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Sweeping Surfaces of Polynomial Curves in Euclidean 3-space

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Abstract

In this study, we investigate the surfaces created by the movement of the profile curves through the regular polynomial spine curves. To overcome the restrictions of establishing a frame of the polynomial curves at the points where the second and higher-order derivatives vanish, the Frenet-like curve (Flc) frame is considered. In this way, by introducing sweeping surfaces defined based on the Flc frame, we analyze their parameter curves to determine conditions to be geodesics, asymptotics, and principal curvature lines. Furthermore, we derive conditions of these sweeping surfaces to be minimal, developable, and Weingarten surfaces. Lastly, we provide some examples of these sweeping surfaces and illustrate their graphical representations.

1 Introduction

Sweeping surfaces are the result of the act of forming complex shapes by sweeping planar curves along any path in 3-dimensional space. Sweeping is a frequently referenced technique that is extensively utilized, especially when creating objects with precise forms in computer-aided design or 3D modeling software. The essence of this technique lies in selecting a geometric object known as a generator and sweeping it along a specified curve, referred to as the spine curve. In substance, a sweeping surface is formed by a plane curve,

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commonly referred to as a profile curve or generatrix, that moves continuously along the plane's normal vector in the same direction. The most popular varieties of sweeping surfaces include the strings, pipe, canal, and tube surfaces. The fundamental mathematical characteristics of a particular kind of sweeping surface called canal surfaces were examined in [1]. Additionally, the tubes that satisfy some specific equations in terms of the curvatures of the surfaces were researched by [2]. The sweeping surfaces produced by both the rotation-minimizing frames and the Darboux frames were researched in [3] and [4], respectively. Furthermore, in Minkowski 3-space, the timelike sweeping surfaces with the Bishop frames and their singularity properties were surveyed in [5]. Köseoğlu and Bilici defined involutive sweeping surfaces as a new surface form and explored their singularity [6]. The characteristic properties and singularities of these types of surfaces have attracted the attention of various researchers [7, 8, 9, 10, 11, 12, 13, 14, 15]. As well as the theoretical findings on sweeping surfaces, there are also continuing improvements in applications based on the modeling of these surfaces from the viewpoint of mechanical design or computer-aided geometric design [16, 17, 18, 19].

As a novelty, our investigation focuses on sweeping surfaces formed by the motion of a regular polynomial curve with its well-defined Flc frame. The main reason for this is that this frame is an alternative to the Frenet frame to get over constraints at the locations where a curve's second and higher-order derivatives vanish. The Flc frame for polynomial curves in motion was first presented to solve this problem in [20, 21]. Numerous researchers explored studies on curves [22, 23, 24, 25] and surfaces [26, 27] using the Flc frame of polynomial curves from different aspects.

In this study, we explore sweeping surfaces with the Frenet-like curve frame of regular polynomial spine curves in Euclidean 3-space. The new equations of the parameter curves of these sweeping surfaces allow us to determine conditions for them to be geodesics, asymptotics, and principal curvature lines. Furthermore, we derive conditions for these sweeping surfaces to be minimal, developable, and Weingarten surfaces. Finally, some sweeping surfaces generated by the Flc frame elements are modeled as examples, and their graphical representations are detailed with their spine curves and the Flc frame elements.

2 Preliminaries

Let $\sigma = \sigma(t)$ be a regular n. order polynomial space curve where $n \geq 2$ in Euclidean 3-space. The elements of the Flc frame along the curve σ are presented by

$$T(t) = \frac{\sigma'(t)}{\|\sigma'(t)\|},\ D_1(t) = \frac{\sigma'(t) \times \sigma^{(n)}(t)}{\left\|\sigma'(t) \times \sigma^{(n)}(t)\right\|},\ D_2(t) = D_1(t) \times T(t),$$

respectively, where $'$ and $^{(n)}$ denote the first and the n. order derivatives of the curve in terms of t. It should be noted that the Flc frame and the Frenet frame are obviously coincident when $n = 2$ [21]. In the case of $n \geq 3$, even if the second and higher-order derivatives of the regular polynomial space curve are zero, and it is not possible to construct a well-defined Frenet frame, this approach enables us to establish a frame with new Frenet-like vectors D_1 and D_2 called binormal-like and normal-like vectors, respectively. Moreover, the curvatures of the Flc frame d_1, d_2, and d_3 are given by

$$d_1 = \frac{\langle T', D_2 \rangle}{\nu},\ d_2 = \frac{\langle T', D_1 \rangle}{\nu},\ d_3 = \frac{\left\langle {D_2}', D_1 \right\rangle}{\nu},$$

where $\|\sigma'\| = \nu$. The derivatives of the Frenet-like curve elements are given the following matrix form:

$$\begin{bmatrix} T' \\ D'_2 \\ D'_1 \end{bmatrix} = \nu \begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix},$$

see [20, 21, 22] for more details.

