c^2_{1,1}v_2$, $[v_1, v_2] = [v_2, v_1] = c^1_{1,2}v_1 + c^2_{1,2}v_2$, $[v_2, v_2] = c^1_{2,2}v_1 + c^2_{2,2}v_2$. Considering a basis change similar to the one used in the proof of Proposition 17, we obtain the law $[e_1, e_1] = e_1 + \alpha e_2$, $[e_2, e_2] = \beta e_1 + e_2$, $[e_1, e_2] = [e_2, e_1] = e_1$ for $\mathcal{F}_1$.

For flexible algebras $\mathcal{F}_2$ and $\mathcal{F}_3$, the basis change $\phi : \mathcal{F}_i \to \mathcal{F}_i$ given by $e_1 = \phi(v_1) = \frac{1}{c^1_{1,1}} v_1$, $e_2 = \phi(v_2) = \frac{1}{c^2_{2,2}} v_2$, for $i = 2, 3$ leads to the law $[e_1, e_1] = e_1$, $[e_2, e_2] = e_2$, $[e_1, e_2] = (1 - \frac{c^2_{1,2}}{c^1_{1,1}} e_1 + \frac{c^2_{1,2}}{c^1_{1,1}} e_2$, $[e_2, e_1] = (1 - \frac{c^2_{2,1}}{c^1_{1,1}} e_1 + \frac{c^2_{2,1}}{c^1_{1,1}} e_2$.

□

7 Algorithmic methods

In this section two algorithmic methods are introduced. The first one checks if a fixed directed and weighted CW complex is associated or not with a flexible algebra. The second procedure obtains all the (pseudo)digraphs associated with a parametric family of flexible algebras starting from its law when contains no full triangles. It also draws all those (pseudo)digraphs. Notice that these algorithms have been used in order to achieve all the results of Section 5 and 6.

7.1 Checking if a fixed directed and weighted CW complex is associated with a flexible algebra

This algorithmic procedure has been implemented by using the symbolic computation package Maple, working the implementation in version 18. To do this, the libraries `linalg` and `combinat` have to be used in order to activate commands related to Linear and Combinatorial Algebra. This algorithmic procedure consists of the following three steps:

a) Obtaining the values of the structure constants according to the directed and weighted CW complex.

b) Defining the law which should be fulfilled by the flexible algebra, starting from the structure constants.

c) Checking if the flexible identity is satisfied for this law.

In order to develop the implementation, three subprocedures for the two first steps and one main procedure for the last one are required. Before running the procedure, one need the command `restart` to reset all the variables and delete all the computations saved in the kernel. The first step of this algorithm is executed by the subprocedure `assignment`, which allows to define the dimension and the value of the structure constants of the vector space associated with the directed and weighted CW complex and to determine the candidate for the bracket product. To do so, `assignment` receives the following two inputs: The list `V` with the vertices of the directed and weighted CW complex as natural numbers, and the set `E` with its weighted, directed edges. The elements of the set E are inserted as $[[i, j, k], l]$, denoting $c^k_{i,j} = l$. The output is the value of the variable dim with the dimension of the directed and weighted CW complex and also the value of all the non-zero structure constants.

```
> restart:
> assignment:=proc(V,E)
>  local B,L;
>  B:=[];L:=[];
>  for x from 1 to nops(V) do
>   B:=[op(B),e[x]];
>  od;
>  assign(dim,nops(V));
>  for i from 1 to nops(E) do
>   assign(c[E[i][1][1],E[i][1][2],E[i][1][3]],E[i][2]);
>  od;
> end proc:
```

Next, one can run the second subprocedure, named law, which receives two natural numbers as inputs. These numbers represent the subindexes of two vectors in the endowed vector space or, equivalently, two vertices from the directed and weighted CW complex. The subroutine computes the bracket of these two vectors. In the implementation, a local variable, v, is used to save the value of the bracket, which is computed by using the structure constants defined in the previous subprocedure.

```
> law:=proc(i,j)
>   local v;
>   v:=0;
>   for k from 1 to dim do
>    if type(c[i,j,k],numeric)=true then
>       v:=v+c[i,j,k]*e[k];
>    fi;
>   od;
>  return v;
> end proc:
```

Now, the implementation of the subprocedure called bracket is shown. This subroutine is devoted to computing the bracket product between two arbitrary vectors expressed as linear combinations of the basis vectors used in the previous subprocedure.

```
> bracket:=proc(u,v,n)
>   local exp; exp:=0;
>   for i from 1 to n do
>      for j from 1 to n do
>         exp:=exp + coeff(u,e[i])*coeff(v,e[j])*law(i,j);
>      od;
>   od;
>   exp;
> end proc:
```

Finally, let us proceed with the implementation of the main procedure called `flexible`, which checks if the vector space is or is not a flexible algebra. This procedure receives as input the dimension n of the vector space $\mathcal{F}$ and returns the message "True" in case that the vector space $\mathcal{F}$ is a flexible algebra and "False" otherwise.

```
> flexible:=proc(n)
>   local L,M,N,P;
>   L:=[];M:=[];N:=[];P:=[];
>    for i from 1 to n do
>     L:=[op(L),i,i];
>    od;
>   M:=permute(L,2);
>   for j from 1 to nops(M) do
>    eq[j]:=bracket(e[M[j][1]],bracket(e[M[j][2]],e[M[j][1]],n),n)-
bracket(bracket(e[M[j][1]],e[M[j][2]],n),e[M[j][1]],n);
>   od;
>  N:=[seq(eq[k], k=1..nops(M))];
>  for i from 1 to nops(N) do
>   if N[i]<>0 then
>     P:=[op(P),N[i]];
>   fi;
>  od;
>  if P=[] then return "True"
>   else return "False";
>  fi;
> end proc:
```

Example 3. *The following example is shown in order to illustrate the algorithmic procedure. It corresponds to the directed and weighted CW complex of Figure 6.*

According to the notation followed in this algorithm, one has to consider

```
> V=[1,2,3];
> E={[[1,2,1],1],[[2,1,1],1],[[2,2,2],1],[[1,3,2],1],[[3,1,2],1],
[[2,3,2],2],[[3,2,2],2],[[3,3,3],1]};
```

Now, by running the remaining procedure, the following is obtained

```
> assignment(V,E);
> flexible(dim);
>     "True"
```

Therefore, the directed and weighted CW complex is associated with a 3*-dimensional flexible algebra.*

7.2 Obtaining the (pseudo)digraph associated with a flexible algebra

In this subsection, a parametric family of flexible algebras is considered. The algorithmic method computes and draws all the (pseudo)digraphs associated

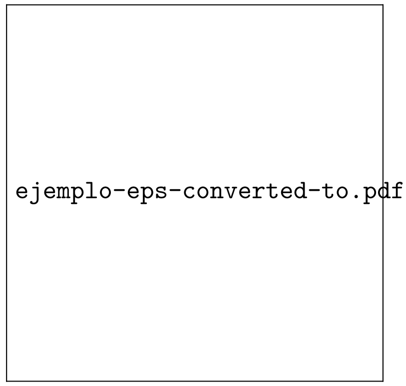

Figure 12: Example.

with flexible algebras obtained from the previous family. Under the same notation of the previous section, let $\mathcal{F}$ be an n-dimensional flexible algebra with basis $\mathcal{B}$ and law given in (1) in order to avoid the presence of full triangles. The algorithm is structured in four steps

1. Obtaining the bracket product between two arbitrary basis vectors in $\mathcal{B}$.

2. Computing the bracket between two vectors expressed as a linear combination of vectors from basis $\mathcal{B}$.

3. Imposing flexible identity and solving the corresponding system of equations.

4. Drawing the (pseudo)digraph associated with the flexible algebra $\mathcal{F}$.

This algorithm is implemented by using the symbolic computation package MAPLE 18 loading the libraries `linalg`, `combinat`, `GraphTheory` and `Maplets[Elements]`. The first three libraries provide commands of Linear Algebra, Combinatorics and Graph Theory, respectively; whereas the last is used to display a message so that the user introduces the required input in the first subprocedure, corresponding to the definition of the law of the algebra $\mathcal{F}$.

The first subprocedure, named `law2`, receives two natural numbers as inputs. These numbers represent the subindexes of two basis vectors in $\mathcal{B}$. The subprocedure returns the result of the bracket between these two vectors. In addition, conditional sentences are inserted to determine the non-zero brackets. Since the user has to complete the subprocedure inserting the non-zero brackets of $\mathcal{F}$, a sentence at the beginning of the implementation has been included reminding this fact. Note that before running any other sentence, all the variables and computation saved in the kernel must be deleted. Additionally, the value of variable `dim` has to be updated with the dimension of $\mathcal{F}$.

```
> restart:
> maplet:=Maplet(AlertDialog("Don't forget to introduce non-zero brackets
of the algebra and its dimension in subprocedure law",
'onapprove'=Shutdown("Continue"),'oncancel'=Shutdown("Aborted"))):
> Maplets[Display](maplet):
> assign(dim,...):
> law2:=proc(i,j)
>  if (i,j)=... then ...;
>  elif ....
>  else 0; fi;
> end proc;
```

The ellipsis in command `assign` corresponds to write the dimension of $\mathcal{F}$. The following two suspension points are associated with the computation of $[e_i, e_j]$: first, the value of the subindexes (i, j) and second, the result of $[e_i, e_j]$ with respect to $\mathcal{B}$. The last ellipsis denotes the rest of non-zero brackets. For each non-zero bracket, a new sentence `elif` has to be included in the cluster.

Now, one must consider the subprocedure `bracket` implemented in the previous subsection.

