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On the Hilbert depth of the Hilbert function of a finitely generated graded module

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Abstract

Let K be a field, A a standard graded K-algebra and M a finitely generated graded A-module. Inspired by our previous works, see [2] and [3], we study the invariant called *Hilbert depth* of h_M, that is

$$\operatorname{hdepth}(h_M) = \max\{d \ : \ \sum_{j\leq k}(-1)^{k-j}\binom{d-j}{k-j}h_M(j) \geq 0 \text{ for all } k\leq d\},$$

where $h_M(-)$ is the Hilbert function of M, and we prove basic results regard it. Using the theory of hypergeometric functions, we prove that $\operatorname{hdepth}(h_S)=n$, where $S=K[x_1,\ldots,x_n]$.

We show that $\operatorname{hdepth}(h_{S/J})=n$, if $J=(f_1,\ldots,f_d)\subset S$ is a complete intersection monomial ideal with $\deg(f_i)\geq 2$ for all $1\leq i\leq d$. Also, we show that $\operatorname{hdepth}(h_{\overline{M}})\geq \operatorname{hdepth}(h_M)$ for any finitely generated graded S-module M, where $\overline{M}=M\otimes_S S[x_{n+1}]$.

Introduction

Let $S=K[x_1,\ldots,x_n]$ be the ring of polynomials in n variables over a field K. The Hilbert depth of a finitely graded S-module M is the maximal depth of a finitely graded S-module N with the same Hilbert series as M; see [7] for further details.

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In [2] we proved a new formula for the Hilbert depth of a quotient J/I of two squarefree monomial ideals $I \subset J \subset S$. This allowed us, in [3], to extend the definition of Hilbert depth to any (numerical) function $h : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ with the property that $h(j) = 0$ for $j \ll 0$.

More precisely, we set

$$\operatorname{hdepth}(h) := \max\{d \;:\; \sum_{j\leq k}(-1)^{k-j}\binom{d-j}{k-j}h(j) \geq 0 \text{ for all } k \leq d\}.$$

Let A be a standard graded K-algebra and M a finitely generated graded A-module. Since $h_M(-)$, the Hilbert function of M, has the property $h_M(j) = 0$ for $j \ll 0$, it makes sense to consider its Hilbert depth, as was defined above; see also Definition 1.1.

In some regards, this new invariant seems unnatural, as it is associated to the Hilbert series of M, seen as a numerical function, and not to M itself. However, we believe that the study of this invariant could be interesting as it reflects the growing pattern of the Hilbert series of M. We mention that in [3], we studied the Hilbert depth for polynomial functions with integer coefficients and other numerical functions.

Another fact that determined us to study this invariant, is the following: Let $I \subset J \subset S$ be squarefree monomial ideals. Although $\operatorname{hdepth}(J/I)$ and $\operatorname{hdepth}(h_{J/I})$ are not the same, there is the following subtle connection between these invariants: If

$$M(J/I) = (J + (x_1^2, \ldots, x_n^2))/(I + (x_1^2, \ldots, x_n^2)),$$

then $\operatorname{hdepth}(J/I) = \operatorname{hdepth}(h_{M(J/I)})$; see Proposition 1.4. Note that $M(J/I)$ is an Artinian S-module.

In Proposition 1.5 we prove that

$$k_0 \leq \operatorname{hdepth}(h_M) \leq k_0 + \frac{h_1}{h_0},$$

where $k_0 = k_0(M) = \min\{k \;:\; M_k \neq 0\}$, $h_0 = h_M(k_0)$ and $h_1 = h_M(k_0 + 1)$.

In Proposition 1.6 we prove that if M is of finite length, then

$$\operatorname{hdepth}(h_M) \leq k_f(M) := \max\{k \;:\; M_k \neq 0\}.$$

In particular, if $M = S/J$ where J is a graded Artinian ideal, we note in Corollary 1.7 that

$$\operatorname{hdepth}(h_{S/J}) \leq \operatorname{reg}(S/J).$$

In Proposition 1.9 we show that if $0 \to U \to M \to N \to 0$ is a short exact sequence of (nonzero) finitely generated graded A-modules, then

$$\operatorname{hdepth}(h_M) \geq \min\{\operatorname{hdepth}(h_U), \operatorname{hdepth}(h_N)\}.$$

In Proposition 1.11 we prove that

$$\operatorname{hdepth}(h_{M(m)}) = \operatorname{hdepth}(h_M) - m,$$

where $M(m)$ is the m-th shift module of M.

