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# A Special Universe and its Massless Scalar Field Mathematical Structure

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**Abstract**

The present paper highlights an exotic universe having a metric which is partially suggested by the metric of an universe without time seen in [2]. The new metric is globally defined and depends on an exotic matter created by some classical waves. The massless scalar field (MSF) of this universe is also analyzed. Its existence depends on some constants $k, k_1$ and $k_2$, called MSF constants, which appear when we try to solve some ordinary differential equations deduced from the Klein-Gordon PDE attached to its metric. The MSF factor is the number $\alpha := -k_1 + k_2$. We prove that the massless scalar field of this exotic universe exists if and only if $\alpha = 0$.

## 1 Introduction

We know that it is possible to find solutions of the Einstein field equations which describe solar systems, black holes, expanding universes, but also solutions which describe exotic universes models, see ([1], [3], [4], [5], [6], [7], [8]). Even if exotic universes are part of possible imaginary mathematical worlds, they offer us ideas about characteristics that our universe cannot have.

The exotic universe we present is related to the one seen in [2].
It is created starting from the metric

$$ds^2 = e^{x^3}(dx^0)^2 + (dx^1)^2 + (dx^2)^2 - (dx^3)^2$$

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after we consider an expansion of the last three terms of factor $e^{x^3}$. It is a sort of Lemître expansion, which, we prove that leads to a globally defined Einstein solution created by an exotic matter depending on classical waves.

In this paper we also analyze the massless scalar field of the studied exotic universe. The way we solve in Theorem 5 the Klein-Gordon PDE leads to two possible massless scalar field type solutions, each one corresponding to a special number $\alpha$. The study made in Theorem 4 bellow allows to show that only one between these two numbers generates the "real massless scalar field" which acts in this universe.

## 2 A wave type geometric matter and its corresponding global exotic universe

As we explained above, we consider a sort of Lemaître type metric attached to the metric seen in [2], i.e.

$$ds^2 = e^{x^3}(dx^0)^2 + e^{x^0}[(dx^1)^2 + (dx^2)^2 - (dx^3)^2]$$

on $M = \mathbf{R}^4$.
The computations show that the only nonzero Christoffel symbols are

$$\Gamma^0_{03} = \Gamma^0_{30} = \frac{1}{2};\ \Gamma^0_{11} = \Gamma^0_{22} = -\frac{1}{2}e^{x^0-x^3};\ \Gamma^0_{33} = \frac{1}{2}e^{x^0-x^3},$$

$$\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \frac{1}{2},$$

$$\Gamma^3_{00} = \frac{1}{2}e^{x^3-x^0};\ \Gamma^3_{03} = \Gamma^3_{30} = \frac{1}{2},$$

i.e. the only nonzero Ricci tensor components are

$$R_{00} = \frac{1}{4}e^{x^3-x^0} - \frac{3}{4};\ R_{11} = -\frac{3}{4}e^{x^0-x^3} = R_{22};\ R_{33} = -\frac{1}{4} + \frac{3}{4}e^{x^0-x^3}.$$

It results

$$R = \frac{1}{2}e^{-x^0} - 3e^{-x^3}.$$

Replacing, the Einstein's field equations are

$$R_{ij} - \frac{1}{2}R\ g_{ij} = 8\pi G\ T_{ij}$$

where the exotic matter is represented by the tensor

$$T_{ij} = \frac{1}{32\pi G}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3e^{x^0-x^3}-1 & 0 & 0 \\ 0 & 0 & 3e^{x^0-x^3}-1 & 0 \\ 0 & 0 & 0 & -3e^{x^0-x^3} \end{pmatrix}.$$

Therefore the metric

$$ds^2 = e^{x^3}(dx^0)^2 + e^{x^0}[(dx^1)^2 + (dx^2)^2 - (dx^3)^2]$$

satisfies Einstein's field equations on $M = \mathbf{R}^4$. If we generic denote by $\Psi$ the non-zero terms of the energy-momentum tensor $T_{ij}$, we observe that a classical wave equation is satisfied,

$$\frac{\partial^2\Psi}{(\partial x^0)^2} - \frac{\partial^2\Psi}{(\partial x^3)^2} = 0, .$$

We can say that the "exotic matter" is created by classical waves.