Now, let us review some well-known basic ideas in differential geometry that are applied to the study of surfaces in order to explain various aspects of their curvature behaviors. Let $\Psi(t,u)$ denote a surface and Ψ_t and Ψ_u be tangent vectors of it, then the equation of the normal vector field of the surface is

$$U(t,u) = \frac{\Psi_t \times \Psi_u}{\|\Psi_t \times \Psi_u\|}, \tag{2.1}$$

where $\Psi_t = \frac{\partial \Psi}{\partial t}$ and $\Psi_u = \frac{\partial \Psi}{\partial u}$. The first and second order fundamental magnitudes of the surface $\Psi(t,u)$ are, respectively, given by

$$E = \left\langle \frac{\partial \Psi}{\partial t}, \frac{\partial \Psi}{\partial t} \right\rangle,\ F = \left\langle \frac{\partial \Psi}{\partial t}, \frac{\partial \Psi}{\partial u} \right\rangle,\ G = \left\langle \frac{\partial \Psi}{\partial u}, \frac{\partial \Psi}{\partial u} \right\rangle \tag{2.2}$$

and

$$k = \left\langle \frac{\partial^2 \Psi}{\partial t^2}, U \right\rangle,\ l = \left\langle \frac{\partial^2 \Psi}{\partial t \partial u}, U \right\rangle,\ m = \left\langle \frac{\partial^2 \Psi}{\partial u^2}, U \right\rangle. \tag{2.3}$$

Based on these functions, the Gaussian and mean curvatures of a surface are formulated as:

$$K = \frac{km - l^2}{EG - F^2} \text{ and } H = \frac{1}{2} \frac{Em - 2El + Gk}{EG - F^2}, \tag{2.4}$$

respectively. It is well-known that, a surface $\Psi(t,u)$ refers to

i. developable surface iff the Gaussian curvature vanishes,

ii. minimal surface iff the mean curvature vanishes,

iii. Weingarten surface iff $K_t H_u - K_u H_t = 0$,

at each point of the surface.

3 Sweeping Surfaces Generated by Regular Polynomial Spine Curves

In this section, we introduce the parametric expression for a sweeping surface along a regular polynomial spine curve $\sigma(t)$ with its well-defined Flc frame. A sweeping surface linked with $\sigma(t)$ constitutes the envelope of a one-parameter set of unit spheres, each with its center residing on the curve $\sigma(t)$. The intersection between the spheres from this collection and the sweeping surface occurs precisely at the great circle of the unit sphere, positioned within the subspace spanned by Frenet-like vectors $\{D_2, D_1\}$ of the regular polynomial spine curve $\sigma(t)$. Now, let us outline a simple method for illustrating the sweeping surface. Choose the parameter along $\sigma(t)$ as one of the variables and establish the position vector Ψ, connecting a point on the curve $\sigma(t)$ to another point on the sweeping surface. The parameterization of a sweeping surface formed by a regular polynomial spine curve $\sigma(t)$ and a planar profile (cross-section) curve $r(u) = (0, \eta(u), \mu(u))^t$, where the symbol ^{t} represents the transpose of the vector, is

$$M : \Psi(t,u) = \sigma(t) + r(u)\Gamma(t) = \sigma(t) + \eta(u) D_2(t) + \mu(u) D_1(t), \quad (3.1)$$

where $\Gamma(t)$ is a 3×3 orthogonal matrix along $\sigma(t)$ such as

$$\Gamma(t) = \begin{pmatrix} T(t) & D_2(t) & D_1(t) \end{pmatrix}.$$

The partial derivatives of the sweeping surface $\Psi(t,u)$ using the Flc frame with respect to t and u are found

$$\Psi_t = -\nu(-1 + \eta d_1 + \mu d_2) T - \nu d_3 \mu D_2 + \nu d_3 \eta D_1$$

and

$$\Psi_u = D_2 \eta_u + D_1 \mu_u,$$

respectively. Thus, by a straightforward computation from the last equations and Eq. (2.1), the normal vector field of the sweeping surface is obtained as follows:

$$U(t,u) = \pm \frac{\mu_u D_2 - \eta_u D_1}{\sqrt{{\eta_u}^2 + {\mu_u}^2}}.$$

Without loss of generality, the positive normal vector is taken into consideration in the operations of this study.