Using the sign of the hypergeometric function ${}_2F_1(-k, n, -n; -1)$, see Lemma 2.1, we prove in Theorem 2.2 that $\operatorname{hdepth}(h_S) = n$. Consequently, in Corollary 2.3 we prove that if

$$F = S(a)^{n_1} \oplus S(a-1)^{n_2} \oplus S(a_1) \oplus \cdots \oplus S(a_r),$$

where n_1, n_2, a, a_j are integers such that $n_1 > n_2 \geq 0$ and $a \geq a_j + 2$ for all $1 \leq j \leq r$, then

$$\operatorname{hdepth}(h_F) = n - a.$$

In Theorem 3.1 and Corollary 3.2 we prove that if $J = (f_1, \ldots, f_r) \subset S$ is a graded complete intersection with $\deg(f_i) \geq 2$ for all $1 \leq i \leq r$, where $0 \leq r \leq n$, then

$$\operatorname{hdepth}(h_{S/J}) = n.$$

In particular, for $r = 0$ we obtain a new proof of the fact that $\operatorname{hdepth}(h_S) = n$.

Finally, in Theorem 4.3 we show that $\operatorname{hdepth}(h_{\overline{M}}) \geq \operatorname{hdepth}(h_M)$, where $\overline{S} = S[x_{n+1}]$, M is a finitely generated S-module and $\overline{M} = M \otimes_S \overline{S}$.

1 Basic properties

Let K be a field and let

$$A = \bigoplus_{n \geq 0} A_n,$$

be a standard graded K-algebra, i.e. A is finitely generated, $A_0 = K$ and A_1 generates A.

Let

$$M = \bigoplus_{k \in \mathbb{Z}} M_k,$$

be a nonzero graded finitely generated A-module.

Since M is finitely generated, $\dim_K(M_k) < \infty$ for all $k \in \mathbb{Z}$ and $M_k = 0$ for $k \ll 0$. In particular, there exists $k_0(M) \in \mathbb{Z}$ such that

$$k_0(M) := \min\{k \ : \ M_k \neq 0\}.$$

We consider the *Hilbert function* of M, that is

$$h_M(-) : \mathbb{Z} \to \mathbb{Z}_{\geq 0},\ h_M(k) := \dim_K M_k,\ \text{for all } k \in \mathbb{Z}.$$

We recall the definition of the Hilbert depth of h_M from [3].

Let d be an integer and let

$$\beta_k^d(h_M) := \begin{cases} \sum\limits_{j=k_0(M)}^{k} (-1)^{k-j}\binom{d-j}{k-j} h_M(j), & k_0(M) \leq k \leq d \\ 0, & \text{otherwise} \end{cases}. \tag{1}$$

From (1) we deduce that

$$h_M(k) := \sum_{j=k_0(M)}^{k} \binom{d-j}{k-j} \beta_j^d(h_M) \text{ for all } k_0(M) \leq k \leq d. \tag{2}$$

With the above notation, we have:

Definition 1.1. *The* Hilbert depth *of h_M is*

$$\operatorname{hdepth}(h_M) := \max\{d \in \mathbb{Z} \ : \ \beta_h^d(h_M) \geq 0 \textit{ for all } k \leq d\}.$$

Note that (1), (2) and Definition 1.1 hold for any function $h : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ with $h(j) = 0$ for $j \ll 0$.

Remark 1.2. If $\beta_k^d(h_M) \geq 0$ for all $k_0(M) \leq k \leq d$, then, from [3, Corollary 1.4], it follows that $\beta_k^{d'}(h_M) \geq 0$ for all $d \leq d'$ and $k_0(M) \leq k \leq d'$. Also, it is clear that $k_0(M) = k_0(h_M)$.

