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Quasi-Statistical Manifolds with Almost Hermitian and Almost Anti-Hermitian Structures

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Abstract

Let (M, g, ∇) be a $2n$-dimensional quasi-statistical manifold that admits a pseudo-Riemannian metric g (or h) and a linear connection ∇ with torsion. This paper aims to study an almost Hermitian structure (g, J) and an almost anti-Hermitian structure (h, J) on a quasi-statistical manifold that admit an almost complex structure J. Firstly, under certain conditions, we present the integrability of the almost complex structure J. We show that when $d^{\nabla} J = 0$ and the condition of torsion-compatibility are satisfied, (M, g, ∇, J) turns into a Kähler manifold. Secondly, we give necessary and sufficient conditions under which (M, h, ∇, J) is an anti-Kähler manifold, where h is an anti-Hermitian metric. Moreover, we search the necessary conditions for (M, h, ∇, J) to be a quasi-Kähler-Norden manifold.

1 Introduction

In recent times, there has been a growing interest in the study of spaces composed of probability measures. One powerful tool employed in exploring such spaces is information geometry, a well-established theory within the field of geometry. Information geometry is a theory that seamlessly combines elements

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of both differential geometry and statistics, and it holds significant importance across various scientific disciplines. Its applications extend to fields such as image processing, physics, computer science, and machine learning, making it a versatile and influential area of study ([1, 4, 6, 7]). Within the framework of information geometry, two geometric quantities known as dual connections take center stage, as they provide crucial insights into the behavior of statistical manifolds. These dual connections are fundamental in characterizing how statistical properties change concerning vector fields. Investigating these dual elements and unraveling the intricate relationships between them constitutes a core focus in the ongoing research on statistical manifolds [5]. Notably, the concept of statistical manifolds has recently garnered substantial attention from mathematicians and researchers alike. This surge of interest has led to a burgeoning body of work exploring various aspects of statistical manifolds, with contributions from scholars in diverse fields ([3, 9, 10, 19, 23]).

The concept of a Statistical Manifold Admitting Torsion (SMAT), also referred to as quasi-statistical manifolds, was initially introduced by Kurose [16]. This type of manifold naturally emerges in the context of quantum statistical models and can be thought of as the quantum analog of statistical manifolds. A Statistical Manifold Admitting Torsion (SMAT) is essentially a pseudo-Riemannian manifold equipped with a pair of dual connections, where only one of these connections needs to be torsion-free, while the other is not necessarily so. The introduction of SMAT was originally motivated by the desire to examine and understand this geometric structure from a mathematical perspective ([2, 16]). It provides a mathematical framework for studying and analyzing quantum state spaces, particularly in the context of statistical models. In this framework, the interplay between the different connections and their properties yields valuable insights into the geometry of quantum statistical systems and their underlying structures.

Let M be a $2n$-dimensional differentiable manifold and g be a pseudo Riemannian metric on M. An almost complex structure on M is a tensor field J of type $(1,1)$ such that $J^2 = -id$. An almost complex manifold is such a manifold with a fixed almost complex structure. Note that almost complex structures exist only when M is of even dimension. Ensuring compatibility of J with g, $g(JX,Y) + g(X,JY) = 0$, for any vector fields X and Y on M, leads to an almost Hermitian manifold (M,g,J). The compatible metric is also called a Hermitian metric. If J is integrable, the manifold (M,g,J) becomes a Hermitian manifold. Moreover, the fundamental 2-form ω can be described $\omega(X,Y) = g(JX,Y)$ and performs to satisfy $\omega(JX,Y) + \omega(X,JY) = 0$. An almost Kähler manifold is an almost Hermitian manifold whose fundamental 2-form ω is closed. In other words, an almost Kähler manifold is a symplectic manifold equipped with a compatible metric. With integrability of J, the almost Kähler manifold (M, g, J) rises to a Kähler manifold. Also, it is well known that the almost Hermitian manifold (M, g, J) is Kähler manifold if and only if the almost complex structure J is covariantly constant with respect to the Levi-Civita connection ∇^g, that is, $\nabla^g J = 0$. Fei and Zhang presented an alternative characterization for Kähler manifolds by taking any torsion-free linear connection ∇ instead of the Levi-Civita connection ∇^g [9]. Firstly, they showed that Codazzi coupling of a torsion-free linear connection ∇ with J implies the integrability of J. Furthermore, they proved that a torsion-free linear connection ∇ is Codazzi-coupled with both g and J, then the triple (M, g, J) turns into a Kähler manifold. Such a Kähler manifold is called Codazzi Kähler manifold [9].

An anti-Kähler (Norden-Kähler) manifold means a manifold (M, h, J) which consists of a differentiable manifold of dimension $2n$, an almost complex structure J and an anti-Hermitian metric h such that $\nabla^h J = 0$, where ∇^h is the Levi-Civita connection of h [11, 18]. The metric h is called an anti-Hermitian (Norden) metric if it satisfies $h(JX, Y) - h(X, JY) = 0$ for all vector fields X and Y on M. Then the metric h has necessarily a neutral signature (n, n). By $\hbar(X, Y) = h(JX, Y)$, the twin metric $\hbar$ can be defined and it is symmetric and satisfies $\hbar(JX, Y) - \hbar(X, JY) = 0$ for any vector fields X, Y on M. Consequently, this twin metric is another anti-Hermitian (Norden) metric. Since a pair of anti-Hermitian (or Norden) metrics exists on anti-Hermitian (or Norden) manifolds, it becomes crucial to consider dual (conjugate) connections corresponding to each of these metric tensors and their relations with dual connections associated with the almost complex structure. Therefore, exploring statistical structures on these manifolds holds significant importance.

Hermitian manifolds as well as Norden manifolds have been subject to analysis from various perspectives, as referenced in ([9, 12, 14]). In [15], the authors presented a novel approach to extend almost anti-Hermitian manifolds to anti-Kähler manifolds. They demonstrated that the anti-Kähler condition is equivalent to the $\mathbb{C}$-analyticity of the anti-Hermitian metric h, denoted as $\Phi_J h = 0$, where Φ_J represents the Tachibana operator. Furthermore, by considering the Codazzi coupling of (∇, J), Gezer and Cakicioglu provided an alternative characterization for anti-Kähler manifolds with respect to a torsion-free linear connection ∇ [12]. Following this, taking into account the presence of the Tachibana operator and the Codazzi coupling of (∇, g) with a torsion-free linear connection ∇, Durmaz and Gezer demonstrated the possibility of classifying locally metallic pseudo-Riemannian manifolds [8].

Now, it is natural to pose the following question: "Can we classify Kähler and anti-Kähler manifolds by considering any linear connection ∇ with a torsion tensor T^∇ instead of the Levi-Civita connection ∇^g (or ∇^h) associated with the metrics g (or h)? Alternatively, does the torsion tensor of the linear connection ∇ have to vanish to classify these manifolds?" This paper seeks to provide answers to these questions.

The organization of this paper is as follows: In section 2, we explore the integrability of the almost complex structure J for any linear connection ∇ with a torsion tensor T^{∇}. This investigation is presented in Lemma 1 and Proposition 2. By considering the definitions of $d^{\nabla}J$ and the Vishnevskii operator Ψ_J, we derive distinct results concerning the integrability of J and $d^{\nabla}J$. In Section 3, we examine the concept of quasi-statistical structure (∇, g) defined with a pseudo-Riemannian metric g. We establish intriguing relationships between the quasi-structures of conjugate connections $\nabla, \nabla^{\dagger}$, and ∇^{J}, as well as $d^{\nabla(\nabla,\nabla^{\dagger},\nabla^{J})}$-closedness of J (refer to Proposition 5). Furthermore, we demonstrate that the triple (M, g, J) constitutes a Kähler manifold when $d^{\nabla}J = 0$ and $T^{\nabla}(JX, Y) = -T^{\nabla}(X, JY)$ on a quasi-statistical manifold (M, g, ∇), where ∇ is any linear connection with a torsion tensor T^{∇} (see Theorem 7). Finally, in the last section, by considering an anti-Hermitian metric h in place of a Hermitian metric g, we revisit the properties of quasi-statistical structures. By utilizing any linear connection ∇ with a torsion tensor T^{∇} instead of the Levi-Civita connection ∇^{h} associated with h, we establish new classifications for anti-Kähler and quasi-Kähler-Norden manifolds (see Theorem 17 and Theorem 18).

2 J-Conjugate of ∇ and d^{∇}-closed endomorphisms

In this section, we delve into the study of a linear connection defined on a differentiable manifold M in conjunction with a $(1,1)$-tensor field J. This tensor field is subject to the condition $d^{\nabla}J = 0$. We refer to this structure as an almost complex structure if $J^2 = -\mathrm{id}$, or alternatively, an almost paracomplex structure if $J^2 = \mathrm{id}$. In the context of this paper, we will focus on the almost complex structure J. It is important to note that similar results can be derived if one chooses to work with the almost paracomplex structure instead of the almost complex structure.

Starting from a linear connection ∇ on M, we can apply an J-conjugate transformation to achieve a new connection
$\nabla^{J} := J^{-1}\nabla J$ or $\nabla^{J}_{X}Y = J^{-1}(\nabla_X JY)$ for any vector fields X and Y, where J^{-1} identifies the inverse isomorphism of J. It can be confirmed that indeed ∇^{J} is a linear connection.

Definition 1. *A linear connection ∇ and a $(1,1)$-tensor field J are called Codazzi-coupled if the following equality exists*

$$(\nabla_X J)Y = (\nabla_Y J)X,$$

where $(\nabla_X J)Y = \nabla_X JY - J\nabla_X Y$.