Theorem 3.1. *Let $\Psi(t,u)$ be a sweeping surface constructed by a regular polynomial spine curve with the Flc frame, then the Gaussian and mean curvatures of the sweeping surfaces are*

$$K(t,u)=\frac{\omega\eta_u\left(d_2\varepsilon+\mu {d_3}^2\right)-\omega\mu_u\left(d_1\varepsilon+\eta {d_3}^2\right)-{d_3}^2\delta^2}{\delta^2\left(\varepsilon^2+{d_3}^2\right)-{d_3}^2\rho^2\delta}$$

and

$$H(t,u)=\frac{\delta\eta_u\left(d_2\varepsilon+\mu {d_3}^2\right)-\delta\mu_u\left(d_1\varepsilon+\eta {d_3}^2\right)+\omega\left(\varepsilon^2+{d_3}^2\right)-2{d_3}^2\rho\delta}{2\sqrt{\delta}\left(-{d_3}^2\rho^2+\delta\left(\varepsilon^2+{d_3}^2\right)\right)}$$

where

$$\delta(u)={\eta_u}^2+{\mu_u}^2\neq 0,\quad \varepsilon(t,u)=\eta d_1+\mu d_2-1,$$
$$\rho(u)=\mu\eta_u-\eta\mu_u,\quad \omega(u)=\mu_u\eta_{uu}-\eta_u\mu_{uu}$$

and

$$\delta\left(\varepsilon^2+{d_3}^2\right)-{d_3}^2\rho^2\neq 0.$$

Proof. Let $\Psi(t,u)$ be a sweeping surface constructed by a regular polynomial spine curve with the Flc frame. From Eqs. (2.2) and (2.3), the coefficients of the first and second fundamental quadratic forms of the sweeping surfaces are

$$\begin{cases} E(t,u)=\nu^2\left(\varepsilon^2+{d_3}^2\right), \\ F(t,u)=-\nu d_3\rho, \\ G(t,u)=\delta, \end{cases}$$

and

$$\begin{cases} k(t,u)=\dfrac{\nu^2\eta_u\left(d_2\varepsilon+\mu {d_3}^2\right)-\nu^2\mu_u\left(d_1\varepsilon+\eta {d_3}^2\right)}{\sqrt{\delta}}, \\ l(t,u)=-\nu d_3\sqrt{\delta}, \\ m(t,u)=\dfrac{\omega}{\sqrt{\delta}}, \end{cases}$$

respectively, such that

$$\begin{aligned} \Psi_{tt}&=\left(\nu^2 d_3\left(\mu d_1-\eta d_2\right)-\nu\left(\eta {d_1}'+\mu {d_2}'\right)-\varepsilon\nu'\right)T \\ &\quad-\left(\nu^2\left(d_1\varepsilon+\eta {d_3}^2\right)+\mu(d_3\nu)'\right)D_2 \\ &\quad-\left(\nu^2\left(\varepsilon d_2+\mu {d_3}^2\right)-\eta(d_3\nu)'\right)D_1, \\ \Psi_{tu}&=-\nu\left(d_1\eta_u+d_2\mu_u\right)T-\nu d_3\mu_u D_2+\nu d_3\eta_u D_1, \\ \Psi_{uu}&=D_2\eta_{uu}+D_1\mu_{uu}, \end{aligned} \tag{3.2}$$

for $\delta(u) = {\eta_u}^2 + {\mu_u}^2,\ \varepsilon(t,u) = \eta d_1 + \mu d_2 - 1,\ \rho(u) = \mu\eta_u - \eta\mu_u,\ \omega(u) = \mu_u\eta_{uu} - \eta_u\mu_{uu}$. If the first and second-order fundamental magnitudes of the sweeping surface are substituted in Eq. (2.4), the Gaussian and mean curvatures of the sweeping surface are found as in the hypothesis. □

Corollary 1. *Let $\Psi(t,u)$ be a sweeping surface constructed by a regular polynomial spine curve with the Flc frame, then the sweeping surface is*

i. developable surface if and only if

$$\omega\eta_u\left(d_2\varepsilon + \mu {d_3}^2\right) = \omega\mu_u\left(d_1\varepsilon + \eta {d_3}^2\right) + {d_3}^2\delta^2,$$

ii. minimal surface if and only if

$$\delta\eta_u\left(d_2\varepsilon + \mu {d_3}^2\right) + \omega\left(\varepsilon^2 + {d_3}^2\right) = 2{d_3}^2\rho\delta + \delta\mu_u\left(d_1\varepsilon + \eta {d_3}^2\right).$$