Let $0 \subset I \subsetneq J \subset S = K[x_1, \ldots, x_n]$ be two squarefree monomial ideals. We recall the method of computing Hilbert depth of J/I given in [2]. For $0 \leq k \leq n$, we let

$$\alpha_k(J/I) = |\{u \in S \text{ is a squarefree monomial with } u \in J \setminus I\}|.$$

For all $0 \leq d \leq n$ and $0 \leq k \leq d$, we consider the integers:

$$\beta_k^d(J/I) := \sum_{j=0}^{k} (-1)^{k-j} \binom{d-j}{k-j} \alpha_j(J/I). \tag{3}$$

We recall the following result:

Theorem 1.3. *([2, Theorem 2.4]) With the above notations, the Hilbert depth of J/I is*

$$\operatorname{hdepth}(J/I) := \max\{d \ : \ \beta_k^d(J/I) \geq 0 \textit{ for all } 0 \leq k \leq d\}.$$

We consider the S-module

$$M(J/I) := (J + (x_1^2, \dots, x_n^2))/(I + (x_1^2, \dots, x_n^2)).$$

It is easy to see that

$$M(J/I) = \bigoplus_{\substack{u \in S \text{ squarefree monomial} \\ u \in J \setminus I}} Ku. \tag{4}$$

From (4) and the definition of $\alpha_k(J/I)$'s it follows that

$$\alpha_k(J/I) = h_{M(J/I)}(k) \text{ for all } 0 \leq k \leq n. \tag{5}$$

From Theorem 1.3, Definition 1.1 and (5) we get the following result:

Proposition 1.4. *With the above notations, we have*

$$\operatorname{hdepth}(J/I) = \operatorname{hdepth}(h_{M(J/I)}).$$

In the following, all modules are assumed finitely generated over a standard graded K-algebra A, unless it is stated otherwise:

Proposition 1.5. *Let M be a nonzero graded A-module, $k_0 = k_0(M)$, $h_0 = h_M(k_0)$, $h_1 := h_M(k_0 + 1)$. Then:*

$$k_0 \leq \operatorname{hdepth}(h_M) \leq k_0 + \frac{h_1}{h_0}.$$

Proof. It follows from [3, Proposition 1.5]. □

Let M be a nonzero graded A-module of finite length, i.e. $\dim_K(M) < \infty$. It follows that there exists $k_f(M) \geq k_0(M)$ such that

$$k_f(M) := \max\{k \ : \ M_k \neq 0\}.$$

Note that $k_f(M) \leq k_f(h_M)$. Hence, from [3, Proposition 1.5] we conclude that:

Proposition 1.6. *If M is a nonzero graded A-module of finite length, then*

$$\operatorname{hdepth}(h_M) \leq k_f(M).$$

Corollary 1.7. *Let $I \subset S = K[x_1, \dots, x_n]$ be an Artinian homogeneous ideal. Then*

$$\operatorname{hdepth}(h_{S/I}) \leq \operatorname{reg}(S/I).$$

Proof. According to [6, Theorem 18.4], we have that

$$\operatorname{reg}(S/I) = \max\{k \;:\; (S/I)_k \neq 0\}.$$

The conclusion follows from Proposition 1.6. □

Remark 1.8. Assume K is a field of characteristic zero and let $I \subset S = K[x_1, \ldots, x_n]$ be a homogeneous ideal. Let $J := \operatorname{Gin}(I)$ be the generic initial ideal of I with respect to the reverse lexicographic order. It is well known that S/I and S/J have the same Hilbert function. Moreover, according to Bayer and Stillman [1], we have

$$\operatorname{reg}(S/I) = \operatorname{reg}(S/J).$$

On the other hand, according to Galligo [5], $J := \operatorname{Gin}(I)$ is strongly stable, hence from the well known result of Eliahou and Kervaire [4], it follows that

$$\operatorname{reg}(S/I) = \operatorname{reg}(S/J) = \max\{\deg(u) \;:\; u \in G(J)\} - 1,$$

where $G(J)$ is the minimal set of monomial generators of J.

In conclusion, $\operatorname{hdepth}(h_{S/I}) \leq \operatorname{reg}(S/I)$ if and only if $\operatorname{hdepth}(h_{S/J}) \leq \operatorname{reg}(S/J)$.

The result from Corollary 1.7 cannot be extended in general. For instance, the ideal $J = (x_1^3) \subset S = K[x_1, x_2, x_3]$ is strongly stable and its regularity is $\operatorname{reg}(S/J) = 3 - 1 = 2$, while $\operatorname{hdepth}(h_{S/J}) = 3$.