As a linear connection, ∇ yields a map $d^{\nabla} : \Omega^{i}(TM) \to \Omega^{i+1}(TM)$, where $\Omega^{i}(TM)$ is the space of smooth i-forms with value in TM. Regarding J as an element of $\Omega^{1}(TM)$, it is easy to see

$$\left(d^{\nabla} J\right)(X,Y) = (\nabla_X J) Y - (\nabla_Y J) X + JT^{\nabla}(X,Y),$$

where the torsion tensor is given by $T^{\nabla}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$. Hence, J is called d^{∇}-closed if $d^{\nabla} J = 0$.

Any $(1,1)$-tensor field J is called a quadratic operator if there exists $\alpha \neq \beta \in \mathbb{C}$ such that $\alpha+\beta$ and $\alpha\beta$ are real numbers and $J^2 - (\alpha+\beta) J + \alpha\beta.id = 0$. Note that J is an isomorphism, so $\alpha\beta \neq 0$.

The Nijenhuis tensor N_J associated with J is described as

$$N_J(X,Y) = -J^2[X,Y] + J[X,JY] + J[JX,Y] - [JX,JY].$$

When $N_J = 0$, J is said to be integrable.

Proposition 1. *Given a linear connection ∇ and a $(1,1)$-tensor field J on a manifold M, we can express the following expressions:*
(i) $d^{\nabla} J = 0 \Leftrightarrow T^{\nabla^J} = 0$;
(ii) $d^{\nabla^J} J = 0 \Leftrightarrow T^{\nabla} = 0$;
(iii) $d^{\nabla} J = d^{\nabla^J} J \Leftrightarrow (\nabla, J)$ is Codazzi-coupled.

Proof. (i) For any vector fields X and Y, we have

$$\begin{aligned}
\left(d^{\nabla} J\right)(X,Y) &= (\nabla_X J) Y - (\nabla_Y J) X + JT^{\nabla}(X,Y) \\
&= \nabla_X JY - \nabla_Y JX - J[X,Y] \\
&= J\left(J^{-1}\nabla_X JY - J^{-1}\nabla_Y JX - [X,Y]\right) \\
&= J\left(\nabla^J_X Y - \nabla^J_Y X - [X,Y]\right) \\
&= JT^{\nabla^J}(X,Y).
\end{aligned}$$

(ii) The result can be proved the same as in (i).
(iii) Due to (ii), it can be easily checked that

$$\begin{aligned}
\left(d^{\nabla} J\right)(X,Y) &= (\nabla_X J) Y - (\nabla_Y J) X + JT^{\nabla}(X,Y) \\
&= (\nabla_X J) Y - (\nabla_Y J) X + \left(d^{\nabla^J} J\right)(X,Y).
\end{aligned}$$

It is straightforward to obtain

$$\left(d^{\nabla} J\right)(X,Y) - \left(d^{\nabla^J} J\right)(X,Y) = (\nabla_X J) Y - (\nabla_Y J) X$$

for any vector fields X, Y, that is, $d^{\nabla} J = d^{\nabla^J} J \Leftrightarrow (\nabla, J)$ is Codazzi coupled. □

Fei and Zhang [9] demonstrated that when a quadric operator J and a linear connection ∇, which has no torsion, are Codazzi-coupled by the Codazzi condition, then the quadric operator J becomes integrable. For a linear connection ∇ with torsion tensor T^{∇}, it is possible to give the following lemma.

Lemma 1. *If ∇ is a linear connection with torsion tensor T^{∇}, and J is a $(1,1)$-tensor field on M such that $d^{\nabla}J = 0$, then the Nijenhuis tensor N_J associated with J becomes*

$$N_J(X,Y) = -J\left(T^{\nabla}(X,JY) + T^{\nabla}(JX,Y)\right).$$

Proof. Since $d^{\nabla}J = 0$, there exists the equality

$$[X,Y] = J^{-1}(\nabla_X JY - \nabla_Y JX). \tag{1}$$

Using the definition of the Nijenhuis tensor N_J associated with J and the given equality (1), we can compute it as follows:

$$\begin{aligned} N_J(X,Y) &= -J^2[X,Y] + J[X,JY] + J[JX,Y] - [JX,JY] \\ &= -J^2\left(J^{-1}(\nabla_X JY - \nabla_Y JX)\right) + \left(\nabla_X J^2Y - \nabla_{JY} JX\right) \\ &\quad + \left(\nabla_{JX} JY - \nabla_Y J^2X\right) - J^{-1}\left(\nabla_{JX} J^2Y - \nabla_{JY} J^2X\right) \\ &= -J(\nabla_X J)Y + J(\nabla_Y J)X - (\nabla_{JY} J)X + (\nabla_{JX} J)Y \\ &= J^2T^{\nabla}(X,Y) - J\left(T^{\nabla}(X,JY) + T^{\nabla}(JX,Y)\right) \\ &\quad + J((\nabla_X J)Y - (\nabla_Y J)X) \\ &= -J\left(T^{\nabla}(X,JY) + T^{\nabla}(JX,Y)\right). \end{aligned}$$

□

Consider the condition $T^{\nabla}(X,JY) = -T^{\nabla}(JX,Y)$ which may be called torsion-compatibility. Then we have the following result.

Proposition 2. *An almost complex structure J is integrable if $d^{\nabla}J = 0$ and $T^{\nabla}(JX,Y) = -T^{\nabla}(X,JY)$ (torsion-compatibility condition).*

There is another way to understand the relationship between $d^{\nabla}J$ and integrability of the structure J. Using the definition of $d^{\nabla}J$, it is possible to write the following equality

$$\left(d^{\nabla}J\right)(JX,Y) + \left(d^{\nabla}J\right)(X,JY) = T^{\nabla}(JX,JY) - T^{\nabla}(X,Y) - N_J(X,Y).$$

If the almost complex structure J is integrable, then there exists the equality $\left(d^{\nabla}J\right)(JX,Y) = -\left(d^{\nabla}J\right)(X,JY)$ provided that $T^{\nabla}(JX,JY) = T^{\nabla}(X,Y)$. Also, via Proposition 1, we can give the following results.

Corollary 1. *Assume that the torsion tensor T^{∇} of a linear connection ∇ satisfies the torsion-compatibility condition. If $T^{\nabla^J} = 0$, then the almost complex structure J is integrable.*

It is possible to give an alternative conclusion related to $d^{\nabla}J$ and T^{∇}. From the definition of $d^{\nabla}J$, we get

$$\left(d^{\nabla}J\right)(JX,Y) = (\nabla_{JX}J)\,Y - (\nabla_{Y}J)\,JX + JT^{\nabla}(JX,Y)$$

and

$$\left(d^{\nabla}J\right)(X,JY) = (\nabla_{X}J)\,JY - (\nabla_{JY}J)\,X + JT^{\nabla}(X,JY).$$

If $\Psi_{JX}Y = \nabla_{JX}Y - J(\nabla_X Y) = 0$ for any vector fields X and Y, where Ψ is the Vishnevskii operator [20], we have

$$\left(d^{\nabla}J\right)(JX,Y) + \left(d^{\nabla}J\right)(X,JY) = J\left(T^{\nabla}(JX,Y) + T^{\nabla}(X,JY)\right),$$

from which we immediately obtain

$$\left(d^{\nabla}J\right)(JX,Y) = -\left(d^{\nabla}J\right)(X,JY)$$

if and only if the torsion tensor T^{∇} satisfies the torsion-compatibility condition.

3 Quasi-statistical structures with a Hermitian metric g

In this section, our primary focus will be on the examination of quasi-statistical structures that admit a linear connection ∇ with torsion tensor T^{∇}, along with the inclusion of a pseudo-Riemannian metric g and an almost complex structure J. We aim to derive noteworthy findings and results related to these structures. Additionally, for Kähler manifolds, we will present a fresh alternative classification.

Definition 2. *Let M be a differentiable manifold with an almost complex structure J. A Hermitian metric on M is a pseudo-Riemannian metric g such that*

$$g(JX,JY) = g(X,Y)$$

or equivalently

$$g(JX,Y) = -g(X,JY) \tag{2}$$

for any vector fields X and Y on M. Then the triple (M,g,J) is an almost Hermitian manifold. The fundamental 2-form ω is given by $\omega(X,Y) = g(JX,Y)$ for any vector fields X, Y on M. (g,J,ω) is known as the "compatible triple." If the almost complex structure J is integrable, the triple (M,g,J)

is a Hermitian manifold. The triple (M, ω, J) *is a Kähler manifold if the structure* J *is integrable and* ω *is closed, that is,* $d\omega = 0$ *or equivalent to these two conditions is that the structure* J *is covariantly constant with respect to the Levi-Civita connection* ∇^g *of* g *[13].*

The well-known formula of the covariant derivative of g with respect to ∇ is as follow

$$(\nabla_Z g)(X, Y) = Zg(X, Y) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y).$$

Clearly $(\nabla_Z g)(X, Y) = (\nabla_Z g)(Y, X)$, due to symmetry of g. It is clear that g is parallel under ∇ if $\nabla g = 0$.

Given a pair (∇, g), we can also construct ∇^*, called a g-conjugate connection by

$$Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla^*_Z Y).$$

It is easy to see that ∇^* is a linear connection and a g-conjugate of a connection ∇ is involutive, that is, $(\nabla^*)^* = \nabla$. These two constructions from an arbitrary pair (∇, g) are related via $(\nabla_Z g)(X, Y) = g((\nabla^* - \nabla)_Z X, Y)$, which satisfy

$$(\nabla^*_Z g)(X, Y) = -(\nabla_Z g)(X, Y).$$

Therefore, we say that $(\nabla^*_Z g)(X, Y) = (\nabla_Z g)(X, Y) = 0$ if and only if $\nabla^* = \nabla$, that is, ∇ is g-self conjugate. A linear connection that is both g-self conjugate and torsion free is the Levi Civita connection ∇^g of g.