Theorem 3.2. *Let $\Psi(t,u)$ be a sweeping surface constructed by a regular polynomial spine curve with the Flc frame, then the $t-$parameter curves of the sweeping are*

i. geodesic curves if and only if

$$\rho(d_3\nu)' = \nu^2\eta_u\left(d_1\varepsilon + \eta {d_3}^2\right) + \nu^2\mu_u\left(d_2\varepsilon + \mu {d_3}^2\right)$$

and

$$\nu'\varepsilon = \nu^2 d_3\left(\mu d_1 - \eta d_2\right) - \nu\left(\eta {d_1}' + \mu {d_2}'\right),$$

ii. asymptotic curves if and only if

$$\nu^2\left(\eta_u\left(d_2\varepsilon + \mu {d_3}^2\right) - \mu_u\left(d_1\varepsilon + \eta {d_3}^2\right)\right) = \left(\eta\eta_u + \mu\mu_u\right)(d_3\nu)'$$

for $\delta \neq 0$.

Proof. To ensure that the parameter curves meet the criteria for being geodesic curves, it is a requirement for the acceleration vector of these curves to be perpendicular to the surface, thereby parallel to the surface's normal vector. Moreover, for the parameter curves to be classified as asymptotic, the inner product of the velocity vectors and the normal vector of the surface must be zero. From Eq. (3.2) and the normal vector of the sweeping, the following equations are obtained;

$$\begin{aligned} U \times \Psi_{tt} = &\frac{1}{\sqrt{\delta}}\left(d_3\nu'\rho + \nu^2\left(\left(d_1\varepsilon + \eta {d_3}^2\right)\eta_u + \left(d_2\varepsilon + \mu {d_3}^2\right)\mu_u\right) + \nu\rho {d_3}'\right)T \\ &+\frac{\eta_u}{\sqrt{\delta}}\left(\nu^2\left(\mu d_1 - \eta d_2\right)d_3 - \varepsilon\nu' - \nu\left(\eta {d_1}' + \mu {d_2}'\right)\right)D_2 \\ &+\frac{\mu_u}{\sqrt{\delta}}\left(\nu^2\left(\mu d_1 - \eta d_2\right)d_3 - \varepsilon\nu' - \nu\left(\eta {d_1}' + \mu {d_2}'\right)\right)D_1, \end{aligned}$$

where $\delta \neq 0$. The linear independence of the vector fields T, D_2, and D_1 gives us

$$\begin{aligned} & d_3\nu'\rho + \nu^2\left(\left(d_1\varepsilon + \eta {d_3}^2\right)\eta_u + \left(d_2\varepsilon + \mu {d_3}^2\right)\mu_u\right) + \nu\rho {d_3}' = 0, \\ & \eta_u\left(\nu^2\left(\mu d_1 - \eta d_2\right)d_3 - \varepsilon\nu' - \nu\left(\eta {d_1}' + \mu {d_2}'\right)\right) = 0, \\ & \mu_u\left(\nu^2\left(\mu d_1 - \eta d_2\right)d_3 - \varepsilon\nu' - \nu\left(\eta {d_1}' + \mu {d_2}'\right)\right) = 0, \end{aligned}$$

and

$$\langle \Psi_{tt}, U\rangle = \frac{1}{\sqrt{\delta}}\left(\nu^2\left(\eta_u\left(d_2\varepsilon + \mu {d_3}^2\right) - \mu_u\left(d_1\varepsilon + \eta {d_3}^2\right)\right) - \left(\eta\eta_u + \mu\eta_u\right)\left(d_3\nu\right)'\right).$$

i. For $U \times \Psi_{tt} = 0$, we can easily see that the $t-$parameter curves are geodesic under the conditions stated in the hypothesis.

ii. Obviously, $\nu^2\left(\eta_u\left(d_2\varepsilon + \mu {d_3}^2\right) - \mu_u\left(d_1\varepsilon + \eta {d_3}^2\right)\right) = \left(\eta\eta_u + \mu\mu_u\right)\left(d_3\nu\right)'$ for $\delta \neq 0$ iff $\langle \Psi_{tt}, U\rangle = 0$. Thus, we can say that the $t-$parameter curves are asymptotic curves under the conditions stated in the hypothesis.

□

Theorem 3.3. *Let $\Psi(t,u)$ be a sweeping surface constructed by a regular polynomial spine curve with the Flc frame, then the $u-$parameter curves of the sweeping are*

i. geodesic curves if and only if $\eta_u\eta_{uu} + \mu_u\mu_{uu} = 0$,

ii. not asymptotic curves if and only if $\omega = 0$.