Proposition 1.9. *Let $0 \to U \to M \to N \to 0$ be a short exact sequence of (nonzero) graded A-modules. Then:*

$$\operatorname{hdepth}(h_M) \geq \min\{\operatorname{hdepth}(h_U), \operatorname{hdepth}(h_N)\}.$$

Proof. It follows from the fact that $h_M(k) = h_U(k) + h_N(k)$ for all $k \in \mathbb{Z}$ and [3, Proposition 1.10]. □

Proposition 1.10. *Let M be a graded A-module and let $r > 0$ be an integer. Then*

$$\operatorname{hdepth}(h_{M^{\oplus r}}) = \operatorname{hdepth}(h_M).$$

Proof. Since $h_{M^{\oplus r}}(k) = r \cdot h_M(k)$ for all $k \in \mathbb{Z}$, the conclusion follows from [3, Proposition 1.11]. □

If M is a graded A-module and m is an integer, then

$$M(m) = \bigoplus_{k \in \mathbb{Z}} M(m)_k = \bigoplus_{k \in \mathbb{Z}} M_{m+k}$$

is the m-th shift module of M.

Proposition 1.11. *Let M be a nonzero graded A-module and $m \in \mathbb{Z}$. Then:*

(1) $k_0(M(m)) = k_0(M) - m$.

(2) If $\dim_K(M) < \infty$ *then* $k_f(M(m)) = k_f(M) - m$.

(3) $\operatorname{hdepth}(h_{M(m)}) = \operatorname{hdepth}(h_M) - m$.

Proof. (1) and (3) Since $h_{M(m)}(k) = h_M(k+m)$ for all $k \in \mathbb{Z}$, the conclusion follows from [3, Proposition 1.12].

(2) It is obvious. $\square$

Remark 1.12. Let $h : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ such that $h(j) = 0$ for $j \ll 0$. Let $k_0 = \min\{j : h(j) > 0\}$ and $c = \left\lfloor \frac{h(k_0+1)}{h(k_0)} \right\rfloor$. Let $S := K[x_1, \ldots, x_n]$ and $\mathbf{m} = (x_1, \ldots, x_n)$. We claim that there exists an Artinian S-module M such that

$$h_M(j) = \begin{cases} h(j), & k_0 \leq j \leq k_0 + c \\ 0, & \text{otherwise} \end{cases}. \tag{6}$$

Indeed, we can take

$$M := (S/\mathbf{m})(-k_0)^{h(k_0)} \oplus (S/\mathbf{m})(-k_0-1)^{h(k_0+1)} \oplus \cdots \oplus (S/\mathbf{m})(-k_0-c)^{h(k_0+c)}.$$

From (6) it is easy to deduce that

$$\operatorname{hdepth}(h) = \operatorname{hdepth}(h_M).$$

Note that M is in fact a graded K-vector space of finite dimension.

2 Hilbert depth of the Hilbert series of a free S-module

Let $a \in \mathbb{C}$ and j a nonnegative integer. We denote $(a)_j = a(a+1)\cdots(a+j-1)$, the *Pochhammer symbol*. The *hypergeometric function* is

$${}_2F_1(a, b, c; z) = \sum_{j \geq 0} \frac{(a)_j (b)_j}{(c)_j} \cdot \frac{z^j}{j!}.$$

First, we prove the following lemma:

Lemma 2.1. *Let $n \geq 1$ be an integer. Then:*

(1) ${}_2F_1(0, n, -n; -1) = 1$ *and* ${}_2F_1(-1, n, -n; -1) = 0$.

(2) $(-1)^k {}_2F_1(-k, n, -n; -1) > 0$ *for any* $2 \leq k \leq n$.

Proof. (1) It is obvious from the definition of the hypergeometric function.

(2) Since ${}_2F_1(-k,n,-n;-1)=\sum\limits_{j=0}^{k}(-1)^j\binom{k}{j}\frac{(n)_j}{(n-j+1)_j}$, in order to prove (2), it is enough to show that for any $n\geq k\geq 2$ we have that:

$$E(n,k):=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(n)_j(n-k+1)_{k-j}>0. \tag{7}$$

We consider the functions

$$f_{1,k},f_{2,k}:(0,2)\to\mathbb{R},\ f_{1,k}(x)=\frac{1}{(2-x)^n},\ f_{2,k}(x)=\frac{1}{x^{n-k+1}}.$$