Definition 3. *Let* ∇ *be a torsion free linear connection on the pseudo-Riemannian manifold* (M, g) *with a pseudo-Riemannian metric* g*. We can say that* (M, g, ∇) *is a statistical manifold if the following equation is satisfied [17]*

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z).$$

We will consider an extension of the notion of a statistical structure. We can say that (M, g, ∇) *is a statistical manifold admitting torsion (SMAT) if* $d^\nabla g = 0$*, where*

$$\left(d^\nabla g\right)(X, Y, Z) = (\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) + g\left(T^\nabla(X, Y), Z\right)$$

for any vector fields X, Y *and* Z*. Also, it is called a statistical manifold admitting torsion (SMAT) as a quasi-statistical manifold [10].*

The fundamental 2-form ω on M is also an almost symplectic structure. Let us introduce the ω-conjugate transformation $\nabla^\dagger$ of ∇ by

$$Z\omega(X, Y) = \omega(\nabla_Z X, Y) + \omega\left(X, \nabla^\dagger_Z Y\right),$$

where conjugation is invariantly defined with respect to either the first or the second entry of ω despite of the skew-symmetric nature of ω [9] . The covariant derivative of ω with respect to ∇ is the following $(0,3)$-tensor field

$$(\nabla_Z\omega)(X,Y) = Z\omega(X,Y) - \omega(\nabla_Z X, Y) - \omega(X, \nabla_Z Y),$$

which is skew-symmetric in X,Y : $(\nabla_Z\omega)(X,Y) = -(\nabla_Z\omega)(Y,X)$. Imposing the Codazzi coupling condition of (∇,ω), that is, $(\nabla_Z\omega)(X,Y) = (\nabla_X\omega)(Z,Y)$ leads to $(\nabla_Z\omega)(X,Y) = 0$.

Lemma 2. *[14] Let (M,ω) be an almost symplectic manifold with the fundamental 2-form ω. Then, for any vector fields X, Y and Z,*

$$\begin{aligned} d\omega(X,Y,Z) &= (\nabla_Z\omega)(X,Y) + (\nabla_X\omega)(Y,Z) + (\nabla_Y\omega)(Z,X) \\ &\quad + \omega\left(T^\nabla(X,Y), Z\right) + \omega\left(T^\nabla(Y,Z), X\right) + \omega\left(T^\nabla(Z,X), Y\right), \end{aligned}$$

where ω is the fundamental 2-form of the almost Hermitian manifold (M,g,J).

Proposition 3. *Let (M,g,J) be an almost Hermitian manifold and let ∇ be a linear connection with torsion tensor T^∇. Let ω be the fundamental 2-form on (M,g,J). Then, there exist the following expressions*

(i) Assume that (∇,J) is Codazzi-coupled. $d^{\nabla^*}\omega = 0 \Leftrightarrow (\nabla^*,g)$ is a quasi-statistical structure.

(ii) Assume that (∇,J) is Codazzi-coupled. $d^{\nabla^\dagger}\omega = 0 \Leftrightarrow \left(\nabla^\dagger,g\right)$ is a quasi-statistical structure.

(iii) Assume that (∇^*,J) is Codazzi-coupled. $d^{\nabla}\omega = 0 \Leftrightarrow (\nabla,g)$ is a quasi-statistical structure.

(iv) Assume that $\left(\nabla^\dagger,J\right)$ is Codazzi-coupled. $d^{\nabla}\omega = 0 \Leftrightarrow (\nabla,g)$ is a quasi-statistical structure.

(v) Assume that $\left(\nabla^\dagger,J\right)$ is Codazzi-coupled. $d^{\nabla^J}\omega = 0 \Leftrightarrow d^{\nabla}\omega = 0$.

(vi) Assume that (∇^*,J) is Codazzi-coupled. $\left(\nabla^J,g\right)$ is a quasi-statistical structure $\Leftrightarrow (\nabla,g)$ is a quasi-statistical structure.

Proof. (i) We can write

$$\begin{aligned} \left(d^{\nabla^*}\omega\right)(X,Y,Z) &= (\nabla^*_X\omega)(Y,Z) - (\nabla^*_Y\omega)(X,Z) + \omega\left(T^{\nabla^*}(X,Y), Z\right) \\ &= X\omega(Y,Z) - \omega(\nabla^*_X Y, Z) - \omega(Y, \nabla^*_X Z) - Y\omega(X,Z) \\ &\quad + \omega(\nabla^*_Y X, Z) + \omega(X, \nabla^*_Y Z) + \omega\left(T^{\nabla^*}(X,Y), Z\right) \\ &= Xg(JY,Z) - g(J\nabla^*_X Y, Z) - g(JY, \nabla^*_X Z) - Yg(JX,Z) \\ &\quad + g(J\nabla^*_Y X, Z) + g(JX, \nabla^*_Y Z) + g\left(JT^{\nabla^*}(X,Y), Z\right) \\ &= -Xg(Y,JZ) + g(\nabla^*_X Y, JZ) + g(Y, J\nabla^*_X Z) + Yg(X,JZ) \\ &\quad - g(\nabla^*_Y X, JZ) - g(X, J\nabla^*_Y Z) - g\left(T^{\nabla^*}(X,Y), JZ\right) \end{aligned}$$

$$\begin{aligned} &= -\left(\nabla^*_X g\right)(Y, JZ) + \left(\nabla^*_Y g\right)(X, JZ) - g\left(T^{\nabla^*}(X,Y), JZ\right) \\ &\quad -g\left(Y, \left(\nabla^*_X J\right) Z\right) + g\left(X, \left(\nabla^*_Y J\right) Z\right) \\ &= -\left(d^{\nabla^*} g\right)(X,Y,JZ) - g\left(Y, \left(\nabla^*_X J\right) Z\right) + g\left(X, \left(\nabla^*_Y J\right) Z\right). \end{aligned}$$

Since $g\left(X, \left(\nabla^*_Y J\right) Z\right) = -g\left(Z, \left(\nabla_Y J\right) X\right)$ and the pair (∇, J) is Codazzi-coupled, we get

$$\left(d^{\nabla^*}\omega\right)(X,Y,Z) = -\left(d^{\nabla^*} g\right)(X,Y,JZ),$$

that is, $d^{\nabla^*}\omega = 0 \Leftrightarrow (\nabla^*, g)$ is a quasi-statistical structure.

(ii) We obtain the following

$$\begin{aligned} &\left(d^{\nabla^\dagger}\omega\right)(X,Y,Z) \\ &= \left(\nabla^\dagger_X \omega\right)(Y,Z) - \left(\nabla^\dagger_Y \omega\right)(X,Z) + \omega\left(T^{\nabla^\dagger}(X,Y), Z\right) \\ &= X\omega(Y,Z) - \omega\left(Y, \nabla^\dagger_X Z\right) - Y\omega(X,Z) + \omega\left(X, \nabla^\dagger_Y Z\right) \\ &\quad -\omega\left([X,Y], Z\right) \\ &= Xg(JY,Z) - g\left(JY, \nabla^\dagger_X Z\right) - Yg(JX,Z) + g\left(JX, \nabla^\dagger_Y Z\right) \\ &\quad -g\left(J[X,Y], Z\right) \\ &= -Xg(Y,JZ) + g\left(Y, J\nabla^\dagger_X Z\right) + Yg(X,JZ) - g\left(X, J\nabla^\dagger_Y Z\right) \\ &\quad +g\left([X,Y], JZ\right) \\ &= -\left(\nabla^\dagger_X g\right)(Y,JZ) + \left(\nabla^\dagger_Y g\right)(X,JZ) - g\left(T^{\nabla^\dagger}(X,Y), JZ\right) \\ &\quad -g\left(Y, \left(\nabla^\dagger_X J\right) Z\right) + g\left(X, \left(\nabla^\dagger_Y J\right) Z\right) \\ &= -\left(d^{\nabla^\dagger} g\right)(X,Y,JZ) - g\left(Y, \left(\nabla^\dagger_X J\right) Z\right) + g\left(X, \left(\nabla^\dagger_Y J\right) Z\right) \end{aligned}$$

such that, from the hypothesis, we have

$$\left(d^{\nabla^\dagger}\omega\right)(X,Y,Z) = -\left(d^{\nabla^\dagger} g\right)(X,Y,JZ),$$

that is, $d^{\nabla^\dagger}\omega = 0 \Leftrightarrow \left(\nabla^\dagger, g\right)$ is a quasi-statistical structure.

(iii) The result can be proved the same as in (i).

(iv) The result can be proved the same as in (ii).

(v) For any vector fields X, Y and Z, we have the following

$$\left(d^{\nabla^J}\omega\right)(X,Y,Z) = \left(\nabla^J_X \omega\right)(Y,Z) - \left(\nabla^J_Y \omega\right)(X,Z) + \omega\left(T^{\nabla^J}(X,Y), Z\right)$$

$$
\begin{aligned}
&= X\omega(Y,Z) - \omega\left(Y, \nabla^J_X Z\right) - Y\omega(X,Z) \\
&\quad + \omega\left(X, \nabla^J_Y Z\right) - \omega([X,Y], Z) \\
&= X\omega(Y,Z) - \omega\left(Y, J^{-1}\nabla_X JZ\right) - Y\omega(X,Z) \\
&\quad + \omega\left(X, J^{-1}\nabla_Y JZ\right) - \omega([X,Y], Z) \\
&= (\nabla_X \omega)(Y,Z) - (\nabla_Y \omega)(X,Z) + \omega\left(T^{\nabla}(X,Y), Z\right) \\
&\quad - \omega\left(Y, J^{-1}(\nabla_X J) Z\right) + \omega\left(X, J^{-1}(\nabla_Y J) Z\right) \\
&= \left(d^{\nabla}\omega\right)(X,Y,Z) - \omega\left(Y, J^{-1}(\nabla_X J) Z\right) + \omega\left(X, J^{-1}(\nabla_Y J) Z\right).
\end{aligned}
$$