Next, the main procedure `flexible-parameters` is implemented. This procedure receives as input the dimension n of the vector space $\mathcal{F}$ with parameters in the bracket products and the output is the solution of a system of equations obtained from imposing the flexible identity. If the system has no solution, then one can conclude that the vector space $\mathcal{F}$ is not a flexible algebra. Otherwise, one obtains the conditions over the structure constants $c_{i,j}^k$ so that $\mathcal{F}$ is a flexible algebra.

```
> flexible-paramters:=proc(n)
> local L,M,N,P;
> L:=[];M:=[];N:=[];P:=[];
> for i from 1 to n do
>  L:=[op(L),i,i];
> od;
> M:=permute(L,2);
> for j from 1 to nops(M) do
eq[j]:=bracket(e[M[j][1]],bracket(e[M[j][2]],e[M[j][1]],n),n)-
bracket(bracket(e[M[j][1]],e[M[j][2]],n),e[M[j][1]],n);
>   od;
>  N:=[seq(eq[k], k=1..nops(M))];
> for k from 1 to nops(N) do
>  for h from 1 to n do
```

```
>   P:=[op(P),coeff(N[k],e[h])];
>  od;
> od;
> return solve(P);
> end proc:
```

Finally, the last step of this algorithmic procedure consists of the implementation to draw the (pseudo)digraph associated with each flexible algebra obtained in the previous step. To do so, first, one of the solutions generated by the main procedure `flexible-parameters` is considered. For this solution the command `associate` is executed in order to define the values of the structure constants. After that, let us proceed with the implementation of the procedure `drawing`. It receives as unique input the dimension `n` of the flexible algebra and draws the associated (pseudo)digraph. To implement the procedure, five local variables `E`, `G`, `L`, `S` and `V` have to be considered. The list `E` saves all the edges of the (pseudo)digraph, `G` is the variable used to generate such a (pseudo)digraph, list `L` will save the vertices with loops, list `V` consists of the list of vertices in `G`, and `S` is used to save the permutations of vertices in `V` chosen two by two. The general idea of the implementation is to evaluate which edges appear in the (pseudo)digraph studying if their weight is zero or not.

Since Maple cannot draw loops within a pseudodigraph, the routine uses the command `HighlightVertex` to change colour to dark blue for vertices with loops, different from the light yellow for the rest.

```
>drawing:=proc(n)
>  local E,G,L,S,V;
>  L:=[];E:=[];
>  V:=[seq(i,i=1..n)];
>  S=permute(V,2);
>  for i from 1 to nops(S) do
>    if law(S[i][1],S[i][2])<>0 then
>      E:={op(E),S[i]};
>    fi;
>  od;
>  for j from 1 to n do
>    if law(j,j)<>0 then
>      L:={op(L),[j,j]};
>    fi;
>  od;
>  G:=Digraph(V,E);
>  for k from 1 to nops(L) do
>     HighlightVertex(G,L[k]);
>  od;
>  DrawGraph(G);
>end proc:
```

Example 4. *Now, let us consider the 3-dimensional flexible algebra given by* *the law*

$$[e_1,e_1]=c_{1,1}^1e_1+c_{1,1}^2e_2+c_{1,1}^3e_3;\ [e_2,e_3]=c_{2,3}^2e_2+c_{2,3}^3e_3;\ [e_3,e_2]=c_{3,2}^2e_2+c_{3,2}^3e_3;$$

$$[e_2,e_2]=c_{2,2}^1e_1+c_{2,2}^2e_2+c_{2,2}^3e_3,\ [e_3,e_3]=c_{3,3}^1e_1+c_{3,3}^2e_2+c_{3,3}^3e_3.$$

First, the implementation of the subprocedure `law2` *must be completed as follows*

```
> if (i,j)=(1,1) then c111*e[1]+c112*e[2]+c113*e[3];
> elif (i,j)=(2,2) then c221*e[1]+c222*e[2]+c223*e[3];
> elif (i,j)=(2,3) then c232*e[2]+c233*e[3];
> elif (i,j)=(3,2) then c322*e[2]+c323*e[3];
> elif (i,j)=(3,3) then c331*e[1]+c332*e[2]+c333*e[3];
> else 0; fi;
```

After that, one must run the subprocedure `bracket` *and the procedure* `flexible-parameters`*. Now, evaluating the main procedure over the variable* `dim`*, one obtains the restrictions*

```
{c232=c322,c233=c323}, {c221=0,c222=-(-c232*c323+c233*c322)/(c232-c322),c223=0,
c331=0,c332=0,c333=-(c232*c323-c233*c322)/(c233-c323)}, {c221=0,c222=0,c223=0,
c232=c232,c233=0,c322=c322,c323=0,c332=0},{c223=0,c232=0,c322=0,c331=0,c332=0,
c333=0}.
```

From the previous output, several families of flexible algebras associated with (pseudo)digraphs can be obtained. For example, the family

$$[e_1,e_1]=e_1;\ [e_2,e_2]=e_2;\ [e_2,e_3]=e_2+e_3=[e_3,e_2];\ [e_3,e_3]=e_3$$

Therefore the following order must be executed

```
> assign({c111=1,c112=0,c113=0,c221=0,c222=1,c223=0,c331=0,c332=0,c333=1,c232=1,
c233=1,c322=1,c323=1});
```

Finally, the procedure `drawing` *is evaluated obtaining Figure 13, which corresponds to Configuration xv) from Figure 9.*

```
> drawing(dim);
```

7.3 Computational and complexity study

Now, a computational study and statistics related to the algorithmic procedure of Subsection 7.2 is shown. The algorithm has been implemented with MAPLE 18, in an Intel(R) Core(TM) i7-4510U CPU with a 2.60 GHz processor and 12.00 GB of RAM. Table 1 shows some computational data about both the computing time and the memory used to return the output of the whole procedure according to the value of the dimension n of the algebra.

Figure 13: Digraph corresponding to Configuration xv).

Input	**Computing time**	**Used memory**
$n = 3$	0.16 s	4.82 MB
$n = 4$	0.28 s	4.95 MB
$n = 5$	0.43 s	5.18 MB
$n = 6$	0.59 s	5.30 MB
$n = 7$	1.14 s	6.54 MB
$n = 8$	1.62 s	7.61 MB
$n = 9$	2.67 s	10.05 MB
$n = 10$	4.02 s	12.61 MB
$n = 11$	6.45 s	14.63 MB
$n = 12$	9.95 s	20.91 MB
$n = 13$	15.64 s	26.68 MB

Table 1: Computing time and used memory.

For this computational study, a general family of 2-step nilpotent flexible algebras has been considered. This family is associated with the generalization of Configuration $xvii$) from Figure 9.

Next, brief statistics about the relation between the computing time and the memory used by the implementation of the previous procedure is shown. Figures 14 and 15 show, respectively, the behavior of the computing time (C.T.) and used memory (U.M.) with respect to the dimension n. We can see how the computing time increases faster than the used memory and both of them fit a positive exponential model. Figure 16 represents a frequency diagram for the quotient between used memory and computing time. In this case, the behavior corresponds to a negative exponential model.

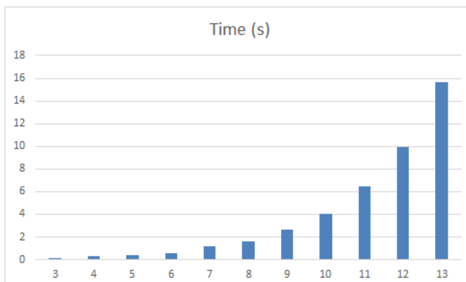

Figure 14: Graph for the C.T. with respect to dimension.

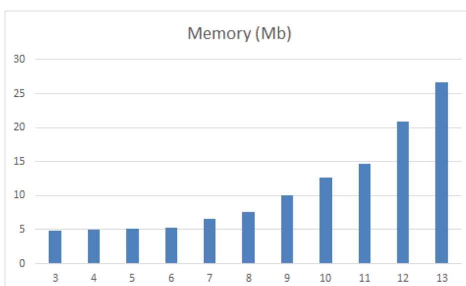

Figure 15: Graph for the U.M. with respect to dimension.

Finally, the complexity of the algorithm is computed considering the number of operations carried out in the worst case. In order to express the complexity, the big O notation is used (see [26]): Fixed two functions $f, g : \mathbb{R} \to \mathbb{R}$,

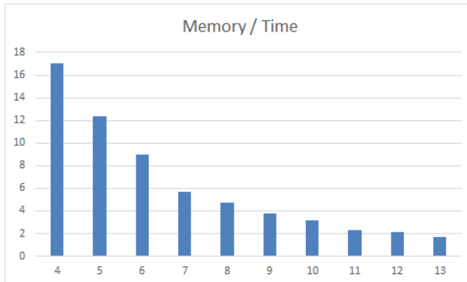

Figure 16: Graph for quotients U.M./C.T. with respect to dimension.

$f(x) = O(g(x))$ if and only if there exist $M \in \mathbb{R}^+$ and $x_0 \in \mathbb{R}$ such that $|f(x)| < M \cdot g(x)$, for all $x > x_0$.

Let us denote by $N_i(n)$ the number of operations for Step i. This function depends on the dimension n of the flexible algebra. Table 2 shows the number of computations and the complexity of each step.

Step	Routine	Complexity	Operations
1	`law2`	$O(n^2)$	$N_1(n) = 2 + \frac{n(n-1)}{2}$
2	`bracket`	$O(n^4)$	$N_2(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} N_1(n)$
3	`flexible-parameters`	$O(n^7)$	$N_3(n) = O(n) + O(n^3)$ $+2\sum_{i=1}^{n^3} N_2(n)+$ $2\sum_{j=1}^{n^3}\sum_{k=1}^{n} 1$
4	`drawing`	$O(n^4)$	$N_4(n) = O(n) + O(n^2)+$ $2\sum_{i=1}^{n^2} N_1(n)$

Table 2: Complexity and number of operations.

8 Conclusions

The tools and results shown in this paper may be useful and helpful for understanding the link between flexible algebras and directed and weighted CW complexes. In addition, this link may provide new methods to deal with open problems such as the classification of flexible algebras by means of the classification of their associated directed and weighted CW complexes.