By straightforwards computation, for all $0\leq j\leq k$ we have that

$$f_{1,k}^{(j)}(x)=\frac{(n)_j}{(2-x)^{n+j}}\text{ and }f_{2,k}^{(k-j)}(x)=\frac{(-1)^{k-j}(n-k+1)_{k-j}}{x^{n-j+1}}, \tag{8}$$

where $f^{(j)}$ denotes the j-th derivative of the function f. Let

$$f_k:(0,2)\to\mathbb{R},\ f_k(x):=f_{1,k}(x)f_{2,k}(x),\ x\in(0,2). \tag{9}$$

From (7), (8), (9) and the chain rule of derivatives, it follows that

$$E(n,k)=f_k^{(k)}(1). \tag{10}$$

We consider the function

$$g_k:(-1,1)\to\mathbb{R},\ g_k(x)=f_k(1-x)=\frac{1}{(1+x)^n(1-x)^{n-k+1}}=\frac{(1-x)^{k-1}}{(1-x^2)^n}.$$

Since $g_k^{(k)}(x)=(-1)^kf_k^{(k)}(1-x)$, from (7) and (10), in order to complete the proof, it is enough to prove that

$$(-1)^kg_k^{(k)}(0)>0\text{ for all }k\geq 2. \tag{11}$$

If $k\geq 2$ and $j\geq 1$, then, using the identity $g_k(x)=(1-x)g_{k-1}(x)$, we deduce that

$$g_k^{(j)}(x)=(1-x)g_{k-1}^{(j)}(x)-jg_{k-1}^{(j-1)}(x)\text{ for all }x\in(-1,1). \tag{12}$$

For $k\geq 1$ and $j\geq 0$ we denote $c_k^{(j)}:=g_k^{(j)}(0)$. Since

$$g_1(x)=\frac{1}{(1-x^2)^n}=\sum_{\ell=0}^{\infty}\binom{n+\ell-1}{\ell}x^{2\ell},$$

it follows that

$$c_1^{(j)} = \begin{cases} 0, & j = 2\ell+1 \\ \binom{n+\ell-1}{\ell}(2\ell)!, & j = 2\ell \end{cases} \tag{13}$$

Also, it is clear that

$$c_k^{(0)} = 1 \text{ for all } k \geq 1. \tag{14}$$

On the other hand, from (12) it follows that

$$c_k^{(j)} = c_{k-1}^{(j)} - jc_{k-1}^{(j-1)} \text{ for all } k \geq 2,\ j \geq 1. \tag{15}$$

From (13), (14) and (15), using induction on $k \geq 2$, we can easily deduce that

$$(-1)^j c_k^{(j)} > 0 \text{ for all } k \geq 2,\ j \geq 0.$$

In particular, it follows that

$$(-1)^k c_k^{(k)} = (-1)^k g_k^{(k)}(0) > 0 \text{ for all } k \geq 2,$$

hence the proof is complete. □

Theorem 2.2. *Let* $S := K[x_1, \dots, x_n]$*. Then* $\operatorname{hdepth}(h_S) = n$.

Proof. The Hilbert function of S is $h_S(k) = \binom{n-1+k}{k}$ for all $k \geq k_0(S) = 0$. Therefore, from (1), we have

$$\beta_k^d(h_S) = \sum_{j=0}^{k} (-1)^{k-j} \binom{d-j}{k-j} \binom{n-1+j}{j} \text{ for all } 0 \leq k \leq d. \tag{16}$$

Since $k_0(S) = 0$, $h_S(0) = 1$ and $h_S(1) = n$, from Proposition 1.5 we get that $\operatorname{hdepth}(h_S) \leq n$. From (16) we have that

$$\beta_k^n(h_S) = \sum_{j=0}^{k} (-1)^{k-j} \binom{n-j}{k-j} \binom{n-1+j}{j} \text{ for all } 0 \leq k \leq n. \tag{17}$$

From (17) if follows that

$$\beta_k^n(h_S) = (-1)^k \binom{n}{k} {}_2F_1(-k, n, -n; -1) \text{ for all } 0 \leq k \leq n, \tag{18}$$

From (18) and Lemma 2.1 it follows that $\operatorname{hdepth}(h_S) \geq n$, as required. □

Corollary 2.3. *Let* $F = S(a)^{n_1} \oplus S(a-1)^{n_2} \oplus S(a_1) \oplus \cdots \oplus S(a_r)$ *where* n_1, n_2, a, a_j *are some integers such that* $n_1 > n_2 \geq 0$ *and* $a \geq a_j + 2$ *for all* $1 \leq j \leq r$*. Then* $\operatorname{hdepth}(h_F) = n - a$.