From the equation $\omega(X, (\nabla_Y J) Z) = \omega\left(Z, \left(\nabla^{\dagger}_Y J\right) X\right)$, it is easy to see that

$$\left(d^{\nabla^J}\omega\right)(X,Y,Z) = \left(d^{\nabla}\omega\right)(X,Y,Z),$$

that is,

$$d^{\nabla^J}\omega = 0 \Leftrightarrow d^{\nabla}\omega = 0.$$

(vi) From the definition of the J-conjugate transformation, we write the following

$$
\begin{aligned}
&\left(d^{\nabla^J} g\right)(X,Y,Z) \\
&= \left(\nabla^J_X g\right)(Y,Z) - \left(\nabla^J_Y g\right)(X,Z) + g\left(T^{\nabla^J}(X,Y), Z\right) \\
&= Xg(Y,Z) - g\left(Y, \nabla^J_X Z\right) - Yg(X,Z) \\
&\quad + g\left(X, \nabla^J_Y Z\right) - g([X,Y], Z) \\
&= (\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) + g\left(T^{\nabla}(X,Y), Z\right) \\
&\quad - g\left(Y, J^{-1}(\nabla_X J) Z\right) + g\left(X, J^{-1}(\nabla_Y J) Z\right) \\
&= \left(d^{\nabla} g\right)(X,Y,Z) - g\left(Y, J^{-1}(\nabla_X J) Z\right) + g\left(X, J^{-1}(\nabla_Y J) Z\right).
\end{aligned}
$$

To complete the proof, one needs to note

$$g(X, (\nabla^*_Y J) Z) = -g(Z, (\nabla_Y J) X).$$

Immediately, from the hypothesis, we say that

$$\left(d^{\nabla^J} g\right)(X,Y,Z) = \left(d^{\nabla} g\right)(X,Y,Z),$$

i.e., $\left(\nabla^J, g\right)$ is a quasi-statistical structure if and only if (∇, g) is a quasi-statistical structure. □

Remark 1. *Let (M, g) be a pseudo-Riemannian manifold and let ∇ be a linear connection with torsion tensor T^{∇}. Let ω be the fundamental 2-form on M. Denote by $\nabla^{*}, \nabla^{\dagger}$ and ∇^{J}, g-conjugate, ω-conjugate and J-conjugate transformations of a linear connection ∇. These transformations are involutive $(\nabla^{*})^{*} = (\nabla^{\dagger})^{\dagger} = (\nabla^{J})^{J} = \nabla$. When the equation (2) is satisfied, these transformations of ∇ are commutative*

$$\nabla^{*} = \left(\nabla^{\dagger}\right)^{J} = \left(\nabla^{J}\right)^{\dagger},$$

$$\nabla^{\dagger} = \left(\nabla^{*}\right)^{J} = \left(\nabla^{J}\right)^{*},$$

$$\nabla^{J} = \left(\nabla^{*}\right)^{\dagger} = \left(\nabla^{\dagger}\right)^{*}.$$

Hence, $(id, *, \dagger, J)$ forms a 4-element Klein group of transformation of linear connections on M (for details, see Theorem 2.13 in [9]).

Proposition 4. *Let (M, g) be a pseudo-Riemannian manifold and let ∇ be a linear connection with torsion tensor T^{∇} on M. Let ω be the fundamental 2-form on M. Then, the following expressions hold*
(i) $d^{\nabla^{J}}\omega = 0 \Leftrightarrow (\nabla, g)$ is a quasi-statistical structure;
(ii) $d^{\nabla}\omega = 0 \Leftrightarrow \left(\nabla^{J}, g\right)$ is a quasi-statistical structure;
(iii) $d^{\nabla^{\dagger}}\omega = 0 \Leftrightarrow (\nabla^{*}, g)$ is a quasi-statistical structure;
(iv) $d^{\nabla^{*}}\omega = 0 \Leftrightarrow \left(\nabla^{\dagger}, g\right)$ is a quasi-statistical structure.

Proof. To show (i), one only needs the following equality

$$\begin{aligned}
& \left(d^{\nabla^{J}}\omega\right)(X, Y, Z) \\
= \; & \left(\nabla^{J}_{X}\omega\right)(Y, Z) - \left(\nabla^{J}_{Y}\omega\right)(X, Z) + \omega\left(T^{\nabla^{J}}(X, Y), Z\right) \\
= \; & X\omega(Y, Z) - \omega\left(Y, \nabla^{J}_{X}Z\right) - Y\omega(X, Z) \\
& +\omega\left(X, \nabla^{J}_{Y}Z\right) - \omega([X, Y], Z) \\
= \; & X\omega(Y, Z) - \omega\left(Y, J^{-1}\nabla_{X}JZ\right) - Y\omega(X, Z) \\
& +\omega\left(X, J^{-1}\nabla_{Y}JZ\right) - \omega([X, Y], Z) \\
= \; & Xg(JY, Z) - g\left(JY, J^{-1}\nabla_{X}JZ\right) - Yg(JX, Z) \\
& +g\left(JX, J^{-1}\nabla_{Y}JZ\right) - g(J[X, Y], Z) \\
= \; & -Xg(Y, JZ) + g(Y, \nabla_{X}JZ) + Yg(X, JZ) \\
& -g(X, \nabla_{Y}JZ) + g([X, Y], JZ) \\
= \; & -(\nabla_{X}g)(Y, JZ) + (\nabla_{Y}g)(X, JZ) - g\left(T^{\nabla}(X, Y), JZ\right) \\
= \; & -\left(d^{\nabla}g\right)(X, Y, JZ),
\end{aligned}$$

which completes the proof, i.e., $d^{\nabla^J}\omega = 0 \Leftrightarrow (\nabla, g)$ is a quasi-statistical structure. From Remark 1, the other statements can easily be proved. □

Corollary 2. *Let M be a pseudo-Riemannian manifold equipped with a pseudo Riemannian metric g and a linear connection ∇. Let (g, ω, J) be a compatible triple, and $\nabla^*, \nabla^\dagger$ and ∇^J denote, respectively, g-conjugate, ω-conjugate and J-conjugate transformations of an arbitrary linear connection ∇. Fei and Zhang [9] showed that (∇, g) is a statistical structure if and only if ∇^* is torsion-free. There exist the following expressions*
(i) (∇^*, g) is a quasi-statistical structure if and only if ∇ is torsion-free;
(ii) $\left(\nabla^J, g\right)$ is a quasi-statistical structure if and only if $\nabla^\dagger$ is torsion-free;
(iii) $\left(\nabla^\dagger, g\right)$ is a quasi-statistical structure if and only if ∇^J is torsion-free.

Proof. From 4-element Klein group action $(id, *, \dagger, J)$ on the space of linear connections, the results immediately follow. □

As a corollary of Proposition 4 and Corollary 2, we get the following.

Proposition 5. *Let M be a pseudo-Riemannian manifold equipped with a pseudo-Riemannian metric g and a linear connection ∇. Let (g, ω, J) be a compatible triple, and $\nabla^*, \nabla^\dagger$ and ∇^J denote, respectively, g-conjugate, ω-conjugate and J-conjugate transformations of the linear connection ∇. Then, there exist the followings*
(i) $d^\nabla \omega = 0 \Leftrightarrow T^{\nabla^\dagger} = 0 \Leftrightarrow d^{\nabla^*} J = 0 \Leftrightarrow d^{\nabla^J} g = 0$;
(ii) $d^{\nabla^*}\omega = 0 \Leftrightarrow T^{\nabla^J} = 0 \Leftrightarrow d^\nabla J = 0 \Leftrightarrow d^{\nabla^\dagger} g = 0$;
(iii) $d^{\nabla^\dagger}\omega = 0 \Leftrightarrow T^\nabla = 0 \Leftrightarrow d^{\nabla^J} J = 0 \Leftrightarrow d^{\nabla^*} g = 0$;
(iv) $d^{\nabla^J}\omega = 0 \Leftrightarrow T^{\nabla^*} = 0 \Leftrightarrow d^{\nabla^\dagger} J = 0 \Leftrightarrow d^\nabla g = 0$.

Proof. One can write

$$\begin{aligned}
\left(d^\nabla \omega\right)(X, Y, Z) &= (\nabla_X \omega)(Y, Z) - (\nabla_Y \omega)(X, Z) + \omega\left(T^\nabla (X, Y), Z\right) \\
&= X\omega(Y, Z) - \omega(Y, \nabla_X Z) - Y\omega(X, Z) \\
&\quad + \omega(X, \nabla_Y Z) - \omega([X, Y], Z) \\
&= \omega\left(\nabla^\dagger_X Y, Z\right) - \omega\left(\nabla^\dagger_Y X, Z\right) - \omega([X, Y], Z) \\
&= \omega\left(T^{\nabla^\dagger}(X, Y), Z\right),
\end{aligned}$$

such that $d^{\nabla}\omega = 0 \Leftrightarrow T^{\nabla^{\dagger}} = 0$. Moreover, we have

$$\begin{aligned}
& g\left(\left(d^{\nabla^*}J\right)(X,Y),Z\right) \\
= \; & g\left((\nabla^*_X J)Y - (\nabla^*_Y J)X + JT^{\nabla^*}(X,Y),Z\right) \\
= \; & g\left((\nabla^*_X J)Y,Z\right) - g\left((\nabla^*_Y J)X,Z\right) + g\left(JT^{\nabla^*}(X,Y),Z\right) \\
= \; & g\left(\nabla^*_X JY,Z\right) - g\left(\nabla^*_Y JX,Z\right) - g\left(J[X,Y],Z\right) \\
= \; & Xg(JY,Z) - g(JY,\nabla_X Z) - Yg(JX,Z) \\
& + g(JX,\nabla_Y Z) - g(J[X,Y],Z) \\
= \; & X\omega(Y,Z) - \omega(Y,\nabla_X Z) - Y\omega(X,Z) \\
& + \omega(X,\nabla_Y Z) - \omega([X,Y],Z) \\
= \; & \omega\left(T^{\nabla^{\dagger}}(X,Y),Z\right),
\end{aligned}$$

which implies that $T^{\nabla^{\dagger}} = 0 \Leftrightarrow d^{\nabla^*}J = 0$. From (ii) of Proposition 4, the following last expression is obtained

$$d^{\nabla}\omega = 0 \Leftrightarrow T^{\nabla^{\dagger}} = 0 \Leftrightarrow d^{\nabla^*}J = 0 \Leftrightarrow d^{\nabla^J}g = 0.$$