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Conflict of interests

This manuscript has no conflict of interests.

Data statement

This manuscript has no associated data.

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9 Appendix

In this section, we show some lists of restrictions from Propositions 2, 3

Restrictions for Propositions 2 and 3:

- Configuration a): $(c^1_{1,1}, c^2_{1,1}) \neq (0,0)$.
- Configuration b): $(c^1_{j,j}, c^2_{j,j}) \neq (0,0)$, for $j = 1,2$.
- Configuration c): $(c^2_{1,2}, c^2_{2,1}) \neq (0,0)$.
- Configuration d): $c^1_{1,2}c^2_{2,1} - c^2_{1,2}c^1_{2,1} = 0 \wedge (c^j_{1,2}, c^j_{2,1}) \neq (0,0)$, for $j = 1,2$.
- Configuration e): $c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = 0 \wedge (c^1_{1,1}, c^2_{1,1}), (c^2_{1,2}, c^2_{2,1}) \neq (0,0)$.
- Configuration f): $c^1_{1,2} = c^1_{2,1} \neq 0, \quad \wedge (c^1_{1,1}, c^2_{1,1}), (c^1_{2,2}, c^2_{2,2}) \neq (0,0)$.
- Configuration g): $c^1_{1,2} = c^1_{2,1} \neq 0, \quad \wedge \quad (c^1_{1,1}, c^2_{1,1}) \neq (0,0)$.
- Configuration h): $c^2_{1,1}(c^j_{1,2} - c^j_{2,1}) = 0$, for $j = 1,2$, $c^1_{1,1}(c^1_{2,1} - c^1_{1,2}) = 0$, $c^1_{1,2}c^2_{2,1} - c^2_{1,2}c^1_{2,1} = 0, \quad \wedge \ (c^1_{1,1}, c^2_{1,1}), (c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1}) \neq (0,0)$.
- Configuration i): $c^2_{1,1}(c^j_{1,2} - c^j_{2,1}) = 0, c^1_{2,2}(c^j_{1,2} - c^j_{2,1}) = 0, c^j_{j,j}(c^1_{2,1} - c^1_{1,2}) + c^1_{1,2}c^2_{2,1} - c^2_{1,2}c^1_{2,1} = 0$, for $j = 1,2, \quad \wedge \ (c^1_{j,j}, c^2_{j,j}), (c^j_{1,2}, c^j_{2,1}) \neq (0,0)$ for $j = 1,2$.

Restrictions for Proposition 5:

- Configuration i): $(c^1_{1,1}, c^2_{1,1}, c^3_{1,1}) \neq (0,0,0)$.
- Configuration ii): $(c^1_{j,j}, c^2_{j,j}, c^3_{j,j}) \neq (0,0,0)$, for $j = 1,3$.
- Configuration iii): $(c^1_{j,j}, c^2_{j,j}, c^3_{j,j}) \neq (0,0,0)$, for $j = 1,2,3$.
- Configuration iv): $(c^3_{2,3}, c^3_{3,2}) \neq (0,0)$.
- Configuration v): $(c^3_{2,3}, c^3_{3,2}) \neq (0,0)$, $(c^1_{1,1}, c^2_{1,1}, c^3_{1,1}) \neq (0,0,0)$.
- Configuration vi): $c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = 0$, $(c^3_{2,3}, c^3_{3,2}) \neq (0,0)$, $(c^1_{2,2}, c^2_{2,2}, c^3_{2,2}) \neq (0,0,0)$.
- Configuration vii): $c^3_{2,3} = c^3_{3,2}$, $(c^3_{2,3}, c^3_{3,2}) \neq (0,0)$, $(c^1_{3,3}, c^2_{3,3}, c^3_{3,3}) \neq (0,0,0)$.
- Configuration $viii$): $c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = 0$, $(c^3_{2,3}, c^3_{3,2}) \neq (0,0)$, $(c^1_{j,j}, c^2_{j,j}, c^3_{j,j}) \neq (0,0,0)$, for $j = 1,2$.

- Configuration *ix*): $c_{2,3}^3 = c_{3,2}^3$, $(c_{2,3}^3, c_{3,2}^3) \neq (0,0)$, $(c_{j,j}^1, c_{j,j}^2, c_{j,j}^3) \neq (0,0,0)$, for $j = 1,3$.
- Configuration *x*): $c_{2,3}^3 = c_{3,2}^3$, $(c_{2,3}^3, c_{3,2}^3) \neq (0,0)$, $(c_{j,j}^1, c_{j,j}^2, c_{j,j}^3) \neq (0,0,0)$, for $j = 2,3$.
- Configuration *xi*): $c_{2,3}^3 = c_{3,2}^3$, $(c_{2,3}^3, c_{3,2}^3) \neq (0,0)$, $(c_{j,j}^1, c_{j,j}^2, c_{j,j}^3) \neq (0,0,0)$, for $j = 1,2,3$.
- Configuration *xii*): $c_{2,3}^2 c_{3,2}^3 = c_{2,3}^3 c_{3,2}^2$, $(c_{2,3}^2, c_{3,2}^2), (c_{2,3}^3, c_{3,2}^3) \neq (0,0)$.
- Configuration *xiii*): $c_{2,3}^2 c_{3,2}^3 = c_{2,3}^3 c_{3,2}^2$, $(c_{2,3}^2, c_{3,2}^2), (c_{2,3}^3, c_{3,2}^3) \neq (0,0)$, $(c_{1,1}^1, c_{1,1}^2, c_{1,1}^3) \neq (0,0,0)$.
- Configuration *xiv*): $c_{2,2}^3(c_{2,3}^2 - c_{3,2}^2) = c_{2,2}^2(c_{2,3}^3 - c_{3,2}^3) = c_{2,2}^j(c_{2,3}^2 - c_{3,2}^2) = c_{3,2}^3 c_{2,3}^2 - c_{2,3}^3 c_{3,2}^2 = 0$, for $j = 1,2$, $(c_{2,3}^2, c_{3,2}^2), (c_{2,3}^3, c_{3,2}^3) \neq (0,0)$, $(c_{2,2}^1, c_{2,2}^2, c_{2,2}^3) \neq (0,0,0)$.
- Configuration *xv*): $c_{2,2}^3(c_{2,3}^2 - c_{3,2}^2) = c_{2,2}^2(c_{2,3}^3 - c_{3,2}^3) = c_{2,2}^j(c_{2,3}^2 - c_{3,2}^2) = c_{3,2}^3 c_{2,3}^2 - c_{2,3}^3 c_{3,2}^2 = 0$, $(c_{2,3}^2, c_{3,2}^2), (c_{2,3}^3, c_{3,2}^3) \neq (0,0)$, $(c_{j,j}^1, c_{j,j}^2, c_{j,j}^3) \neq (0,0,0)$, for $j = 1,2$.
- Configuration *xvi*): $c_{2,2}^3(c_{2,3}^3 - c_{3,2}^3) = c_{2,2}^j(c_{2,3}^2 - c_{3,2}^2) = c_{3,3}^k(c_{2,3}^3 - c_{3,2}^3) = c_{3,3}^2(c_{2,3}^2 - c_{3,2}^2) = -c_{2,3}^2 c_{2,2}^2 + c_{3,2}^2 c_{2,2}^2 + c_{3,2}^3 c_{2,3}^2 - c_{2,3}^3 c_{3,2}^2 = c_{3,2}^3 c_{2,3}^2 - c_{2,3}^3 c_{3,2}^2 + c_{2,3}^3 c_{3,3}^3 - c_{3,2}^3 c_{3,3}^3 = 0$, for $j = 1,3$, $k = 1,2$, $(c_{2,3}^2, c_{3,2}^2), (c_{2,3}^3, c_{3,2}^3) \neq (0,0)$, $(c_{\ell,\ell}^1, c_{\ell,\ell}^2, c_{\ell,\ell}^3) \neq (0,0,0)$, for $\ell = 2,3$.
- Configuration *xvii*): $c_{2,2}^3(c_{2,3}^3 - c_{3,2}^3) = c_{2,2}^3(c_{2,3}^2 - c_{3,2}^2) = c_{j+1,j+1}^j(c_{2,3}^2 - c_{3,2}^2) = c_{3,3}^j(c_{2,3}^3 - c_{3,2}^3) = -c_{2,3}^2 c_{2,2}^2 + c_{3,2}^2 c_{2,2}^2 + c_{3,2}^3 c_{2,3}^2 - c_{2,3}^3 c_{3,2}^2 = c_{3,2}^3 c_{2,3}^2 - c_{2,3}^3 c_{3,2}^2 + c_{2,3}^3 c_{3,3}^3 - c_{3,2}^3 c_{3,3}^3 = 0$, for $j = 1,2$ $(c_{2,3}^2, c_{3,2}^2)$,
$(c_{2,3}^3, c_{3,2}^3) \neq (0,0)$, $(c_{\ell,\ell}^1, c_{\ell,\ell}^2, c_{\ell,\ell}^3) \neq (0,0,0)$, for $\ell = 1,2,3$.

Restrictions for Proposition 7:

- Configuration *a*): $(c_{1,2}^2, c_{2,1}^2), (c_{2,3}^2, c_{3,2}^2) \neq (0,0)$.
- Configuration *b*): $(c_{1,2}^2, c_{2,1}^2), (c_{2,3}^3, c_{3,2}^3) \neq (0,0)$.
- Configuration *c*): $(c_{1,2}^1, c_{2,1}^1), (c_{2,3}^3, c_{3,2}^3) \neq (0,0)$.
- Configuration *d*): $c_{1,2}^1 c_{2,1}^2 = c_{1,2}^2 c_{2,1}^1$, $(c_{1,2}^1, c_{2,1}^1), (c_{1,2}^2, c_{2,1}^2), (c_{2,3}^3, c_{3,2}^3) \neq (0,0)$.
- Configuration *e*): $c_{1,2}^1 c_{2,1}^2 = c_{1,2}^2 c_{2,1}^1$, $(c_{1,2}^1, c_{2,1}^1), (c_{1,2}^2, c_{2,1}^2), (c_{2,3}^2, c_{3,2}^2) \neq (0,0)$.