Proof. From Theorem 2.2, Proposition 1.10 and Proposition 1.11 it follows that

$$\begin{aligned}&\operatorname{hdepth}(h_{S(a)^{n_1}}) = n - a,\\ &\operatorname{hdepth}(h_{S(a-1)^{n_2}}) = n - a + 1, \text{ if } n_2 > 0 \text{ and}\\ &\operatorname{hdepth}(h_{S(a_j)}) = n - a_j \text{ for all } 1 \leq j \leq r.\end{aligned}$$

Using Proposition 1.9, we deduce that

$$\operatorname{hdepth}(h_F) \geq \min\{n-a,\ n-a+1,\ n-a_j,\ 1 \leq j \leq r\} = n - a. \tag{19}$$

On the other hand, from hypothesis, we have

$$\begin{aligned}&h_F(-a) = \dim_K S(a)^{n_1}_{-a} = n_1 \dim_K S_0 = n_1 \text{ and}\\ &h_F(-a+1) = \dim_K S(a)^{n_1}_{-a+1} + \dim_K S(a-1)^{n_2}_{-a+1} = n \cdot n_1 + n_2.\end{aligned}$$

Since $n_1 > n_2$, from Proposition 1.5 it follows that $\operatorname{hdepth}(h_F) \leq n-a$. Hence, the conclusion follows from (19). □

3 Hilbert depth of the Hilbert series of a complete intersection

Let $S = K[x_1, \ldots, x_n]$ and $J = (f_1, \ldots, f_n) \subset S$ be a graded complete intersection ideal with $d_i = \deg(f_i) \geq 2$ for all $1 \leq i \leq n$. The Hilbert series of S/J is

$$H_{S/J}(t) = \sum_{k\geq 0} h_{S/J}(k)t^k = (1+t+\cdots+t^{d_1-1})(1+t+\cdots+t^{d_2-1})\cdots(1+t+\cdots+t^{d_n-1}).$$

Theorem 3.1. *With the above notations, we have that*

$$\operatorname{hdepth}(h_{S/J}) = n.$$

Proof. First, note that $\operatorname{hdepth}(h_{S/J}) \leq n$ by Proposition 1.5, since $h_{S/J}(0) = 1$ and $h_{S/J}(1) = n$.

We use induction on $n \geq 1$ and $d := d_1 + \cdots + d_n \geq 2n$. If $n = 1$ then there is nothing to prove. If $d = 2n$, that is $d_i = 2$ for all $1 \leq i \leq n$, then:

$$\beta^n_k(h_{S/J}) = \sum_{j=0}^{k}(-1)^{k-j}\binom{n-j}{k-j}\binom{n}{j} = \sum_{j=0}^{k}(-1)^{k-j}(-1)^{k-j}\binom{n}{k}\binom{k}{j}.$$

Therefore, $\beta^n_0(h_{S/J}) = 1$ and $\beta^n_k(h_{S/J}) = 0$ for $2 \leq k \leq n$. From Remark 1.2, it follows that $\operatorname{hdepth}(h_{S/J}) \geq n$ and thus $\operatorname{hdepth}(h_{S/J}) = n$.

Assume $d > 2n$. Without any loss of generality, we may assume that $d_n \geq 3$. Let $I = (g_1, \ldots, g_n)$ be a graded complete intersection ideal with $\deg(g_i) = \deg(f_i) = d_i$ for $1 \leq i \leq n-1$ and $\deg(g_n) = d_n - 1$. Let $J' = (f'_1, \ldots, f'_{n-1}) \subset S' = K[x_1, \ldots, x_{n-1}]$ be a graded complete intersection ideal with $\deg(f'_i) = \deg(f_i) = d_i$ for $1 \leq i \leq n-1$. We have that that

$$H_{S/J}(t) = (1+t+\cdots+t^{d_1-1})\cdots(1+t+\cdots+t^{d_{n-1}-1})(1+t+\cdots+t^{d_n-2}+t^{d_n-1}) =$$

$$= H_{S/I}(t) + t^{d_n-1}H_{S'/J'}(t). \tag{20}$$

From (20), it follows that for $0 \leq k \leq n$ we have that

$$\beta_k^n(h_{S/J}) = \sum_{j=0}^{k}(-1)^{k-j}\binom{n-j}{k-j}h_{S/J}(j) =$$
$$= \sum_{j=0}^{k}(-1)^{k-j}\binom{n-j}{k-j}(h_{S/I}(j) + h_{S'/J'}(j-d_n+1)) =$$
$$= \sum_{j=0}^{k}(-1)^{k-j}\binom{n-j}{k-j}h_{S/I}(j) + \sum_{j=0}^{k}(-1)^{k-j}\binom{n-j}{k-j}h_{S'/J'}(j-d_n+1) =$$

$$= \beta_k^n(h_{S/I}) + \sum_{j=0}^{k}(-1)^{k-j}\binom{n-j}{k-j}h_{S'/J'}(j-d_n+1). \tag{21}$$