With help of Remark 1, the expressions (ii), (iii) and (iv) can be easily proved. □

Remark 2. *The Proposition 5 says to us that the torsion tensor of the g-conjugate of a linear connection ∇ is always zero on a quasi-statistical manifold. Also, this proposition gives information about the integrability of the structure J. Suppose that (∇, J) is Codazzi-coupled. If the triple (M, g, ∇^*) is a quasi-statistical manifold, then the almost complex structure J is integrable.*

Proposition 6. *Let ∇ be a linear connection with torsion tensor T^{∇}, J be an almost complex structure and g be a pseudo-Riemannian metric on M. If $d^{\nabla}J = 0$ and (M, g, ∇) is a quasi-statistical manifold, then the below equality is satisfied*

$$\begin{aligned}
& \left(d^{\nabla^J}g\right)(X,Y,Z) + \left(d^{\nabla^J}g\right)(Y,Z,X) + \left(d^{\nabla^J}g\right)(Z,X,Y) \\
= \; & g\left(Y,T^{\nabla}(X,Z)\right) + g\left(Z,T^{\nabla}(Y,X)\right) + g\left(X,T^{\nabla}(Z,Y)\right).
\end{aligned}$$

Proof. We calculate

$$\begin{aligned}
\left(d^{\nabla^J}g\right)(X,Y,Z) &= \left(\nabla^J_X g\right)(Y,Z) - \left(\nabla^J_Y g\right)(X,Z) + g\left(T^{\nabla^J}(X,Y),Z\right) \\
&= Xg(Y,Z) - g\left(Y,\nabla^J_X Z\right) - Yg(X,Z) \\
&\quad + g\left(X,\nabla^J_Y Z\right) - g([X,Y],Z)
\end{aligned}$$

$$\begin{aligned}
&= Xg(Y,Z) - g\left(Y, J^{-1}\nabla_X JZ\right) - Yg(X,Z) \\
&\quad + g\left(X, J^{-1}\nabla_Y JZ\right) - g([X,Y],Z) \\
&= Xg(Y,Z) - g(Y,\nabla_X Z) - g\left(Y, J^{-1}(\nabla_X J)Z\right) - Yg(X,Z) \\
&\quad + g(X,\nabla_Y Z) + g\left(X, J^{-1}(\nabla_Y J)Z\right) - g([X,Y],Z) \\
&= \left(d^{\nabla} g\right)(X,Y,Z) - g\left(Y, J^{-1}(\nabla_X J)Z\right) + g\left(X, J^{-1}(\nabla_Y J)Z\right).
\end{aligned}$$

The equality $d^{\nabla} J = 0$ implies that

$$\begin{aligned}
&\left(d^{\nabla^J} g\right)(X,Y,Z) + \left(d^{\nabla^J} g\right)(Y,Z,X) + \left(d^{\nabla^J} g\right)(Z,X,Y) \\
= \; & g\left(Y, T^{\nabla}(X,Z)\right) + g\left(Z, T^{\nabla}(Y,X)\right) + g\left(X, T^{\nabla}(Z,Y)\right).
\end{aligned}$$

Here, we also use $d^{\nabla} g = 0$ and $J^2 = -id$. □

Lemma 3. *Let ∇ be a linear connection with torsion tensor T^{∇}, J be an almost complex structure and g be a pseudo-Riemannian metric on M. Let (g,ω,J) be a compatible triple. If $d^{\nabla} J = 0$ and (M,g,∇) is a quasi-statistical manifold, then ω is closed, that is, $d\omega = 0$.*

Proof. From Lemma 2, we obtain

$$\begin{aligned}
& d\omega(X,Y,Z) \\
= \; & (\nabla_Z \omega)(X,Y) + (\nabla_X \omega)(Y,Z) + (\nabla_Y \omega)(Z,X) \\
& + \omega\left(T^{\nabla}(X,Y), Z\right) + \omega\left(T^{\nabla}(Y,Z), X\right) + \omega\left(T^{\nabla}(Z,X), Y\right) \\
= \; & -(\nabla_Z g)(X, JY) - g(X, (\nabla_Z J)Y) - (\nabla_X g)(Y, JZ) \\
& - g(Y, (\nabla_X J)Z) - (\nabla_Y g)(Z, JX) - g(Z, (\nabla_Y J)X) \\
& + g\left(JT^{\nabla}(X,Y), Z\right) + g\left(JT^{\nabla}(Y,Z), X\right) + g\left(JT^{\nabla}(Z,X), Y\right) \\
= \; & -(\nabla_X g)(Z, JY) - (\nabla_Y g)(X, JZ) - (\nabla_Z g)(Y, JX) \\
& - g(X, (\nabla_Z J)Y) - g(Y, (\nabla_X J)Z) - g(Z, (\nabla_Y J)X)
\end{aligned}$$

and

$$\begin{aligned}
d\omega(Z,Y,X) &= -(\nabla_Z g)(X, JY) - (\nabla_Y g)(Z, JX) - (\nabla_X g)(Y, JZ) \\
&\quad - g(Z, (\nabla_X J)Y) - g(Y, (\nabla_Z J)X) - g(X, (\nabla_Y J)Z).
\end{aligned}$$

Thus, we have the following

$$\begin{aligned}
& d\omega(X,Y,Z) - d\omega(Z,Y,X) \\
= \; & -(\nabla_X g)(Z, JY) - (\nabla_Y g)(X, JZ) - (\nabla_Z g)(Y, JX) \\
& - g(X, (\nabla_Z J)Y) - g(Y, (\nabla_X J)Z) - g(Z, (\nabla_Y J)X)
\end{aligned}$$

$$
\begin{aligned}
& +(\nabla_Z g)(X, JY) + (\nabla_Y g)(Z, JX) + (\nabla_X g)(Y, JZ) \\
& +g(Z, (\nabla_X J)Y) + g(Y, (\nabla_Z J)X) + g(X, (\nabla_Y J)Z) \\
= \; & -(\nabla_X g)(Z, JY) + (\nabla_Z g)(X, JY) \\
& +g(Y, (\nabla_Z J)X - (\nabla_X J)Z) \\
& -(\nabla_Y g)(X, JZ) + (\nabla_X g)(Y, JZ) \\
& +g(Z, (\nabla_X J)Y - (\nabla_Y J)X) \\
& -(\nabla_Z g)(Y, JX) + (\nabla_Y g)(Z, JX) \\
& +g(X, (\nabla_Y J)Z - (\nabla_Z J)Y) \\
= \; & -(\nabla_X g)(Z, JY) + (\nabla_Z g)(X, JY) + g\left(Y, JT^{\nabla}(X, Z)\right) \\
& -(\nabla_Y g)(X, JZ) + (\nabla_X g)(Y, JZ) + g\left(Z, JT^{\nabla}(Y, X)\right) \\
& -(\nabla_Z g)(Y, JX) + (\nabla_Y g)(Z, JX) + g\left(X, JT^{\nabla}(Z, Y)\right) \\
= \; & -\left(d^{\nabla} g\right)(X, Z, JY) - \left(d^{\nabla} g\right)(Y, X, JZ) - \left(d^{\nabla} g\right)(Z, Y, JX) \\
= \; & 0,
\end{aligned}
$$

such that $d\omega(X, Y, Z) = d\omega(Z, Y, X)$. Since $d\omega$ is totally skew-symmetric, we conclude $d\omega = 0$, i.e., ω is closed. □

We are now ready to introduce our first main theorem. As a corollary of Proposition 2 and Lemma 3, we have the following.

Theorem 7. *Let ∇ be a linear connection with torsion tensor T^{∇}, g be a pseudo-Riemannian metric and J be an almost complex structure on M, and (g, ω, J) be a compatible triple. Assume that the torsion tensor T^{∇} of ∇ satisfies the torsion-compatibility condition. If $d^{\nabla} J = 0$ and (M, g, ∇) is a quasi-statistical manifold, then (M, g, J) is a Kähler manifold.*

Remark 3. *We know that for any statistical manifold (M, g, ∇), if there exists Codazzi couplings of ∇ with an almost complex structure J, (M, g, ∇, J) is a Kähler manifold (see Theorem 3.2 in [9]). Theorem 7 says that an alternative characterization can be made for Kähler manifolds by taking a quasi-statistical structure instead of a statistical structure. That is, to make such a characterization, the torsion tensor of a linear connection ∇ need not be zero.*

Theorem 8. *Let ∇ be a linear connection with torsion tensor T^{∇} on M, and (g, ω, J) be a compatible triple. Then, for the following three statements regarding any compatible triple (g, ω, J), any two imply the third*

(i) (M, ∇, g) is a quasi-statistical manifold;
(ii) $d^{\nabla} J = 0$, that is, J is d^{∇}-closed;
(iii) $\nabla^{*} \omega = 0$.