- Configuration f): $c^1_{1,2}c^2_{2,1} = c^2_{1,2}c^1_{2,1}$, $c^2_{2,3}c^3_{3,2} = c^3_{2,3}c^2_{3,2}$, $(c^1_{1,2}, c^1_{2,1})$, $(c^2_{1,2}, c^2_{2,1})$, $(c^2_{2,3}, c^2_{3,2})$, $(c^3_{2,3}, c^3_{3,2}) \neq (0,0)$.
- Configuration g): $(c^1_{1,2}, c^1_{2,1})$, $(c^3_{1,3}, c^3_{3,1})$, $(c^2_{2,3}, c^2_{3,2}) \neq (0,0)$.
- Configuration h): $(c^2_{1,2}, c^2_{2,1})$, $(c^3_{1,3}, c^3_{3,1})$, $(c^2_{2,3}, c^2_{3,2}) \neq (0,0)$.
- Configuration i): $c^2_{2,3}c^3_{3,2} = c^3_{2,3}c^2_{3,2}$, $(c^2_{1,2}, c^2_{2,1})$, $(c^3_{1,3}, c^3_{3,1})$, $(c^2_{2,3}, c^2_{3,2})$, $(c^3_{2,3}, c^3_{3,2}) \neq (0,0)$.
- Configuration j): $c^2_{2,3}c^3_{3,2} = c^3_{2,3}c^2_{3,2}$, $(c^1_{1,2}, c^1_{2,1})$, $(c^3_{1,3}, c^3_{3,1})$, $(c^2_{2,3}, c^2_{3,2})$, $(c^3_{2,3}, c^3_{3,2}) \neq (0,0)$.
- Configuration k): $c^2_{2,3}c^3_{3,2} = c^3_{2,3}c^2_{3,2}$, $(c^1_{1,2}, c^1_{2,1})$, $(c^1_{1,3}, c^1_{3,1})$, $(c^2_{2,3}, c^2_{3,2})$, $(c^3_{2,3}, c^3_{3,2}) \neq (0,0)$.
- Configuration l): $c^1_{1,2}c^2_{2,1} = c^2_{1,2}c^1_{2,1}$, $c^2_{2,3}c^3_{3,2} = c^3_{2,3}c^2_{3,2}$, $(c^1_{1,2}, c^1_{2,1})$,$(c^2_{1,2}, c^2_{2,1})$, $(c^3_{1,3}, c^3_{3,1})$, $(c^2_{2,3}, c^2_{3,2})$, $(c^3_{2,3}, c^3_{3,2}) \neq (0,0)$.
- Configuration m): $c^1_{1,2}c^2_{2,1} = c^2_{1,2}c^1_{2,1}$, $c^1_{1,3}c^3_{3,1} = c^3_{1,3}c^1_{3,1}$, $c^2_{2,3}c^3_{3,2} = c^3_{2,3}c^2_{3,2}$, $(c^1_{1,2}, c^1_{2,1})$, $(c^2_{1,2}, c^2_{2,1})$, $(c^3_{1,3}, c^3_{3,1})$, $(c^2_{2,3}, c^2_{3,2})$, $(c^3_{2,3}, c^3_{3,2}) \neq (0,0)$.

Restrictions for Proposition 9:

- Configuration 1): $c^2_{1,1}c^2_{1,2} - c^2_{1,1}c^2_{2,1} = 0$, $(c^2_{1,2}, c^2_{2,1}), (c^2_{2,3}, c^2_{3,2}) \neq (0,0)$, $(c^1_{1,1}, c^2_{1,1}, c^3_{1,1}) \neq (0,0,0)$.
- Configuration 2): $c^2_{1,2} = c^2_{2,1}$, $(c^2_{1,2}, c^2_{2,1}), (c^2_{2,3}, c^2_{3,2}) \neq (0,0)$, $(c^1_{2,2}, c^2_{2,2}, c^3_{2,2}) \neq (0,0,0)$.
- Configuration 3): $c^2_{1,2} = c^2_{2,1}$, $c^2_{2,3} = c^2_{3,2}$, $(c^2_{1,2}, c^2_{2,1}), (c^2_{2,3}, c^2_{3,2}) \neq (0,0)$, $(c^1_{i,i}, c^2_{i,i}, c^3_{i,i}) \neq (0,0,0)$, for $i = 1,2$.
- Configuration 4): $c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = c^2_{3,3}(c^2_{3,2} - c^2_{2,3}) = 0$, $(c^2_{1,2}, c^2_{2,1}), (c^2_{2,3}, c^2_{3,2}) \neq (0,0)$, $(c^1_{i,i}, c^2_{i,i}, c^3_{i,i}) \neq (0,0,0)$, for $i = 1,3$.
- Configuration 5): $c^2_{1,2} = c^2_{2,1}, c^2_{2,3} = c^2_{3,2}$, $(c^2_{1,2}, c^2_{2,1}), (c^2_{2,3}, c^2_{3,2}) \neq (0,0)$, $(c^1_{i,i}, c^2_{i,i}, c^3_{i,i}) \neq (0,0,0)$, for $i = 1,2,3$.
- Configuration 6): $c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = 0$, $(c^2_{1,2}, c^2_{2,1}), (c^2_{2,3}, c^2_{3,2}) \neq (0,0)$, $(c^1_{1,1}, c^2_{1,1}, c^3_{1,1}) \neq (0,0,0)$.
- Configuration 7): $c^1_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^2_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^3_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = 0$, $(c^2_{1,2}, c^2_{2,1}), (c^2_{2,3}, c^2_{3,2}) \neq (0,0)$, $(c^1_{2,2}, c^2_{2,2}, c^3_{2,2}) \neq (0,0,0)$.

- Configuration 8): $c^3_{2,3}=c^3_{3,2}$, $(c^2_{1,2},c^2_{2,1}),(c^2_{2,3},c^2_{3,2})\neq(0,0)$, $(c^1_{3,3},c^2_{3,3},c^3_{3,3})\neq(0,0,0)$.
- Configuration 9): $c^2_{1,2}=c^2_{2,1}, c^3_{2,3}=c^3_{3,2}$, $(c^2_{1,2},c^2_{2,1}),(c^2_{2,3},c^2_{3,2})\neq(0,0)$, $(c^1_{i,i},c^2_{i,i},c^3_{i,i})\neq(0,0,0)$, for $i=1,2$.
- Configuration 10): $c^2_{1,2}=c^2_{2,1}, c^3_{2,3}=c^3_{3,2}$, $(c^2_{1,2},c^2_{2,1}),(c^2_{2,3},c^2_{3,2})\neq(0,0)$, $(c^1_{i,i},c^2_{i,i},c^3_{i,i})\neq(0,0,0)$, for $i=1,3$.
- Configuration 11): $c^2_{1,2}=c^2_{2,1}, c^3_{2,3}=c^3_{3,2}$, $(c^2_{1,2},c^2_{2,1}),(c^2_{2,3},c^2_{3,2})\neq(0,0)$, $(c^1_{i,i},c^2_{i,i},c^3_{i,i})\neq(0,0,0)$, for $i=2,3$.
- Configuration 12): $c^2_{1,2}=c^2_{2,1}, c^3_{2,3}=c^3_{3,2}$, $(c^2_{1,2},c^2_{2,1}),(c^2_{2,3},c^2_{3,2})\neq(0,0)$, $(c^1_{i,i},c^2_{i,i},c^3_{i,i})\neq(0,0,0)$, for $i=1,2,3$.
- Configuration 13): $c^1_{1,2}=c^1_{2,1}$, $(c^1_{1,2},c^1_{2,1}),(c^3_{2,3},c^3_{3,2})\neq(0,0)$, $(c^1_{1,1},c^2_{1,1},c^3_{1,1})\neq(0,0,0)$.
- Configuration 14): $c^1_{2,2}(c^1_{1,2}-c^1_{2,1})=c^3_{2,2}(c^3_{2,3}-c^3_{3,2})=0$, $(c^1_{1,2},c^1_{2,1}),(c^3_{2,3},c^3_{3,2})\neq(0,0)$, $(c^1_{2,2},c^2_{2,2},c^3_{2,2})\neq(0,0,0)$.
- Configuration 15): $c^1_{1,1}(c^1_{1,2}-c^1_{2,1})=c^2_{1,1}(c^1_{1,2}-c^1_{2,1})=c^3_{1,1}(c^1_{1,2}-c^1_{2,1})=c^1_{2,2}(c^1_{1,2}-c^1_{2,1})=c^3_{2,2}(c^3_{2,3}-c^3_{3,2})=0$, $(c^1_{1,2},c^1_{2,1}),(c^3_{2,3},c^3_{3,2})\neq(0,0)$, $(c^1_{i,i},c^2_{i,i},c^3_{i,i})\neq(0,0,0)$, for $i=1,2$.
- Configuration 16): $c^1_{1,2}=c^1_{2,1}$, $c^3_{2,3}=c^3_{3,2}$, $(c^1_{1,2},c^1_{2,1}),(c^3_{2,3},c^3_{3,2})\neq(0,0)$, $(c^1_{i,i},c^2_{i,i},c^3_{i,i})\neq(0,0,0)$, for $i=1,3$.
- Configuration 17): $c^1_{1,2}=c^1_{2,1}$, $c^3_{2,3}=c^3_{3,2}$, $(c^1_{1,2},c^1_{2,1}),(c^3_{2,3},c^3_{3,2})\neq(0,0)$, $(c^1_{i,i},c^2_{i,i},c^3_{i,i})\neq(0,0,0)$, for $i=1,2,3$.
- Configuration 18): $c^i_{1,1}(c^1_{1,2}-c^1_{2,1})=c^2_{1,1}(c^2_{1,2}-c^2_{2,1})=c^2_{2,1}c^1_{1,2}-c^2_{1,2}c^1_{2,1}=0$, for $i=1,2,3$, $(c^1_{1,2},c^1_{2,1}),(c^2_{1,2},c^2_{2,1}),(c^2_{2,3},c^2_{3,2})\neq(0,0)$, $(c^1_{1,1},c^2_{1,1},c^3_{1,1})\neq(0,0,0)$.
- Configuration 19): $c^2_{2,1}c^1_{1,2}-c^2_{1,2}c^1_{2,1}=c^2_{2,1}c^1_{1,2}-c^2_{1,2}c^1_{2,1}+c^2_{1,2}c^2_{2,2}-c^2_{2,1}c^2_{2,2}=c^3_{2,2}(c^2_{1,2}-c^2_{2,1})=c^1_{2,2}(c^1_{1,2}-c^1_{2,1})=c^1_{2,2}(c^2_{1,2}-c^2_{2,1})=c^3_{2,2}(c^3_{2,3}-c^3_{3,2})=0$, $(c^1_{1,2},c^1_{2,1}),(c^2_{1,2},c^2_{2,1})$, $(c^2_{2,3},c^2_{3,2})\neq(0,0)$, $(c^1_{2,2},c^2_{2,2},c^3_{2,2})\neq(0,0,0)$.
- Configuration 20): $c^1_{1,2}c^2_{2,1}=c^2_{1,2}c^1_{2,1}, c^3_{2,3}=c^3_{3,2}$, $(c^1_{1,2},c^1_{2,1}),(c^2_{1,2},c^2_{2,1})$, $(c^2_{2,3},c^2_{3,2})\neq(0,0)$, $(c^1_{3,3},c^2_{3,3},c^3_{3,3})\neq(0,0,0)$.