## 3 The mathematical structure of the massless scalar field of our exotic universe

To step forward this problem we consider first the massless scalar field mathematical structure in the case when $x^1 = x^2 = 0$. Denote $t := x^0$ and $z := x^3$. The massless scalar field (MSF), here seen through the function $u(t, z)$ is described by the partial differential equation

$$e^z u_{tt} - e^t u_{zz} = 0.$$

To solve it, we propose the "auswahl" $u(t, z) = a(t)b(z)$. Replacing into the above equation we obtain

$$\frac{a''(t)}{e^t a(t)} = \frac{b''(z)}{e^z b(z)},$$

where $a''$ and $b''$ are the notations we use to express the second derivative with respect $t$, respectively $z$. The first member depends only on $t$ and the second one only on $z$, therefore both members are constant.
Denote the constant by $k \in \mathbf{R}$.
**Theorem 4:** i) If $k = 0$, the MSF $u_0(t, z)$ corresponding to the PDE

$$e^z u_{tt} - e^t u_{zz} = 0$$

is described by the product of two first degree polynomials, i.e.

$$u_0(t,z) = (C_1 t + C_2)(C_3 z + C_4),\ C_m \in \mathbf{R}.$$

ii) If $k \neq 0$ the MSF satisfying the previous PDE, now denoted by $u_k(t,z)$, has the form

$$u_k(t,z) = C \sum_{n,m=0}^{\infty} \frac{k^{n+m}}{(n!)^2(m!)^2} e^{nt+mz}.$$

**Proof:** The simple case $k = 0$ leads to $a(t) = C_1 t + C_2$. Therefore the MSF corresponding to the constant $k = 0$, denoted here $u_0$, has the form

$$u_0(t,z) = (C_1 t + C_2)(C_1 z + C_2).$$

We concentrate on the case when $k \neq 0$. Therefore we have to solve

$$a''(t) - k e^t a(t) = 0.$$

Denote $e^t = v^2$. It results $t = 2\ln v$ and $a(t) = a(2\ln v) =: \phi(v)$.
Therefore

$$\phi'(v) = \frac{2}{v} a'(2\ln v),$$

$$\phi''(v) = -\frac{2}{v^2} a'(2\ln v) + \frac{4}{v^2} a''(2\ln v).$$

Replacing in the last expression the equality $a''(t) = k e^t a(t) = k v^2 \phi(v)$, it results a version of Bessel ODE, here

$$v^2 \phi''(v) + v\phi'(v) - 4kv^2\phi(v) = 0,$$

which can be solved via the classical method. We choose the solution described by a formal series

$$\phi(v) = \sum_{n=0}^{\infty} \phi_n v^n,$$

where the coefficients $\phi_n$ are determined after we transform the Bessel ODE into the formal identity

$$\phi_1 v + \sum_{n=2}^{\infty} [n^2 \phi_n - 4k\phi_{n-2}] v^n \equiv 0.$$

We have $\phi_1 = 0$ and the relation $\phi_n = \dfrac{4k}{n^2}\phi_{n-2}$.
Since $\phi_{2n+1} = 0$ for all $n \in \mathbf{N}$ and $\phi_{2n} = \dfrac{k}{n^2}\phi_{2n-2}$ it results $\phi_{2n} = \dfrac{k^n}{(n!)^2}$. The

solution depending on an arbitrary coefficient $\phi_0$ is

$$\phi(v) = \phi_0 \sum_0^\infty \frac{k^n}{(n!)^2} v^{2n},$$

therefore

$$a(t) = \phi_0 \sum_0^\infty \frac{k^n}{(n!)^2} e^{nt}.$$

D'Alembert criterion shows us that this series converges, that is it describes a function. It results the form of the massless scalar field corresponding to the studied domain,

$$u_k(t,z) = C \sum_{n,m=0}^\infty \frac{k^{n+m}}{(n!)^2 (m!)^2} e^{nt+mz}. \qquad q.e.d.$$