Proof. Assume that $d^{\nabla}g = 0$ and $d^{\nabla}J = 0$. From Lemma 3, we have

$$\begin{aligned}
& d\omega(X,Y,Z) \\
= & -(\nabla_Z g)(X,JY) - g(X,(\nabla_Z J)Y) - (\nabla_X g)(Y,JZ) \\
& -g(Y,(\nabla_X J)Z) - (\nabla_Y g)(Z,JX) - g(Z,(\nabla_Y J)X) \\
& +g\left(JT^{\nabla}(X,Y),Z\right) + g\left(JT^{\nabla}(Y,Z),X\right) + g\left(JT^{\nabla}(Z,X),Y\right) \\
= & -(\nabla_Z g)(X,JY) - g(X,(\nabla_Z J)Y) - (\nabla_X g)(Y,JZ) \\
& -g(Y,(\nabla_X J)Z) - (\nabla_Y g)(Z,JX) - g(Z,(\nabla_Y J)X) \\
& -g\left(T^{\nabla}(X,Y),JZ\right) - g\left(T^{\nabla}(Y,Z),JX\right) - g\left(T^{\nabla}(Z,X),JY\right) \\
= & -(\nabla_X g)(Z,JY) - (\nabla_Y g)(X,JZ) - (\nabla_Z g)(Y,JX) \\
& -g(X,(\nabla_Z J)Y) - g(Y,(\nabla_X J)Z) - g(Z,(\nabla_Y J)X) \\
= & -(\nabla_X g)(Z,JY) - (\nabla_Y g)(X,JZ) - (\nabla_Z g)(Y,JX) \\
& -g\left(X,JT^{\nabla}(Y,Z) + (\nabla_Y J)Z\right) \\
& -g\left(Y,JT^{\nabla}(Z,X) + (\nabla_Z J)X\right) \\
& -g\left(Z,JT^{\nabla}(X,Y) + (\nabla_X J)Y\right) \\
= & -(\nabla_Z g)(X,JY) - (\nabla_X g)(Y,JZ) - (\nabla_Y g)(Z,JX) \\
& -g(X,(\nabla_Y J)Z) - g(Y,(\nabla_Z J)X) - g(Z,(\nabla_X J)Y).
\end{aligned}$$

Besides, due to skew-symmetric of ω, we can write

$$\begin{aligned}
& (\nabla_Z \omega)(X,Y) + (\nabla_Z \omega)(Y,X) \\
= & -(\nabla_Z g)(X,JY) - g(X,(\nabla_Z J)Y) \\
& -(\nabla_Z g)(Y,JX) - g(Y,(\nabla_Z J)X) \\
= & -(\nabla_Z g)(X,JY) - g(X,(\nabla_Z J)Y) - (\nabla_Y g)(Z,JX) \\
& -g\left(T^{\nabla}(Y,Z),JX\right) - g(Y,(\nabla_Z J)X) \\
= & -(\nabla_Z g)(X,JY) - (\nabla_Y g)(Z,JX) \\
& -g(Y,(\nabla_Z J)X) - g(X,(\nabla_Y J)Z) \\
= & \ 0.
\end{aligned}$$

With these relations, we get

$$d\omega(X,Y,Z) = -(\nabla_X g)(Y,JZ) - g(Z,(\nabla_X J)Y) = -(\nabla^*_X \omega)(Y,Z) = 0,$$

that is, $\nabla^*\omega = 0$.

Next, let us suppose that $d^{\nabla}g = 0$ and $\nabla^*\omega = 0$. Thus, we get

$$\begin{aligned}
\left(d^{\nabla}g\right)(X,Y,JZ) &= (\nabla_X g)(Y,JZ) - (\nabla_Y g)(X,JZ) \\
&\quad +g\left(T^{\nabla}(X,Y),JZ\right) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned} \left(\nabla^*_X \omega\right)(Y,Z) &= \left(\nabla_X g\right)(Y,JZ) + g\left(Z,\left(\nabla_X J\right)Y\right) \\ &= 0 \end{aligned}$$

from which we immediately see that

$$\begin{aligned} & \left(d^{\nabla} g\right)(X,Y,JZ) \\ = & -g\left(Z,\left(\nabla_X J\right)Y\right) + g\left(Z,\left(\nabla_Y J\right)X\right) - g\left(JT^{\nabla}\left(X,Y\right),Z\right) \\ = & -g\left(Z,\left(d^{\nabla} J\right)\left(X,Y\right)\right) = 0, \end{aligned}$$

such that $d^{\nabla} J = 0$. It is easy to see that if $d^{\nabla} J = 0$ and $\nabla^* \omega = 0$, then $d^{\nabla} g = 0$. □

Now we turn our attention to the linear connection $\widetilde{\nabla}$ which is the average of a linear connection and its J-conjugate connection such that $\widetilde{\nabla} = \frac{1}{2}\left(\nabla^J + \nabla\right)$. The connection $\widetilde{\nabla}$ is a complex connection, that is, $\widetilde{\nabla} J = 0$ [14].

Proposition 9. *Let (M,g) be a pseudo-Riemannian manifold, ∇ be an arbitrary linear connection with torsion tensor T^{∇}, ∇^* be the g-conjugate connection of ∇ and J be an almost complex structure that is compatible with g. Assume that (∇^*, J) is Codazzi-coupled. $\left(\widetilde{\nabla}, g\right)$ is a quasi-statistical structure if and only if (∇, g) is a quasi-statistical structure, where $\widetilde{\nabla} = \frac{1}{2}\left(\nabla^J + \nabla\right)$ and ∇^J is the J-conjugate connection of ∇.*

Proof. We have

$$\begin{aligned} & \left(d^{\widetilde{\nabla}} g\right)(X,Y,Z) \\ = & \left(\widetilde{\nabla}_X g\right)(Y,Z) - \left(\widetilde{\nabla}_Y g\right)(X,Z) + g\left(T^{\widetilde{\nabla}}\left(X,Y\right),Z\right) \\ = & \frac{1}{2}\left(\nabla_X g\right)(Y,Z) + \frac{1}{2}\left(\nabla^J_X g\right)(Y,Z) - \frac{1}{2}\left(\nabla_Y g\right)(X,Z) \\ & -\frac{1}{2}\left(\nabla^J_Y g\right)(X,Z) + \frac{1}{2} g\left(T^{\nabla}\left(X,Y\right),Z\right) \\ & +\frac{1}{2} g\left(T^{\nabla^J}\left(X,Y\right),Z\right). \end{aligned}$$

On considering the following equalities

$$\begin{aligned} \left(\nabla^J_X g\right)(Y,Z) &= \left(\nabla_X g\right)(Y,Z) - g\left(J^{-1}\left(\nabla_X J\right)Y,Z\right) \\ & -g\left(Y,J^{-1}\left(\nabla_X J\right)Z\right) \end{aligned}$$

and

$$g\left(T^{\nabla^J}(X,Y),Z\right) = g\left(T^{\nabla}(X,Y),Z\right) + g\left(J^{-1}(\nabla_X J)Y,Z\right) - g\left(J^{-1}(\nabla_Y J)X,Z\right),$$

we obtain

$$\begin{aligned}&\left(d^{\widetilde{\widetilde{\nabla}}}g\right)(X,Y,Z)\\ = \ &\left(d^{\nabla}g\right)(X,Y,Z) - \frac{1}{2}g\left(Y,J^{-1}(\nabla_X J)Z\right)\\ &+\frac{1}{2}g\left(X,J^{-1}(\nabla_Y J)Z\right).\end{aligned}$$

To complete the proof, we need to note $g\left(X,J^{-1}(\nabla_Y J)Z\right) = -g\left((\nabla^*_Y J)X,JZ\right)$. From the hypothesis, we get $\left(d^{\widetilde{\widetilde{\nabla}}}g\right)(X,Y,Z) = \left(d^{\nabla}g\right)(X,Y,Z)$. □

Via (vi) of Proposition 3, (i) and (iv) of Proposition 5 and Proposition 9, we obtain the following result.

Corollary 3. *Assume that (∇^*, J) is Codazzi-coupled. Then, $\left(\widetilde{\widetilde{\nabla}}, g\right)$ is a quasi-statistical structure if and only if the torsion tensors of ∇^* and $\nabla^\dagger$ are zero.*

Proposition 10. *Let (M,g) be a pseudo-Riemannian manifold equipped with a pseudo-Riemannian metric g and a linear connection ∇ with torsion tensor T^{∇}. Let (g,ω,J) be a compatible triple, and $\nabla^*, \nabla^\dagger$ denote, respectively, g-conjugation, ω-conjugation of the linear connection ∇. Then, $\left(\widetilde{\widetilde{\nabla}}, g\right)$ is a quasi-statistical structure if and only if $T^{\nabla^*} = -T^{\nabla^\dagger}$, where $\widetilde{\widetilde{\nabla}} = \frac{1}{2}\left(\nabla^J + \nabla\right)$.*

Proof. Considering the definition of $d^{\widetilde{\widetilde{\nabla}}}g$, we have

$$\begin{aligned}&\left(d^{\widetilde{\widetilde{\nabla}}}g\right)(X,Y,Z)\\ = \ &\left(\widetilde{\widetilde{\nabla}}_X g\right)(Y,Z) - \left(\widetilde{\widetilde{\nabla}}_Y g\right)(X,Z) + g\left(T^{\widetilde{\widetilde{\nabla}}}(X,Y),Z\right)\\ = \ &Xg(Y,Z) - g\left(Y,\widetilde{\widetilde{\nabla}}_X Z\right) - Yg(X,Z)\\ &+g\left(X,\widetilde{\widetilde{\nabla}}_Y Z\right) - g([X,Y],Z)\\ = \ &\frac{1}{2}g\left(\nabla^*_X Y,Z\right) + \frac{1}{2}g\left(\nabla^\dagger_X Y,Z\right) - \frac{1}{2}g\left(\nabla^*_Y X,Z\right)\\ &-\frac{1}{2}g\left(\nabla^\dagger_Y X,Z\right) - g([X,Y],Z)\end{aligned}$$

$$= \frac{1}{2} g\left(T^{\nabla^*}(X,Y) + T^{\nabla^\dagger}(X,Y), Z\right).$$

□

Proposition 11. *Let (M,g) be a pseudo-Riemannian manifold equipped with a pseudo-Riemannian metric g and a linear connection ∇ with torsion tensor T^{∇}. Let (g,ω,J) be a compatible triple, and $\nabla^*, \nabla^\dagger$ denote, respectively, g-conjugation, ω-conjugation of an arbitrary linear connection ∇. Then, $T^{\nabla^*} = -T^{\nabla^\dagger}$ if and only if $d^{\nabla^*}J = -d^{\nabla^\dagger}J$.*

Proof. We calculate

$$\begin{aligned}
& g\left(\left(d^{\nabla^*}J\right)(X,Y) + \left(d^{\nabla^\dagger}J\right)(X,Y), Z\right) \\
= \; & g\left(\left(d^{\nabla^*}J\right)(X,Y), Z\right) + g\left(\left(d^{\nabla^\dagger}J\right)(X,Y), Z\right) \\
= \; & g\left((\nabla^*_X J) Y - (\nabla^*_Y J) X + JT^{\nabla^*}(X,Y), Z\right) \\
& + g\left(\left(\nabla^\dagger_X J\right) Y - \left(\nabla^\dagger_Y J\right) X + JT^{\nabla^\dagger}(X,Y), Z\right) \\
= \; & g(\nabla^*_X JY, Z) - g(\nabla^*_Y JX, Z) - g(J[X,Y], Z) \\
& + g\left(\nabla^\dagger_X JY, Z\right) - g\left(\nabla^\dagger_Y JX, Z\right) - g(J[X,Y], Z) \\
= \; & \omega\left(T^{\nabla^\dagger}(X,Y) + T^{\nabla^*}(X,Y), Z\right).
\end{aligned}$$

□

Propositions 10 and 11 immediately give the following.