- Configuration 21): $-c^1_{1,2}c^1_{1,1} + c^1_{2,1}c^1_{1,1} + c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} + c^2_{1,2}c^2_{2,2} - c^2_{2,1}c^2_{2,2} = c^{i+1}_{i,i}(c^2_{1,2} - c^2_{2,1}) = c^j_{1,1}(c^1_{1,2} - c^1_{2,1}) = c^1_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^1_{2,2}(c^1_{1,2} - c^1_{2,1}) = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = 0$, for $i = 1, 2$, $j = 2, 3$, $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1}), (c^2_{2,3}, c^2_{3,2}) \neq (0, 0)$, $(c^1_{\ell,\ell}, c^2_{\ell,\ell}, c^3_{\ell,\ell}) \neq (0, 0, 0)$, for $\ell = 1, 2$.
- Configuration 22): $c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^i_{1,1}(c^1_{1,2} - c^1_{2,1}) = c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = c^i_{3,3}(c^3_{2,3} - c^3_{3,2}) = 0$, for $i = 1, 2, 3$ $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1}), (c^2_{2,3}, c^2_{3,2}) \neq (0, 0)$, $(c^1_{\ell,\ell}, c^2_{\ell,\ell}, c^3_{\ell,\ell}) \neq (0, 0, 0)$, for $\ell = 1, 3$.
- Configuration 23): $c^1_{1,2} = c^1_{2,1}, c^2_{1,2} = c^2_{2,1}, c^3_{2,3} = c^3_{3,2}$, $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1}), (c^2_{2,3}, c^2_{3,2}) \neq (0, 0)$, $(c^1_{\ell,\ell}, c^2_{\ell,\ell}, c^3_{\ell,\ell}) \neq (0, 0, 0)$, for $\ell = 2, 3$.
- Configuration 24): $-c^1_{1,2}c^1_{1,1} + c^1_{2,1}c^1_{1,1} + c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} + c^2_{1,2}c^2_{2,2} - c^2_{2,1}c^2_{2,2} = c^3_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^1_{2,2}(c^1_{1,2} - c^1_{2,1}) = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = c^1_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^j_{i,i}(c^1_{1,2} - c^1_{2,1}) = c^k_{3,3}(c^3_{2,3} - c^3_{3,2}) =$, for $i = 1$, $j = 2, 3$, $k = 1, 2, 3$, $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1})$, $(c^2_{2,3}, c^2_{3,2}) \neq (0, 0)$, $(c^1_{\ell,\ell}, c^2_{\ell,\ell}, c^3_{\ell,\ell}) \neq (0, 0, 0)$, for $\ell = 1, 2, 3$.
- Configuration 25): $c^i_{1,1}(c^1_{1,2} - c^1_{2,1}) = c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = 0$, for $i = 1, 2, 3$, $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1}), (c^3_{2,3}, c^3_{3,2}) \neq (0, 0)$, $(c^1_{1,1}, c^2_{1,1}, c^3_{1,1}) \neq (0, 0, 0)$.
- Configuration 26): $c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} + c^2_{1,2}c^2_{2,2} - c^2_{2,1}c^2_{2,2} = c^3_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^1_{2,2}(c^1_{1,2} - c^1_{2,1}) = c^1_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = 0$, $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1})$, $(c^3_{2,3}, c^3_{3,2}) \neq (0, 0)$, $(c^1_{2,2}, c^2_{2,2}, c^3_{2,2}) \neq (0, 0, 0)$.
- Configuration 27): $c^1_{1,2}c^2_{2,1} = c^2_{1,2}c^1_{2,1}, c^3_{2,3} = c^3_{3,2}$, $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1})$, $(c^3_{2,3}, c^3_{3,2}) \neq (0, 0)$, $(c^1_{3,3}, c^2_{3,3}, c^3_{3,3}) \neq (0, 0, 0)$.
- Configuration 28): $-c^1_{1,2}c^1_{1,1} + c^1_{2,1}c^1_{1,1} + c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} + c^2_{1,2}c^2_{2,2} - c^2_{2,1}c^2_{2,2} = c^{i+1}_{i,i}(c^2_{1,2} - c^2_{2,1}) = c^j_{1,1}(c^1_{1,2} - c^1_{2,1}) = c^1_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^1_{2,2}(c^1_{1,2} - c^1_{2,1}) = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = 0$, for $i = 1, 2$, $j = 2, 3$, $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1}), (c^3_{2,3}, c^3_{3,2}) \neq (0, 0)$, $(c^1_{\ell,\ell}, c^2_{\ell,\ell}, c^3_{\ell,\ell}) \neq (0, 0, 0)$, for $\ell = 1, 2$.
- Configuration 29): $c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^i_{1,1}(c^1_{1,2} - c^1_{2,1}) = c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = c^i_{3,3}(c^3_{2,3} - c^3_{3,2}) = 0$, for $i = 1, 2, 3$ $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1}), (c^3_{2,3}, c^3_{3,2}) \neq (0, 0)$, $(c^1_{\ell,\ell}, c^2_{\ell,\ell}, c^3_{\ell,\ell}) \neq (0, 0, 0)$, for $\ell = 1, 3$.
- Configuration 30): $c^1_{1,2} = c^1_{2,1}, c^2_{1,2} = c^2_{2,1}, c^3_{2,3} = c^3_{3,2}$, $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1}), (c^3_{2,3}, c^3_{3,2}) \neq (0, 0)$, $(c^1_{\ell,\ell}, c^2_{\ell,\ell}, c^3_{\ell,\ell}) \neq (0, 0, 0)$, for $\ell = 2, 3$.

- Configuration 31): $-c^1_{1,2}c^1_{1,1} + c^1_{2,1}c^1_{1,1} + c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} + c^2_{1,2}c^2_{2,2} - c^2_{2,1}c^2_{2,2} = c^3_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^1_{2,2}(c^1_{1,2} - c^1_{2,1}) = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = c^1_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^j_{i,i}(c^1_{1,2} - c^1_{2,1}) = c^k_{3,3}(c^3_{2,3} - c^3_{3,2}) =$, for $i = 1$, $j = 2, 3$, $k = 1, 2, 3$, $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1}), (c^3_{2,3}, c^3_{3,2}) \neq (0, 0)$, $(c^1_{\ell,\ell}, c^2_{\ell,\ell}, c^3_{\ell,\ell}) \neq (0, 0, 0)$, for $\ell = 1, 2, 3$.

- Configuration 32): $c^i_{1,1}(c^1_{1,2} - c^1_{2,1}) = c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = 0$, for $i = 1, 2, 3$, $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1}), (c^2_{2,3}, c^2_{3,2})$, $(c^3_{2,3}, c^3_{3,2}) \neq (0, 0)$, $(c^1_{1,1}, c^2_{1,1}, c^3_{1,1}) \neq (0, 0, 0)$.

- Configuration 33): $c^i_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^i_{2,2}(c^2_{2,3} - c^2_{3,2}) = c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^1_{2,2}(c^1_{1,2} - c^1_{2,1}) = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = -c^2_{1,2}c^1_{2,2} + c^2_{2,1}c^1_{2,2} + c^3_{2,2}c^2_{2,3} - c^3_{2,2}c^2_{3,2} = 0$, for $i = 1, 2, 3$, $(c^1_{1,2}, c^1_{2,1})$, $(c^2_{1,2}, c^2_{2,1})$, $(c^2_{2,3}, c^2_{3,2}), (c^3_{2,3}, c^3_{3,2}) \neq (0, 0)$, $(c^1_{2,2}, c^2_{2,2}, c^3_{2,2}) \neq (0, 0, 0)$.