In the general case, the MSF is described by a function $u(t,x,y,z)$ satisfying the PDE

$$e^z u_{tt} + e^t (u_{xx} + u_{yy} - u_{zz}) = 0.$$

To solve it, we propose the same type "auswahl" $u(t,x,y,z) = a(t)b(x)c(y)d(z)$. After simple computations we can arrange the equation in the form

$$\frac{a''(t)}{e^t a(t)} = \frac{-b''(x)c(y)d(z) - b(x)c''(y)d(z) + b(x)c(y)d''(z)}{e^z b(x)c(y)d(z)}.$$

The first member depends on $t$ only and the second member depends on $x$, $y$, $z$, therefore both members are constant. Again the constant can be chosen as $k$. Therefore we obtain two equations; the first one

$$a''(t) - ke^t a(t) = 0$$

was already studied both in the case when $k = 0$ and in the case when $k \neq 0$. In the first case we have a first degree polynomial solution, while in the second case we have obtained the solution

$$a(t) = \phi_0 \sum_0^\infty \frac{k^n}{(n!)^2} e^{nt}.$$

The second equation is deduced from

$$ke^z b(x)c(y)d(z) + b(x)c''(y)d(z) - b(x)c(y)d''(z) = -b''(x)c(y)d(z)$$

arranged as

$$-\frac{b''(x)}{b(x)} = \frac{ke^z c(y)d(z) + c''(y)d(z) - c(y)d''(z)}{c(y)d(z)}.$$

We continue to solve when $k \neq 0$. It results

$$-\frac{b''(x)}{b(x)} = k_1,\ \ k_1 \in \mathbf{R}.$$

The case $k_1 = 0$ is again simple, $b_0(x) = B_1 x + B_2$.
If $k_1 \neq 0$ the solution of this ODE depends on the the sign of $k_1$.
If $k_1 = -l^2$ the equation to solve is $b''(x) - l^2 b(x) = 0$, while if $k_1 = l^2$ the equation is $b''(x) + l^2 b(x) = 0$.
In the first case the solution is $b_-(x) = Ae^{lx} + Be^{-lx}$, while in the second case the solution is $b_+(x) = A\sin lx + B\cos lx$.
If we continue, from

$$k_1 c(y)d(z) = ke^z c(y)d(z) + c''(y)d(z) - c(y)d''(z)$$

we obtain

$$k_1 - \frac{c''(y)}{c(y)} = ke^z - \frac{d''(z)}{d(z)}.$$

Both members are constant, therefore we have to study the following two equations,

$$\frac{c''(y)}{c(y)} = k_2$$

and

$$ke^z - \frac{d''(z)}{d(z)} = k_1 - k_2.$$

The first differential equation is studied exactly as we did it for the function $b(x)$. According to the constant $k_2$ which can be 0, negative or positive, we have the solutions denoted by $c_0(y)$, $c_-(y)$ and $c_+$.

Now, denote $-\alpha := k_1 - k_2$.
In the same way in which the equation

$$a''(t) - ke^t a(t) = 0$$

was transformed into the Bessel ODE

$$v^2\phi''(v) + v\phi'(v) - 4kv^2\phi(v) = 0,$$

the last equation

$$\frac{d''(z)}{d(z)} = \alpha + ke^z$$

can be transformed via $e^z = v^2$ and $d(z) = d(2\ln v) = \phi(v)$ into the more general Bessel ODE

$$v^2\phi''(v) + v\phi'(v) - 4(kv^2 + \alpha)\phi(v) = 0.$$

Again, we choose the solution described by a formal series

$$\phi(v) = \sum_{n=0}^{\infty} \phi_n v^n,$$

where the coefficients $\phi_n$ are determined after we transform the Bessel ODE into the formal identity

$$-4\alpha\phi_0 + (1-4\alpha)\phi_1 v + \sum_{n=2}^{\infty}[(n^2 - 4\alpha)\phi_n - 4k\phi_{n-2}]v^n \equiv 0.$$