Corollary 4. $\left(\widetilde{\nabla}, g\right)$ *is a quasi-statistical structure* $\Leftrightarrow T^{\nabla^*} = -T^{\nabla^\dagger} \Leftrightarrow d^{\nabla^*}J = -d^{\nabla^\dagger}J$.

4 Quasi Statistical Structures with an anti-Hermitian metric h

In this section, we will investigate the properties of quasi-statistical manifolds by taking anti-Hermitian metric h instead of the Hermitian metric g. Moreover, considering any linear connection ∇ with torsion tensor T^{∇} instead of the Levi-Civita connection ∇^h of the anti-Hermitian metric h, we will show that the anti-Kähler and quasi-Kähler-Norden manifolds can be classified under certain conditions.

Definition 4. *On given a pseudo-Riemannian manifold (M, h) endowed with an almost complex structure J, then the triple (M, h, J) is called an almost anti-Hermitian manifold (or Norden manifold) if*

$$h(JX, Y) = h(X, JY)$$

for any vector fields X and Y on M, where the signature of h is (n, n), that is, h is a neutral metric. If the structure J is integrable, then the triple (M, h, J) is called an anti-Hermitian manifold or complex Norden manifold. Also, the twin anti-Hermitian metric is defined by

$$\hbar(X, Y) = h(JX, Y)$$

for any vector fields X and Y. An anti-Kähler manifold is an almost anti-Hermitian manifold such that $\nabla^h J = 0$, where ∇^h is the Levi-Civita connection of the pseudo-Riemannian manifold (M, h) [11, 15, 18].

The Tachibana operator on an almost anti-Hermitian manifold (M, h, J)

$$\Phi_J : \Im_2^0(M) \longrightarrow \Im_3^0(M)$$

which is defined from the set of all $(0, 2)$-tensor fields $(\Im_2^0(M))$ into the set of all $(0, 3)$-tensor fields $(\Im_3^0(M))$ on M is given by [20, 22]

$$\begin{aligned}(\Phi_J h)(X, Y, Z) &= JXh(Y, Z) - Xh(JY, Z) \\ &\quad + h((L_Y J) X, Z) + h(Y, (L_Z J) X),\end{aligned}$$

where $(L_X J) Y = [X, JY] - J[X, Y]$.

Definition 5. *[18] An almost anti-Hermitian manifold is called a quasi-Kähler-Norden manifold if*

$$\underset{X,Y,Z}{\sigma} h\left(\left(\nabla_X^h J\right) Y, Z\right) = 0,$$

where σ is the cyclic sum by three arguments.

Theorem 12. *[21] Let (M, h, J) be a non-integrable almost anti-Hermitian manifold. Then the triple (M, h, J) is quasi-Kähler-Norden if and only if*

$$(\Phi_J h)(X, Y, Z) + (\Phi_J h)(Y, Z, X) + (\Phi_J h)(Z, X, Y) = 0.$$

As is known, the almost complex structure J on an anti-Kähler manifold (M, h) is always integrable.

We will recall notions related to an anti-Hermitian metric.

The covariant derivative of the metrics h and $\hbar$ are defined by

$$(\nabla_Z h)(X,Y) = Zh(X,Y) - h(\nabla_Z X, Y) - h(X, \nabla_Z Y)$$

and

$$(\nabla_Z \hbar)(X,Y) = Z\hbar(X,Y) - \hbar(\nabla_Z X, Y) - \hbar(X, \nabla_Z Y).$$

Clearly $(\nabla_Z h)(X,Y) = (\nabla_Z h)(Y,X)$ and $(\nabla_Z \hbar)(X,Y) = (\nabla_Z \hbar)(Y,X)$ due to symmetry of h and $\hbar$. For any linear connection ∇, its h-conjugate connection $\nabla^\sharp$ and its $\hbar$-conjugate connection $\nabla^\ddagger$ are defined by

$$Zh(X,Y) = h(\nabla_Z X, Y) + h\left(X, \nabla^\sharp_Z Y\right)$$

and

$$Z\hbar(X,Y) = \hbar(\nabla_Z X, Y) + \hbar\left(X, \nabla^\ddagger_Z Y\right),$$

respectively. It can be easily checked that the following conditions are satisfied

$$\left(\nabla^\sharp\right)^\sharp = \left(\nabla^\ddagger\right)^\ddagger = \left(\nabla^J\right)^J = \nabla,$$

$$\nabla^\sharp = \left(\nabla^\dagger\right)^J = \left(\nabla^J\right)^\dagger,$$

$$\nabla^\ddagger = \left(\nabla^\sharp\right)^J = \left(\nabla^J\right)^\sharp,$$

$$\nabla^J = \left(\nabla^\sharp\right)^\ddagger = \left(\nabla^\ddagger\right)^\sharp,$$

which give that $(id, \sharp, \ddagger, J)$ is a 4-element Klein group action on the space of linear connections (also see [12]). Next, we will give some results without proof. These results can be proven by following the proofs of Propositions 3, 4 and 5. Their proofs are used the anti-Hermitian metric h and the twin anti-Hermitian metric $\hbar$ instead of the Hermitian metric g and the fundamental 2-form ω and purity conditions.

Proposition 13. *Let (M,h,J) be an almost anti-Hermitian manifold and let ∇ be a linear connection with torsion tensor T^∇ on M. Let $\hbar$ be the twin anti-Hermitian metric. Then, there exist the following expressions*

(i) Assume that (∇, J) is Codazzi-coupled. $\left(\nabla^\sharp, \hbar\right)$ is a quasi-statistical structure $\Leftrightarrow$ $\left(\nabla^\sharp, h\right)$ is a quasi-statistical structure.

(ii) Assume that (∇, J) is Codazzi-coupled. $\left(\nabla^\ddagger, \hbar\right)$ is a quasi-statistical structure $\Leftrightarrow$ $\left(\nabla^\ddagger, h\right)$ is a quasi-statistical structure.

(iii) Assume that $\left(\nabla^\sharp, J\right)$ is Codazzi-coupled. $(\nabla, \hbar)$ is a quasi-statistical structure $\Leftrightarrow$ (∇, h) is a quasi-statistical structure.

(iv) Assume that $\left(\nabla^\ddagger, J\right)$ is Codazzi-coupled. $(\nabla, \hbar)$ is a quasi-statistical structure $\Leftrightarrow$ (∇, h) is a quasi-statistical structure.

(v) Assume that $\left(\nabla^{\ddagger}, J\right)$ *is Codazzi-coupled.* $\left(\nabla^{J}, \hbar\right)$ *is a quasi statistical structure* $\Leftrightarrow (\nabla, \hbar)$ *is a quasi statistical structure.*

(vi) Assume that $\left(\nabla^{\sharp}, J\right)$ *is Codazzi-coupled.* $\left(\nabla^{J}, h\right)$ *is a quasi-statistical structure* $\Leftrightarrow (\nabla, h)$ *is a quasi-statistical structure.*

Proposition 14. *Let* (M, h, J) *be an anti-Hermitian manifold and let* ∇ *be a linear connection with torsion tensor* T^{∇} *on* M*. Let* $\hbar$ *be the twin anti-Hermitian metric. Then, the following expressions hold*

(i) $\left(\nabla^{J}, \hbar\right)$ *is a quasi-statistical structure if and only if* (∇, h) *is a quasi-statistical structure.*

(ii) $(\nabla, \hbar)$ *is a quasi-statistical structure if and only if* $\left(\nabla^{J}, h\right)$ *is a quasi-statistical structure.*

(iii) $\left(\nabla^{\ddagger}, \hbar\right)$ *is a quasi-statistical structure if and only if* $\left(\nabla^{\sharp}, h\right)$ *is a quasi-statistical structure.*

(iv) $\left(\nabla^{\sharp}, \hbar\right)$ *is a quasi-statistical structure if and only if* $\left(\nabla^{\ddagger}, h\right)$ *is a quasi-statistical structure.*

Corollary 5. *Let* M *be a manifold equipped with an anti-Hermitian metric* h*, a linear connection* ∇ *with torsion tensor* T^{∇} *and the twin anti-Hermitian metric* $\hbar$*. Denote by* $\nabla^{\sharp}, \nabla^{\ddagger}$ *and* ∇^{J}*, respectively,* h*-conjugation,* $\hbar$*-conjugation and* J*-conjugate transformations of an arbitrary linear connection* ∇*. From 4-element Klein group action on the space of linear connections, we have*

(i) (∇, h) *is a quasi-statistical structure if and only if the linear connection* $\nabla^{\sharp}$ *is torsion-free.*

(ii) $\left(\nabla^{\sharp}, h\right)$ *is a quasi-statistical structure if and only if the linear connection* ∇ *is torsion-free.*

(iii) $\left(\nabla^{J}, h\right)$ *is a quasi-statistical structure if and only if the linear connection* $\nabla^{\ddagger}$ *is torsion-free.*

(iv) $\left(\nabla^{\ddagger}, h\right)$ *is a quasi-statistical structure if and only if the linear connection* ∇^{J} *is torsion-free.*

From Proposition 14 and Corollary 5, we have the following result.