- Configuration 34): $c^i_{1,1}(c^1_{1,2} - c^1_{2,1}) = c^1_{2,2}(c^1_{1,2} - c^1_{2,1}) = c^j_{2,2}(c^2_{2,3} - c^2_{3,2}) = c^k_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = c^1_{2,2}(c^2_{2,3} - c^2_{3,2}) = -c^1_{1,2}c^1_{1,1} + c^1_{2,1}c^1_{1,1} + c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} + c^2_{1,2}c^2_{2,2} - c^2_{2,1}c^2_{2,2} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = 0$, for $i = 2, 3$, $j = 2, 3$, $k = 1, 3$, $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1}), (c^2_{2,3}, c^2_{3,2}), (c^3_{2,3}, c^3_{3,2}) \neq (0, 0)$, $(c^1_{\ell,\ell}, c^2_{\ell,\ell}, c^3_{\ell,\ell}) \neq (0, 0, 0)$, for $\ell = 1, 2$.

- Configuration 35): $-c^1_{1,2}c^1_{1,1} + c^1_{2,1}c^1_{1,1} + c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = -c^2_{1,2}c^1_{2,2} + c^2_{2,1}c^1_{2,2} + c^3_{2,2}c^2_{2,3} - c^3_{2,2}c^2_{3,2} = -c^2_{2,3}c^2_{2,2} + c^2_{3,2}c^2_{2,2} + c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} + c^2_{1,2}c^2_{2,2} - c^2_{2,1}c^2_{2,2} = c^i_{1,1}(c^1_{1,2} - c^1_{2,1}) = c^1_{2,2}(c^1_{1,2} - c^1_{2,1}) = c^j_{2,2}(c^2_{2,3} - c^2_{3,2}) = c^2_{3,3}(c^2_{2,3} - c^2_{3,2}) = c^k_{3,3}(c^3_{2,3} - c^3_{3,2}) = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = c^j_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} + c^3_{2,3}c^3_{3,3} - c^3_{3,2}c^3_{3,3} = 0$, for $i = 2, 3$, $j = 1, 3$, $k = 1, 2$, $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1}), (c^2_{2,3}, c^2_{3,2})$, $(c^3_{2,3}, c^3_{3,2}) \neq (0, 0)$, $(c^1_{\ell,\ell}, c^2_{\ell,\ell}, c^3_{\ell,\ell}) \neq (0, 0, 0)$, for $\ell = 1, 2, 3$.

- Configuration 36): $c^3_{1,1}(c^3_{1,3} - c^3_{3,1}), c^i_{1,1}(c^1_{2,1} - c^1_{1,2}) = 0$, for $i = 1, 2, 3$, $(c^1_{1,2}, c^1_{2,1})$, $(c^3_{1,3}, c^3_{3,1}), (c^2_{2,3}, c^2_{3,2}) \neq (0, 0), (c^1_{1,1}, c^2_{1,1}, c^3_{1,1}) \neq (0, 0, 0)$

- Configuration 37): $c^3_{1,1}(c^3_{1,3} - c^3_{3,1}), c^i_{1,1}(c^1_{2,1} - c^1_{1,2}) = c^i_{2,2}(c^2_{3,2} - c^2_{2,3}) = c^1_{2,2}(c^1_{2,1} - c^1_{1,2}) = 0$, for $i = 1, 2, 3$, $(c^1_{1,2}, c^1_{2,1}), (c^3_{1,3}, c^3_{3,1}), (c^2_{2,3}, c^2_{3,2}) \neq (0, 0), (c^1_{j,j}, c^2_{j,j}, c^3_{j,j}) \neq (0, 0, 0)$, for $j = 1, 2$.

- Configuration 38): $c^1_{1,2} = c^1_{2,1}, c^3_{1,3} = c^3_{3,1}, c^2_{2,3} = c^2_{3,2}, (c^1_{1,2}, c^1_{2,1}), (c^3_{1,3}, c^3_{3,1})$, $(c^2_{2,3}, c^2_{3,2}) \neq (0, 0), (c^1_{j,j}, c^2_{j,j}, c^3_{j,j}) \neq (0, 0, 0)$, for $j = 1, 2, 3$.

- Configuration 39): $c^i_{1,1}(c^2_{1,2} - c^2_{2,1}) = 0$, for $i = 2, 3$, $(c^2_{1,2}, c^2_{2,1}), (c^3_{1,3}, c^3_{3,1})$, $(c^2_{2,3}, c^2_{3,2}) \neq (0, 0), (c^1_{1,1}, c^2_{1,1}, c^3_{1,1}) \neq (0, 0, 0)$.

- Configuration 40): $c^2_{1,2} = c^2_{2,1}$, $c^2_{2,3} = c^2_{3,2}$, $(c^2_{1,2}, c^2_{2,1}), (c^3_{1,3}, c^3_{3,1})$, $(c^2_{2,3}, c^2_{3,2}) \neq (0,0)$, $(c^1_{2,2}, c^2_{2,2}, c^3_{2,2}) \neq (0,0,0)$.

- Configuration 41): $-c^2_{1,2}c^1_{2,2} + c^2_{2,1}c^1_{2,2} + c^2_{2,3}c^3_{2,2} - c^2_{3,2}c^3_{2,2} = c^i_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^i_{2,2}(c^2_{3,2} - c^2_{2,3}) = c^j_{1,1}(c^j_{1,j} - c^j_{j,1}) = 0$, for $i = 1,2,3$, $j = 2,3$, $(c^2_{1,2}, c^2_{2,1}), (c^3_{1,3}, c^3_{3,1}), (c^2_{2,3}, c^2_{3,2}) \neq (0,0), (c^1_{k,k}, c^2_{k,k}, c^3_{k,k}) \neq (0,0,0)$, for $k = 1,2$.

- Configuration 42): $c^2_{1,2} = c^2_{2,1}, c^3_{1,3} = c^3_{3,1}, c^2_{2,3} = c^2_{3,2}, (c^2_{1,2}, c^2_{2,1}), (c^3_{1,3}, c^3_{3,1})$, $(c^2_{2,3}, c^2_{3,2}) \neq (0,0), (c^1_{j,j}, c^2_{j,j}, c^3_{j,j}) \neq (0,0,0)$, for $j = 1,2,3$.

- Configuration 43): $c^1_{1,2} = c^1_{2,1}, c^1_{1,3} = c^1_{3,1}$, $(c^1_{1,2}, c^1_{2,1})$, $(c^1_{1,3}, c^1_{3,1})$, $(c^2_{2,3}, c^2_{3,2}) \neq (0,0)$, $(c^1_{1,1}, c^2_{1,1}, c^3_{1,1}) \neq (0,0,0)$.

- Configuration 44): $c^1_{2,2}(c^1_{1,2} - c^1_{2,1}) = c^i_{2,2}(c^2_{3,2} - c^2_{2,3}) = 0$, for $i = 1,2,3$, $(c^1_{1,2}, c^1_{2,1})$, $(c^1_{1,3}, c^1_{3,1})$, $(c^2_{2,3}, c^2_{3,2}) \neq (0,0)$, $(c^1_{2,2}, c^2_{2,2}, c^3_{2,2}) \neq (0,0,0)$.

- Configuration 45): $c^1_{1,2} = c^1_{2,1}, c^1_{1,3} = c^1_{3,1}, c^2_{2,3} = c^2_{3,2}, (c^1_{1,2}, c^1_{2,1}), (c^1_{1,3}, c^1_{3,1})$, $(c^2_{2,3}, c^2_{3,2}) \neq (0,0), (c^1_{i,i}, c^2_{i,i}, c^3_{i,i}) \neq (0,0,0)$, for $i = 1,2$.

- Configuration 46): $c^1_{1,2} = c^1_{2,1}, c^1_{1,3} = c^1_{3,1}, c^2_{2,3} = c^2_{3,2}, (c^1_{1,2}, c^1_{2,1}), (c^1_{1,3}, c^1_{3,1})$, $(c^2_{2,3}, c^2_{3,2}) \neq (0,0), (c^1_{i,i}, c^2_{i,i}, c^3_{i,i}) \neq (0,0,0)$, for $i = 1,2,3$.

- Configuration 47): $c^i_{1,1}(c^2_{1,2} - c^2_{2,1}) = c^2_{2,3}c^3_{3,2} - c^3_{2,3}c^2_{3,2} = 0$, for $i = 2,3$, $(c^2_{1,2}, c^2_{2,1})$, $(c^3_{1,3}, c^3_{3,1})$, $(c^2_{2,3}, c^2_{3,2}), (c^3_{2,3}, c^3_{3,2}) \neq (0,0), (c^1_{1,1}, c^2_{1,1}, c^3_{1,1}) \neq (0,0,0)$.

- Configuration 48): $c^i_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^k_{2,2}(c^2_{2,3} - c^2_{3,2}) = c^2_{1,2}c^1_{2,2} + c^2_{2,1}c^1_{2,2} = -c^2_{2,3}c^2_{2,2} + c^2_{3,2}c^2_{2,2} + c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = 0$, for $i = 1,2,3$, $k = 1,3$ $(c^2_{1,2}, c^2_{2,1})$, $(c^3_{1,3}, c^3_{3,1}), (c^2_{2,3}, c^2_{3,2}), (c^3_{2,3}, c^3_{3,2}) \neq (0,0)$, $(c^1_{2,2}, c^2_{2,2}, c^3_{2,2}) \neq (0,0,0)$.

- Configuration 49): $c^i_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^j_{1,1}(c^j_{1,j} - c^j_{j,1}) = c^k_{2,2}(c^2_{2,3} - c^2_{3,2}) = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = c^2_{1,2}c^1_{2,2} + c^2_{2,1}c^1_{2,2} = -c^2_{2,3}c^2_{2,2} + c^2_{3,2}c^2_{2,2} + c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = 0$, for $i = 1,2,3$, $j = 2,3$, $k = 1,3$, $(c^2_{1,2}, c^2_{2,1}), (c^3_{1,3}, c^3_{3,1}), (c^2_{2,3}, c^2_{3,2}), (c^3_{2,3}, c^3_{3,2}) \neq (0,0)$, $(c^1_{\ell,\ell}, c^2_{\ell,\ell}, c^3_{\ell,\ell}) \neq (0,0,0)$, for $\ell = 1,2$.