From induction hypothesis, it follows that $\beta_k^n(h_{S/I}) \geq 0$ for all $0 \leq k \leq n$. If $k < d_n - 1$ then from (21) it follows that

$$\beta_k^n(h_{S/J}) = \beta_k^n(h_{S/I}) \geq 0. \tag{22}$$

If $k \geq d_n - 1$ then

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{n-j}{k-j}h_{S'/J'}(j-d_n+1) = \sum_{j=d_n-1}^{k}(-1)^{k-j}\binom{n-j}{k-j}h_{S'/J'}(j-d_n+1)$$
$$= \sum_{j'=0}^{k-d_n+1}(-1)^{(k-d_n+1)-j'}\binom{(n-d_n+1)-j'}{(k-d_n+1)-j'}h_{S'/J'}(j') = \beta_{k-d_n+1}^{n-d_n+1}(h_{S'/J'}).$$

Therefore, from (21) and the induction hypothesis it follows that

$$\beta_k^n(h_{S/J}) = \beta_k^n(h_{S/I}) + \beta_{k-d_n+1}^{n-d_n+1}(h_{S'/J'}) \geq 0. \tag{23}$$

The conclusion follows from (22) and (23). $\square$

Corollary 3.2. *If $J = (f_1, \dots, f_r) \subset S$ is a graded complete intersection with $\deg(f_i) \geq 2$ for all $1 \leq i \leq r$, where $0 \leq r \leq n$, then*

$$\operatorname{hdepth}(h_{S/J}) = n.$$

In particular, we reobtain the result $\operatorname{hdepth}(h_S) = n$.

Proof. Let $\overline{J} = (g_1, \dots, g_n)$ be a graded complete intersection with $\deg(g_i) = \deg(f_i)$ for all $1 \leq i \leq r$ and $\deg(g_i) = n+1$ for $r+1 \leq i \leq n$. Since

$$H_{S/J}(t) = (1+t+\cdots+t^{d_1-1})\cdots(1+t+\cdots+t^{d_r-1})\cdot(1+t+t^2+\cdots)^{n-r} \text{ and}$$
$$H_{S/\overline{J}}(t) = (1+t+\cdots+t^{d_1-1})\cdots(1+t+\cdots+t^{d_r-1})\cdot(1+t+t^2+\cdots+t^n)^{n-r},$$

it follows that $h_{S/J}(j) = h_{S/\overline{J}}(j)$ for all $0 \leq j \leq n$. Therefore

$$\beta_k^n(h_{S/J}) = \beta_k^n(h_{S/\overline{J}}) \text{ for all } 0 \leq k \leq n,$$

hence the result follows from Theorem 3.1. □

4 Hilbert depth of the Hilbert series of a tensor product of modules

As in the beginning of the section, K is a field, A is a standard graded K-algebra and the modules over A are considered finitely generated and graded unless is stated otherwise.

We recall the following well known lemma, regarding the Hilbert series of a tensor product of modules, for which we sketch a proof in order of completion.

Lemma 4.1. *Let M, N be two A-modules such that N is flat. Then:*

$$\mathrm{H}_{M \otimes_A N}(t) = \frac{\mathrm{H}_M(t)\,\mathrm{H}_N(t)}{\mathrm{H}_A(t)}.$$

Proof. Take a free resolution of M,

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0 \tag{24}$$

where each F_n is concentrated in degrees $\geq n$. It follows that

$$\mathrm{H}_M(t) = \sum_{i \geq 0} (-1)^i\,\mathrm{H}_{F_i}(t). \tag{25}$$

Taking $\otimes_A N$ in (24) we get an exact sequence

$$\cdots \to F_2 \otimes_A N \to F_1 \otimes_A N \to F_0 \otimes_A N \to M \otimes_A N \to 0, \tag{26}$$