If $\alpha = 0$, the formal identity above transforms into

$$\phi_1 v + \sum_{n=2}^{\infty}[n^2\phi_n - 4k\phi_{n-2}]v^n \equiv 0,$$

which is exactly the identity we already studied in the previous theorem. In this case the solution depending on $\phi_0 = D$ is

$$\phi(v) = D\sum_{0}^{\infty} \frac{k^n}{(n!)^2} v^{2n},$$

therefore

$$d(z) = D\sum_{0}^{\infty} \frac{k^n}{(n!)^2} e^{nz}.$$

If $\alpha = \frac{1}{4}$ it results $\phi_0 = \phi_2 = ... = \phi_{2n} = ... = 0$ and

$$\phi_{2n+1} = \frac{k}{(n+1)n}\phi_{2n-1},$$

therefore the solution depending on $\phi_1 = E$ is

$$\phi(v) = E\sum_{0}^{\infty} \frac{k^n}{(n+1)(n!)^2} v^{2n+1}.$$

It results

$$d(z) = E\sum_{0}^{\infty} \frac{k^n}{(n+1)(n!)^2} e^{(n+1/2)z}.$$

The same d'Alembert criterion assures us that the series is a convergent one, so it describes a function.

If $\alpha \notin \left\{0, \frac{1}{4}\right\}$ it results that $\phi(v) \equiv 0$.

Let's go back to the case when $k = 0$. In all previous formulas we replace $k$ by 0. Therefore we have to solve the following equations:

$$-\frac{b''(x)}{b(x)} = k_1,\ \ k_1 \in \mathbf{R};$$

$$k_1 - \frac{c''(y)}{c(y)} = -\frac{d''(z)}{d(z)}.$$

The first one was already studied, the second one implies the following two equations

$$\frac{c''(y)}{c(y)} = k_2$$

and

$$-\frac{d''(z)}{d(z)} = k_1 - k_2.$$

Here, we wish to notice that only in the case when $k_1 = k_2$ we have a degree one polynomial solution.

**Definition:** The numbers $k, k_1, k_2$ are called the *MSF numbers* of the studied universe.

**Definition:** $\alpha := -k_1 + k_2$ is called the *MSF type factor.*

Let us denote the possible solutions of the above ODE in $x$ and $y$ by $b_\beta(x)$ and $c_\gamma(y)$, where $\beta$ and $\gamma$ belong to the set $\{0, +, -\}$. So, the indexes $\beta$ and $\gamma$ are determined by the MSF numbers $k_1$ and $k_2$.

We can conclude the previous results in the form of

**Theorem 5:** The Klein-Gordon equation of the universe expressed by the metric

$$ds^2 = e^{x^3}(dx^0)^2 + e^{x^0}[(dx^1)^2 + (dx^2)^2 - (dx^3)^2]$$

is

$$e^z u_{tt} + e^t(u_{xx} + u_{yy} - u_{zz}) = 0.$$

Consider the MSF numbers $k, k_1, k_2$ and the MSF type factor $\alpha = -k_1 + k_2$. The solutions

$$u(t, x, y, z) = a(t)b(x)c(y)d(z)$$

depending on the MSF numbers are:
i) If $k \neq 0$ and $\alpha \notin \left\{0, \frac{1}{4}\right\}$, then $u(t, x, y, z) \equiv 0$, that is the MSF is trivial.
ii) If $k = 0$ and $\alpha = 0$, then