- Configuration 50): $c^i_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^i_{3,3}(c^3_{1,3} - c^3_{3,1}) = c^j_{3,3}(c^3_{2,3} - c^3_{3,2}) = c^k_{2,2}(c^2_{2,3} - c^2_{3,2}) = c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = c^2_{1,2}c^1_{2,2} + c^2_{2,1}c^1_{2,2} = -c^2_{2,3}c^2_{2,2} + c^2_{3,2}c^2_{2,2} + c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = c^2_{3,3}(c^2_{2,3} - c^2_{3,2}) = c^3_{1,1}(c^3_{1,3} - c^3_{3,1}) = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} + c^3_{2,3}c^3_{3,3} - c^3_{3,2}c^3_{3,3} = -c^3_{1,3}c^1_{3,3} - c^3_{2,3}c^2_{3,3} + c^3_{3,1}c^1_{3,3} + c^3_{3,2}c^2_{3,3} = 0$, for $i = 1,2,3$, $j = 1,2$, $k = 1,3$, $c^1_{1,2} = c^1_{2,1}, c^1_{1,3} = c^1_{3,1}, c^2_{2,3} =$

$c^2_{3,2}, (c^2_{1,2}, c^2_{2,1}), (c^3_{1,3}, c^3_{3,1}), (c^2_{2,3}, c^2_{3,2}), (c^3_{2,3}, c^3_{3,2}) \neq (0,0), (c^1_{\ell,\ell}, c^2_{\ell,\ell}, c^3_{\ell,\ell}) \neq (0,0,0)$, for $\ell = 1,2,3$.

- Configuration 51): $c^3_{1,1}(c^3_{1,3} - c^3_{3,1}) = c^i_{1,1}(-c^1_{2,1} + c^1_{1,2}) = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = 0$, for $i = 1,2,3$, $(c^1_{1,2}, c^1_{2,1}), (c^3_{1,3}, c^3_{3,1}), (c^2_{2,3}, c^2_{3,2}), (c^3_{2,3}, c^3_{3,2}) \neq (0,0)$, $(c^1_{1,1}, c^2_{1,1}, c^3_{1,1}) \neq (0,0,0)$.

- Configuration 52): $c^1_{2,2}(c^1_{1,2} - c^1_{2,1}) = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = c^i_{2,2}(c^2_{2,3} - c^2_{3,2}) = -c^2_{2,3}c^2_{2,2} + c^2_{3,2}c^2_{2,2} + c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = 0$, for $i = 1,3$, $(c^1_{1,2}, c^1_{2,1})$, $(c^3_{1,3}, c^3_{3,1})$, $(c^2_{2,3}, c^2_{3,2})$, $(c^3_{2,3}, c^3_{3,2}) \neq (0,0)$, $(c^1_{2,2}, c^2_{2,2}, c^3_{2,2}) \neq (0,0,0)$.

- Configuration 53): $c^3_{1,3} = c^3_{3,1}, c^2_{2,3} = c^2_{3,2}, c^3_{2,3} = c^3_{3,2}, (c^1_{1,2}, c^1_{2,1}), (c^3_{1,3}, c^3_{3,1}), (c^2_{2,3}, c^2_{3,2}), (c^3_{2,3}, c^3_{3,2}) \neq (0,0), (c^1_{3,3}, c^2_{3,3}, c^3_{3,3}) \neq (0,0,0)$.

- Configuration 54): $c^3_{1,1}(c^3_{1,3} - c^3_{3,1}) = c^i_{1,1}(-c^1_{2,1} + c^1_{1,2}) = -c^1_{2,2}(-c^1_{2,1} + c^1_{1,2}) = c^j_{2,2}(c^2_{2,3} - c^2_{3,2}) = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = -c^2_{2,3}c^2_{2,2} + c^2_{3,2}c^2_{2,2} + c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = 0$, for $i = 1,2,3$, $j = 1,3$, $(c^1_{1,2}, c^1_{2,1}), (c^3_{1,3}, c^3_{3,1}), (c^2_{2,3}, c^2_{3,2}), (c^3_{2,3}, c^3_{3,2}) \neq (0,0), (c^1_{i,i}, c^2_{i,i}, c^3_{i,i}) \neq (0,0,0)$, for $i = 1,2$.

- Configuration 55): $c^1_{1,2} = c^1_{2,1}, c^3_{1,3} = c^3_{3,1}, c^2_{2,3} = c^2_{3,2}, c^3_{2,3} = c^3_{3,2}, (c^1_{1,2}, c^1_{2,1}), (c^3_{1,3}, c^3_{3,1})$, $(c^2_{2,3}, c^2_{3,2})$, $(c^3_{2,3}, c^3_{3,2}) \neq (0,0)$, $(c^1_{i,i}, c^2_{i,i}, c^3_{i,i}) \neq (0,0,0)$, for $i = 1,3$.

- Configuration 56): $c^3_{1,1}(c^3_{1,3} - c^3_{3,1}) = -c^1_{2,2}(-c^1_{2,1} + c^1_{1,2}) = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = c^2_{3,3}(c^2_{2,3} - c^2_{3,2}) = -c^2_{2,3}c^2_{2,2} + c^2_{3,2}c^2_{2,2} + c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} + c^3_{2,3}c^3_{3,3} - c^3_{3,2}c^3_{3,3} = -c^3_{1,3}c^1_{3,3} - c^3_{2,3}c^2_{3,3} + c^3_{3,1}c^1_{3,3} + c^3_{3,2}c^2_{3,3} = c^i_{1,1}(c^1_{2,1} - c^1_{1,2}) = c^i_{3,3}(c^3_{1,3} - c^3_{3,1}) = c^j_{3,3}(c^3_{2,3} - c^3_{3,2}) = c^k_{2,2}(c^2_{2,3} - c^2_{3,2}) = 0$, for $i = 1,2,3$, $j = 1,2$, $k = 1,3$, $(c^1_{1,2}, c^1_{2,1})$, $(c^3_{1,3}, c^3_{3,1})$, $(c^2_{2,3}, c^2_{3,2})$, $(c^3_{2,3}, c^3_{3,2}) \neq (0,0)$, $(c^1_{i,i}, c^2_{i,i}, c^3_{i,i}) \neq (0,0,0)$, for $i = 1,2,3$.

- Configuration 57): $c^2_{1,1}c^1_{1,2} - c^2_{1,1}c^1_{2,1} + c^3_{1,1}c^1_{1,3} - c^3_{1,1}c^1_{3,1} = -c^1_{1,2}c^1_{1,1} + c^1_{2,1}c^1_{1,1} + c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = c^j_{1,1}(c^1_{1,2} - c^1_{2,1}) = c^i_{1,1}(-c^1_{3,1} + c^1_{1,3}) = 0$, for $i = 1,2,3, j = 2,3$, $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1})$, $(c^1_{1,3}, c^1_{3,1})$, $(c^2_{2,3}, c^2_{3,2}), (c^3_{2,3}, c^3_{3,2}) \neq (0,0), (c^1_{1,1}, c^2_{1,1}, c^3_{1,1}) \neq (0,0,0)$.

- Configuration 58): $c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} + c^2_{1,2}c^2_{2,2} - c^2_{2,1}c^2_{2,2} = -c^2_{1,2}c^1_{2,2} + c^2_{2,1}c^1_{2,2} + c^3_{2,2}c^2_{2,3} - c^3_{2,2}c^2_{3,2} = -c^2_{2,3}c^2_{2,2} + c^2_{3,2}c^2_{2,2} + c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = c^i_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^i_{2,2}(c^2_{2,3} - c^2_{3,2}) = c^1_{2,2}(c^1_{1,2} - c^1_{2,1}) = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = 0$, for $i = 1,3$, $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1}), (c^1_{1,3}, c^1_{3,1}), (c^2_{2,3}, c^2_{3,2}), (c^3_{2,3}, c^3_{3,2}) \neq (0,0)$, $(c^1_{2,2}, c^2_{2,2}, c^3_{2,2}) \neq (0,0,0)$.

- Configuration 59): $c^1_{1,2}c^2_{2,1} - c^2_{1,2}c^1_{2,1} = c^1_{1,2}c^2_{2,1} - c^2_{1,2}c^1_{2,1} = c^2_{2,3}c^3_{3,2} - c^3_{2,3}c^2_{3,2} = c^2_{2,3}c^3_{3,2} - c^3_{2,3}c^2_{3,2} + c^3_{2,3}c^3_{3,3} - c^3_{3,2}c^3_{3,3} = c^i_{3,3}(c^3_{2,3} - c^3_{3,2}) = -c^1_{3,3}(c^1_{1,3}-c^1_{3,1}) = -c^2_{3,3}(c^2_{2,3}-c^2_{3,2}) = 0$, for $i = 1,2$, $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1}), (c^1_{1,3}, c^1_{3,1}), (c^2_{2,3}, c^2_{3,2}), (c^3_{2,3}, c^3_{3,2}) \neq (0,0)$, $(c^1_{3,3}, c^2_{3,3}, c^3_{3,3}) \neq (0,0,0)$.