Since F_i is free, it follows that

$$\mathrm{H}_{F_i \otimes_A N}(t) = \frac{\mathrm{H}_{F_i}(t)\,\mathrm{H}_N(t)}{\mathrm{H}_A(t)} \text{ for all } i,$$

and thus from (25) we get

$$\mathrm{H}_{M \otimes_A N}(t) = \sum_{i \geq 0} (-1)^i \frac{\mathrm{H}_{F_i}(t)\,\mathrm{H}_N(t)}{\mathrm{H}_A(t)} = \frac{\mathrm{H}_N(t)}{\mathrm{H}_A(t)} \sum_{i \geq 0} (-1)^i\, \mathrm{H}_{F_i}(t) = \frac{\mathrm{H}_M(t)\,\mathrm{H}_N(t)}{\mathrm{H}_A(t)},$$

as required. □

Lemma 4.2. *Let $S = K[x_1, \ldots, x_n]$, $\overline{S} = S[x_{n+1}]$ and M be a S-module. If $\overline{M} = M[x_{n+1}] := M \otimes_S \overline{S}$, then*

$$\mathrm{H}_{\overline{M}}(t) = \frac{\mathrm{H}_M(t)}{(1-t)}.$$

In particular, $h_{\overline{M}}(j) = \sum_{\ell \leq j} h_M(\ell)$.

Proof. Since $H_S(t) = \frac{1}{(1-t)^n}$, $H_{\overline{S}}(t) = \frac{1}{(1-t)^{n+1}}$ and $\overline{S}$ is flat over S, the conclusion follows from Lemma 4.1. □

Theorem 4.3. *Let $S = K[x_1, \ldots, x_n]$, $\overline{S} = S[x_{n+1}]$, M be a S-module and $\overline{M} = M[x_{n+1}]$. Then*

$$\operatorname{hdepth}(h_{\overline{M}}) \geq \operatorname{hdepth}(h_M).$$

Proof. Let $d = \operatorname{hdepth}(h_M)$, $k_0 = k_0(M)$ and $k_0 \leq k \leq d$. By (1) we have that

$$\beta_k^d(h_M) = \sum_{j=k_0}^{k} (-1)^{k-j} \binom{d-j}{k-j} h_M(j) \geq 0. \tag{27}$$

By (1), Remark 1.2 and Lemma 4.2 it follows that

$$\beta_k^d(h_{\overline{M}}) = \sum_{j=k_0}^{k} (-1)^{k-j} \binom{d-j}{k-j} h_{\overline{M}}(j) = \sum_{j=k_0}^{k} (-1)^{k-j} \binom{d-j}{k-j} \sum_{\ell=k_0}^{j} h_M(\ell) =$$

$$= \sum_{t=0}^{k-k_0} \sum_{j=k_0}^{k} (-1)^{k-j} \binom{d-j}{k-j} h_M(\ell - t) = (j' = j - t) =$$

$$= \sum_{t=0}^{k-k_0} \sum_{j'=k_0}^{k-t} (-1)^{(k-t)-j'} \binom{(d-t)-j'}{(k-t)-j'} h_M(j') = \sum_{t=0}^{k-k_0} \beta_{k-t}^{d-t}(M) \geq 0,$$

as required. □

Remark 4.4. Let $k_0 := k_0(M)$ and let $k_0 \leq k \leq d$ be some integers. We have that

$$\beta_k^d(h_{\overline{M}}) = \sum_{j=k_0}^{k} (-1)^{k-j}\binom{d-j}{k-j}\sum_{\ell=k_0}^{j} h_M(\ell) =$$
$$= \sum_{\ell=k_0}^{k} (-1)^{k-\ell}\left(\sum_{j=\ell}^{k}(-1)^{j-\ell}\binom{d-j}{k-j}\right) h_M(\ell) =$$
$$= \sum_{\ell=k_0}^{k} (-1)^{k-\ell}\binom{d-\ell}{k-\ell} {}_2F_1(1,-k+\ell,-d+\ell;-1)h_M(\ell).$$

On the other hand, for any nonnegative integer s, it holds that

$${}_2F_1(1,-s,-s;-1) = \sum_{\ell=0}^{s}(-1)^{\ell} = \begin{cases} 1, & s \text{ is even} \\ 0, & s \text{ is odd} \end{cases}.$$

Therefore, we get

$$\beta_d^d(h_{\overline{M}}) = \sum_{\ell=k_0}^{k} h_M(\ell|d) \geq 0, \text{ where } h_M(\ell|d) = \begin{cases} h_M(\ell), & d \equiv \ell (\text{mod } 2) \\ 0, & \text{otherwise} \end{cases}.$$

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