Proposition 15. *Let* (M, h, J) *be an almost anti-Hermitian manifold,* ∇ *be an arbitrary linear connection,* $\nabla^{\sharp}$ *be the* h*-conjugate connection of* ∇ *and* $\nabla^{\ddagger}$ *be the* $\hbar$*-conjugate connection of* ∇*. Then, there exist the below expressions*

(i) $d^{\nabla}\hbar = 0 \Leftrightarrow T^{\nabla^{\ddagger}} = 0 \Leftrightarrow d^{\nabla^{\sharp}}J = 0 \Leftrightarrow d^{\nabla^{J}}h = 0;$

(ii) $d^{\nabla^{\sharp}}\hbar = 0 \Leftrightarrow T^{\nabla^{J}} = 0 \Leftrightarrow d^{\nabla}J = 0 \Leftrightarrow d^{\nabla^{\ddagger}}h = 0;$

(iii) $d^{\nabla^{\ddagger}}\hbar = 0 \Leftrightarrow T^{\nabla} = 0 \Leftrightarrow d^{\nabla^{J}}J = 0 \Leftrightarrow d^{\nabla^{\sharp}}h = 0;$

(iv) $d^{\nabla^{J}}\hbar = 0 \Leftrightarrow T^{\nabla^{\sharp}} = 0 \Leftrightarrow d^{\nabla^{\ddagger}}J = 0 \Leftrightarrow d^{\nabla}h = 0.$

Proposition 16. *Let (M, h, J) be an almost anti-Hermitian manifold and ∇ be a linear connection with torsion tensor T^{∇} on M. If (∇, h) is a quasi-statistical structure, that is, $d^{\nabla}h = 0$, we get*

$$\begin{aligned}(\Phi_J h)(X,Y,Z) &= (\nabla_Y h)(JX,Z) - (\nabla_Y h)(X,JZ) + h((\nabla_Y J)X,Z) \\ &\quad + h(Y,(\nabla_Z J)X) - h(Y,(\nabla_X J)Z) \\ &\quad + h\left(Y, T^{\nabla}(JX,Z) - JT^{\nabla}(X,Z)\right).\end{aligned}$$

Proof. Using the definition of the Tachibana operator Φ_J, we have

$$\begin{aligned}(\Phi_J h)(X,Y,Z) &= JXh(Y,Z) - Xh(JY,Z) \\ &\quad + h((L_Y J)X,Z) + h(Y,(L_Z J)X),\end{aligned}$$

where $(L_X J)Y = [X, JY] - J[X,Y]$. Then, we obtain

$$\begin{aligned}&(\Phi_J h)(X,Y,Z) \\ = \;&(\nabla_{JX} h)(Y,Z) - (\nabla_X h)(Y,JZ) + h\left(T^{\nabla}(JX,Y),Z\right) \\ &- h\left(T^{\nabla}(X,Y),JZ\right) + h((\nabla_Y J)X,Z) + h(Y,(\nabla_Z J)X) \\ &- h(Y,(\nabla_X J)Z) + h\left(Y, T^{\nabla}(JX,Z) - JT^{\nabla}(X,Z)\right).\end{aligned}$$

Since (∇, h) is a quasi-statistical structure, it follows that

$$\begin{aligned}(\Phi_J h)(X,Y,Z) &= (\nabla_Y h)(JX,Z) - (\nabla_Y h)(X,JZ) \\ &\quad + h((\nabla_Y J)X,Z) + h(Y,(\nabla_Z J)X) \\ &\quad - h(Y,(\nabla_X J)Z) + h\left(Y, T^{\nabla}(JX,Z) - JT^{\nabla}(X,Z)\right).\end{aligned}$$

□

Let us sign that $h\left(T^{\nabla}(JX,Y),Z\right) = T^{\nabla}(JX,Y,Z)$ and $h((\nabla_X J)Y,Z) = B(X,Y,Z)$. Hence, we say that if $T^{\nabla}(JX,Z,Y) = -B(Y,Z,X)$, then we have $T^{\nabla}(JX,Y) = -T^{\nabla}(X,JY)$. Hence, we are ready to give the second main theorem of this paper.

Theorem 17. *Let (M, h, J) be an almost anti-Hermitian manifold and ∇ be a linear connection with torsion tensor T^{∇} on M. Suppose that $d^{\nabla}J = 0$ and $d^{\nabla}h = 0$. Then, the triple (M, h, J) is an anti-Kähler manifold if and only if the condition $T^{\nabla}(JX,Z,Y) = -B(Y,Z,X)$ for any vector fields X, Y, Z on M holds.*

Proof. From the Proposition 16, if (∇, h) is a quasi-statistical structure, then we get

$$\begin{aligned}(\Phi_J h)(X,Y,Z) &= (\nabla_Y h)(JX,Z) - (\nabla_Y h)(X,JZ) + h((\nabla_Y J)X,Z) \\ &\quad + h(Y,(\nabla_Z J)X) - h(Y,(\nabla_X J)Z) \\ &\quad + h\left(Y, T^{\nabla}(JX,Z) - JT^{\nabla}(X,Z)\right).\end{aligned}$$

Considering the condition $d^{\nabla}J=0$, we have

$$(\Phi_J h)(X,Y,Z)=T^{\nabla}(JX,Z,Y)+B(Y,Z,X),$$

from which we immediately say that the triple (M,h,J) is an anti-Kähler manifold if and only if the condition $T^{\nabla}(JX,Z,Y)=-B(Y,Z,X)$ holds. □

Remark 4. *The Theorem 17 says that for any quasi-statistical manifold (M,h,∇), if the almost complex structure J is d^{∇}-closed and the condition $T^{\nabla}(JX,Z,Y)+B(Y,Z,X)=0$ is satisfied, (M,h,∇,J) is an anti-Kähler manifold. By taking any linear connection ∇ with torsion tensor T^{∇} instead of Levi-Civita connection ∇^h of h or torsion-free linear connection, it is also possible to make a characterization for anti-Kähler manifolds.*

Let (M,h,J) be an almost anti-Hermitian manifold and ∇ be a linear connection with torsion tensor T^{∇} on M. If (∇,h) is a quasi-statistical structure, with help of Proposition 16 we obtain

$$\begin{aligned}&(\Phi_J h)(X,Y,Z)+(\Phi_J h)(Y,Z,X)+(\Phi_J h)(Z,X,Y)\\=\;&h\left(Y,T^{\nabla}(JX,Z)\right)+h\left(Z,T^{\nabla}(JY,X)\right)+h\left(X,T^{\nabla}(JZ,Y)\right)\\&+h(Y,(\nabla_Z J)X)+h(Z,(\nabla_X J)Y)+h(X,(\nabla_Y J)Z).\end{aligned}$$

Hence, the last equality and Theorem 12 give the following result.

Theorem 18. *Let (M,h,J) be an almost anti-Hermitian manifold and ∇ be a linear connection with torsion tensor T^{∇} on M. Under the assumption that (∇,h) is a quasi-statistical structure, the triple (M,h,J) is a quasi-Kähler-Norden manifold if and only if*

$$\begin{aligned}&h(Y,(\nabla_Z J)X)+h(Z,(\nabla_X J)Y)+h(X,(\nabla_Y J)Z)\\=\;&-\left(h\left(Y,T^{\nabla}(JX,Z)\right)+h\left(Z,T^{\nabla}(JY,X)\right)+h\left(X,T^{\nabla}(JZ,Y)\right)\right).\end{aligned}$$

5 Conclusion

In the realm of differential geometry, a Kähler manifold is a geometric structure characterized by a triple (M,g,J), where M represents the manifold itself, g is a pseudo-Riemannian metric, and J denotes an almost complex structure. A crucial property of Kähler manifolds is that the almost complex structure J must satisfy the condition of being parallel under the Levi-Civita connection ∇^g associated with the pseudo-Riemannian metric g.

In [9], a novel alternative characterization for Kähler manifolds is introduced by leveraging the Codazzi couplings of the linear connection ∇ (torsion-free) with both the pseudo-Riemannian metric g and the almost complex structure J. Notably, this characterization expands the understanding of Kähler manifolds beyond the traditional requirement of having a Levi-Civita connection. This expansion is made possible by considering any linear connection ∇, regardless of whether it possesses torsion.

In this paper, under the assumption that the linear connection ∇ with torsion tensor T^{∇} satisfies the conditions $d^{\nabla}g = 0$, $d^{\nabla}J = 0$ and $T^{\nabla}(JX, Y) = -T^{\nabla}(X, JY)$, it is proven that the almost complex structure J is integrable, and the Kähler form ω is closed. Consequently, the almost Hermitian manifold (M, g, J) rises to a Kähler manifold. This demonstration illustrates that torsion-free connections are not an absolute requirement for such characterizations.

Furthermore, the paper goes beyond Kähler manifolds and extends its findings to anti-Kähler manifolds. It reveals that, under certain conditions, anti-Kähler manifolds can also be characterized using any linear connection ∇ with torsion tensor T^{∇}, rather than being limited to the Levi-Civita connection ∇^{h} associated with a pseudo-Riemannian metric h or torsion-free linear connections. In essence, this paper opens up new avenues for characterizing Kähler, anti-Kähler, and quasi-Kähler-Norden manifolds, broadening our understanding of these geometric structures.

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