- Configuration 60): $c^2_{1,1}c^1_{1,2} - c^2_{1,1}c^1_{2,1} + c^3_{1,1}c^1_{1,3} - c^3_{1,1}c^1_{3,1} = -c^1_{1,2}c^1_{1,1} + c^1_{2,1}c^1_{1,1} + c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} + c^2_{1,2}c^2_{2,2} - c^2_{2,1}c^2_{2,2} = -c^2_{2,3}c^2_{2,2} + c^2_{3,2}c^2_{2,2} + c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = -c^2_{1,2}c^1_{2,2} + c^2_{2,1}c^1_{2,2} + c^3_{2,2}c^2_{2,3} - c^3_{2,2}c^2_{3,2} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2}c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = c^i_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^i_{2,2}(c^2_{2,3} - c^2_{3,2}) = c^j_{1,1}(c^1_{1,2} - c^1_{2,1}) = c^1_{2,2}(c^1_{1,2} - c^1_{2,1}) = c^k_{1,1}(-c^1_{3,1} + c^1_{1,3}) = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = 0$, $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1}), (c^1_{1,3}, c^1_{3,1}), (c^2_{2,3}, c^2_{3,2}), (c^3_{2,3}, c^3_{3,2}) \neq (0,0)$, $(c^1_{\ell,\ell}, c^2_{\ell,\ell}, c^3_{\ell,\ell}) \neq (0,0,0)$, for $i = 1,3, j = 2,3, k = 1,2,3, \ell = 1,2$,

- Configuration 61): $c^2_{1,1}c^1_{1,2} - c^2_{1,1}c^1_{2,1} + c^3_{1,1}c^1_{1,3} - c^3_{1,1}c^1_{3,1} = -c^1_{1,2}c^1_{1,1} + c^1_{2,1}c^1_{1,1} + c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} + c^3_{2,3}c^3_{3,3} - c^3_{3,2}c^3_{3,3} = c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = c^2_{3,3}(c^2_{2,3} - c^2_{3,2}) = c^1_{3,3}(-c^1_{3,1} + c^1_{1,3}) = c^i_{1,1}(c^1_{1,2} - c^1_{2,1}) = c^j_{1,1}(-c^1_{3,1} + c^1_{1,3}) = c^k_{3,3}(c^3_{2,3} - c^3_{3,2}) = 0$, $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1}), (c^1_{1,3}, c^1_{3,1}), (c^2_{2,3}, c^2_{3,2}), (c^3_{2,3}, c^3_{3,2}) \neq (0,0)$, $(c^1_{\ell,\ell}, c^2_{\ell,\ell}, c^3_{\ell,\ell}) \neq (0,0,0)$, for $i = 2,3, j = 1,2,3, k = 1,2, \ell = 1,3$,

- Configuration 62): $c^2_{1,1}c^1_{1,2} - c^2_{1,1}c^1_{2,1} + c^3_{1,1}c^1_{1,3} - c^3_{1,1}c^1_{3,1} = -c^1_{1,2}c^1_{1,1} + c^1_{2,1}c^1_{1,1} + c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} + c^2_{1,2}c^2_{2,2} - c^2_{2,1}c^2_{2,2} = -c^2_{2,3}c^2_{2,2} + c^2_{3,2}c^2_{2,2} + c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = -c^2_{1,2}c^1_{2,2} + c^2_{2,1}c^1_{2,2} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^3_{3,2} + c^3_{2,3}c^3_{3,3} - c^3_{3,2}c^3_{3,3} = c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = -c^1_{2,2}(c^1_{1,2} - c^1_{2,1}) = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = c^2_{3,3}(c^2_{2,3} - c^2_{3,2}) = c^i_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^j_{1,1}(c^1_{1,2} - c^1_{2,1}) = c^k_{1,1}(-c^1_{3,1} + c^1_{1,3}) = c^\ell_{3,3}(-c^1_{3,1} + c^1_{1,3}) = c^i_{2,2}(c^2_{2,3} - c^2_{3,2}) = c^\ell_{3,3}(c^3_{2,3} - c^3_{3,2}) = 0$, $(c^1_{1,2}, c^1_{2,1}), (c^2_{1,2}, c^2_{2,1}), (c^1_{1,3}, c^1_{3,1}), (c^2_{2,3}, c^2_{3,2}), (c^3_{2,3}, c^3_{3,2}) \neq (0,0)$, $(c^1_{k,k}, c^2_{k,k}, c^3_{k,k}) \neq (0,0,0)$, for $i = 1,3, j = 2,3, k = 1,2,3, \ell = 1,2$.

- Configuration 63): $c^2_{1,1}c^1_{1,2} - c^2_{1,1}c^1_{2,1} + c^3_{1,1}c^1_{1,3} - c^3_{1,1}c^1_{3,1} = -c^1_{1,2}c^1_{1,1} + c^1_{2,1}c^1_{1,1} + c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = -c^1_{1,3}c^1_{1,1} + c^1_{3,1}c^1_{1,1} + c^3_{3,1}c^1_{1,3} - c^3_{1,3}c^1_{3,1} = c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = c^3_{3,1}c^1_{1,3} - c^3_{1,3}c^1_{3,1} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = c^3_{1,1}(c^3_{1,3} - c^3_{3,1}) = c^k_{1,1}(c^1_{1,2} - c^1_{2,1}) = c^k_{1,1}(c^1_{1,3} - c^1_{3,1}) = 0$, $(c^i_{1,2}, c^i_{2,1}), (c^j_{1,3}, c^j_{3,1}), (c^k_{2,3}, c^k_{3,2}) \neq (0,0), (c^1_{1,1}, c^2_{1,1}, c^3_{1,1}) \neq (0,0,0)$, for $i = 1,2, j = 1,3, k = 2,3$.

- Configuration 64): $c^2_{1,1}c^1_{1,2} - c^2_{1,1}c^1_{2,1} + c^3_{1,1}c^1_{1,3} - c^3_{1,1}c^1_{3,1} = -c^1_{1,2}c^1_{1,1} + c^1_{2,1}c^1_{1,1} + c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = -c^1_{1,3}c^1_{1,1} + c^1_{3,1}c^1_{1,1} + c^3_{3,1}c^1_{1,3} - c^3_{1,3}c^1_{3,1} = c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} + c^2_{1,2}c^2_{2,2} - c^2_{2,1}c^2_{2,2} = -c^2_{1,2}c^1_{2,2} + c^2_{2,1}c^1_{2,2} + c^3_{2,2}c^2_{2,3} - c^3_{2,2}c^2_{3,2} = c^3_{3,1}c^1_{1,3} - c^3_{1,3}c^1_{3,1} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = -c^2_{2,3}c^2_{2,2} + c^2_{3,2}c^2_{2,2} + c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = c^j_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^3_{1,1}(c^3_{1,3} -$

$c^3_{3,1}) = c^k_{1,1}(c^1_{1,2} - c^1_{2,1}) = c^1_{2,2}(c^1_{1,2} - c^1_{2,1}) = c^k_{1,1}(c^1_{1,3} - c^1_{3,1}) = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = c^j_{2,2}(c^2_{2,3} - c^2_{3,2}) = 0$, $(c^i_{1,2}, c^i_{2,1}), (c^j_{1,3}, c^j_{3,1}), (c^k_{2,3}, c^k_{3,2}) \neq (0,0)$, $(c^1_{i,i}, c^2_{i,i}, c^3_{i,i}) \neq (0,0,0)$, for $i = 1,2, j = 1,3, k = 2,3$.

- Configuration 65): $c^2_{1,1}c^1_{1,2} - c^2_{1,1}c^1_{2,1} + c^3_{1,1}c^1_{1,3} - c^3_{1,1}c^1_{3,1} = -c^1_{1,2}c^1_{1,1} + c^1_{2,1}c^1_{1,1} + c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} = c^2_{2,1}c^1_{1,2} - c^2_{1,2}c^1_{2,1} + c^2_{1,2}c^2_{2,2} - c^2_{2,1}c^2_{2,2} = -c^1_{1,3}c^1_{1,1} + c^1_{3,1}c^1_{1,1} + c^3_{3,1}c^1_{1,3} - c^3_{1,3}c^1_{3,1} = -c^2_{2,3}c^2_{2,2} + c^2_{3,2}c^2_{2,2} + c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} = c^3_{3,1}c^1_{1,3} - c^3_{1,3}c^1_{3,1} + c^3_{1,3}c^3_{3,3} - c^3_{3,1}c^3_{3,3} = c^3_{3,2}c^2_{2,3} - c^3_{2,3}c^2_{3,2} + c^3_{2,3}c^3_{3,3} - c^3_{3,2}c^3_{3,3} = -c^2_{1,2}c^1_{2,2} + c^2_{2,1}c^1_{2,2} + c^3_{2,2}c^2_{2,3} - c^3_{2,2}c^2_{3,2} = -c^3_{1,3}c^1_{3,3} - c^3_{2,3}c^2_{3,3} + c^3_{3,1}c^1_{3,3} + c^3_{3,2}c^2_{3,3} = c^2_{1,1}(c^2_{1,2} - c^2_{2,1}) = c^1_{3,3}(c^1_{1,3} - c^1_{3,1}) = c^2_{3,3}(c^2_{2,3} - c^2_{3,2}) = c^3_{1,1}(c^3_{1,3} - c^3_{3,1}) = c^k_{1,1}(c^1_{1,2} - c^1_{2,1}) = c^k_{1,1}(c^1_{1,3} - c^1_{3,1}) = c^j_{2,2}(c^2_{1,2} - c^2_{2,1}) = c^1_{2,2}(c^1_{1,2} - c^1_{2,1}) = c^3_{2,2}(c^3_{2,3} - c^3_{3,2}) = c^j_{2,2}(c^2_{2,3} - c^2_{3,2}) = c^i_{3,3}(c^3_{1,3} - c^3_{3,1}) = c^i_{3,3}(c^3_{2,3} - c^3_{3,2}) = 0$, $(c^i_{1,2}, c^i_{2,1}), (c^j_{1,3}, c^j_{3,1}), (c^k_{2,3}, c^k_{3,2}) \neq (0,0)$, $(c^1_{\ell,\ell}, c^2_{\ell,\ell}, c^3_{\ell,\ell}) \neq (0,0,0)$, for $i = 1,2, j = 1,3, k = 2,3, \ell = 1,2,3$.

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