$$u(t, x, y, z) = (A_1 t + A_2) b_\beta(x) c_\gamma(y) \cdot (D_1 z + D_2).$$

iii) If $k = 0$ and $\alpha \neq 0$, then

$$u(t, x, y, z) = (A_1 t + A_2) b_\beta(x) c_\gamma(y) \cdot d_\delta(z), \;\; \delta \neq 0.$$

iv) If $k \neq 0$ and $\alpha = 0$, then

$$u(t, x, y, z) = \phi_0 b_\beta(x) c_\gamma(y) \cdot D \sum_{n,m=0}^{\infty} \frac{k^{n+m}}{(n!)^2 (m!)^2} e^{nt+mz}.$$

iii) If $k \neq 0$ and $\alpha = \frac{1}{4}$, then

$$u(t, x, y, z) = b_\beta(x) c_\gamma(y) \cdot E\phi_0 \sum_{m,n=0}^{\infty} \frac{k^{m+n}}{(n+1)(m!)^2 (n!)^2} e^{mt+(n+1/2)z}.$$

According to Theorem 5 the most important fact related to the MSF of the studied universe is captured in the following

**Consequence 6:** *The massless scalar field of this universe exists if the MSF type factor $\alpha$ belongs to the set $\left\{0, \frac{1}{4}\right\}$.*

It is important to observe that between the two possible massless scalar fields types for our studied universe only one can be considered (as expected). Let us look at Theorem 5. If we compare the two possible MSF solutions obtained in the case when both $x = 0$ and $y = 0$ with the result obtained in Theorem 4, we deduce

**Theorem 7:** *The massless scalar field of the studied universe is described only by MSF type factor $\alpha = 0$.*

## 4 Conclusions

The metric we studied

$$ds^2 = e^{x^3}(dx^0)^2 + e^{x^0}[(dx^1)^2 + (dx^2)^2 - (dx^3)^2]$$

is related to the metrics seen both in [2] and [3]. It describes an exotic universe determined by a geometric matter determined by some classical waves.

It is important to note that the massless scalar field of this universe depends on three numbers $k$, $k_1$ and $k_2$, called MSF numbers. The massless scalar field of this universe exists if the constants $k_1$ and $k_2$ are well accorded to produce one the two possible MSF type factors, $\alpha = 0$ or $\alpha = \frac{1}{4}$.
There is a mathematical criterion which allows to distinguish between the two possible types of massless scalar fields the one that "functions" in this universe. To do this, we have to compare the results of Theorem 5 when $x = y = 0$ with the result of Theorem 4.
The massless scalar field exists if and only if the MSF type factor $\alpha$ is 0, i.e. $k_1$ and $k_2$ must be equal.

## References

[1] W.G. Boskoff, S. Capozziello, A Mathematical Journey to Relativity, Unitext for Physics, *Springer* (2020).

[2] W.G. Boskoff, A simple Gödel-type Universe, *Int. J. Geom. Meth. Mod. Phys.*, **20**, (1), 2350018, (2023).

[3] W.G. Boskoff, A universe without time produced by a disappearing wormhole, *Int. J. Geom. Meth. Mod. Phys.*, DOI: 10.1142/S021988782450110X (Dec 2023)

[4] L. Flamm, Beitrage zur einsteinschen gravitationsthorie, *Phys. Z.* **18**, 448, (1916).

[5] C. W. Misner, J. A. Wheeler, Classical Physics as Geometry: Gravitation, electromagnetism, unquantized charge and mass as properties of curved empty space, *Annals Phys.*, **2**, 525 (1957).

[6] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt, Exact Solutions of Einstein's Field Equations, Second Edition, *Cambridge University Press*, Cambridge (2003).

[7] K. Gödel, An example of a new type of cosmological solution of Einstein's field equations of gravitation, *Rev. Mod. Phys.*, **21**, 447 (1949).

[8] Bercu, Gabriel; Crasmareanu, Mircea, Classes of harmonic functions in 2D generalized Poincaré geometry,*Filomat*, **35**, No. 1, 287-297 (2021).

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