Proof. From Eq. (3.2) and the normal vector of the sweeping surface, the following equations are obtained;

i.
$$\Psi_{uu} \times U = -\frac{\eta_u\eta_{uu} + \mu_u\mu_{uu}}{\sqrt{\delta}}T.$$

Thus, if $\eta_u\eta_{uu} + \mu_u\mu_{uu} = 0$, then $\Psi_{uu} \times U = 0$, that is, the $u-$parameter curves of the sweeping surface are geodesic curves under the conditions stated in the hypothesis.

ii.
$$\langle \Psi_{uu}, U\rangle = \frac{\omega}{\sqrt{\delta}}.$$

Thus, if $\omega = 0$, then $\langle \Psi_{uu}, U\rangle = 0$. Therefore, the $u-$parameter curves of the sweeping surface cannot be asymptotic under the conditions stated in the hypothesis.

□

Theorem 3.4. *Let $\Psi(t,u)$ be a sweeping surface constructed by a regular polynomial spine curve with the Flc frame, then the t and $u-$parameter curves of the sweeping surface are principal curvature lines if and only if $d_3 = 0$.*

Proof. Let $\Psi(t,u)$ be a sweeping surface constructed by a regular polynomial spine curve with a Flc frame. If the parameter curves of the sweeping surface are principal curvature lines, then $F = l = 0$. From the equations $F(t,u) = -\nu d_3 \rho$ and $l(t,u) = -\nu d_3 \sqrt{\delta}$, we have $d_3 = 0$ for the conditions $F = l = 0$. So, the t and $u-$parameter curves of the sweeping surface are principal curvature lines. □

Now let's take the curve $r(u) = (0, \cos u, \sin u)$ as the planar profile (cross-section) curve. The unit vector γ, where u is the angle between γ and D_2, lies in the subspace $sp\{D_2, D_1\}$. Also, the unit vector γ is perpendicular to the tangent vector T. Therefore, we can write

$$\gamma(t) = \cos u D_2(t) + \sin u D_1(t), \tag{3.3}$$

which is the characteristic circle of $\Psi(t,u)$. Considering Eq. (3.1) and (3.3), we can write the parametric equation of the sweeping surface as follows:

$$\Psi(t,u) = \sigma(t) + \cos u D_2(t) + \sin u D_1(t). \tag{3.4}$$

Considering Eq. (3.4), the above expressions are easily investigated in a similar manner. The sweeping surface is then characterized without proof. The partial derivatives of the the sweeping surface $\Psi(t,u)$ using the Flc frame with respect to t and u are satisfied

$$\Psi_t = -\nu(-1 + \cos u d_1 + \sin u d_2) T - \nu d_3 \sin u D_2 + \nu d_3 \cos u D_1$$

and

$$\Psi_u = \cos u D_1 - \sin u D_2.$$

Thus, by a straightforward computation from the last equations and Eq. (2.1), the normal vector field of the sweeping surface is obtained as follows:

$$U(t,u) = \pm(\cos u D_2 + \sin u D_1)$$

also, the positive normal vector is taken into consideration in the operations of this study.

Theorem 3.5. *Let $\Psi(t,u)$ be a sweeping surface constructed by a regular polynomial spine curve with the Flc frame, then the Gaussian and mean curvatures of the sweeping surfaces are*

$$K(t,u) = \frac{\cos u d_1 + \sin u d_2}{\cos u d_1 + \sin u d_2 - 1}$$

and

$$H(t,u)=\frac{2\cos ud_1+2\sin ud_2-1}{2\left(1-\cos ud_1-\sin ud_2\right)},$$

where $\cos ud_1+\sin ud_2\neq 1$.

Corollary 2. *Let* $\Psi(t,u)$ *be a sweeping surface constructed by regular polynomial spine curve with the Flc frame, then the sweeping surface is*

i. developable surface if and only if $u=\arctan\left(-\frac{d_1}{d_2}\right)$,

ii. minimal surface if and only if $u=\mp\arccos\left(\frac{d_1\mp\sqrt{-d_2^2+4d_1^2d_2^2+4d_2^4}}{2\left(d_1^2+d_2^2\right)}\right)$.

Theorem 3.6. *Let* $\Psi(t,u)$ *be a sweeping surface constructed by a regular polynomial spine curve with the Flc frame, then the sweeping surface is a Weingarten surface.*

Proof. Let $\Psi(t,u)$ be a sweeping surface constructed by a regular polynomial spine curve with a Flc frame. If the partial differentiation of the equations of the Gaussian and mean curvatures of the sweeping surface given by Theorem 3.5 in terms of t and u, we get

$$\begin{cases} K_t=-\dfrac{\cos u{d_1}'+\sin u{d_2}'}{\left(-1+\cos ud_1+\sin ud_2\right)^2}, \\ K_u=\dfrac{\csc u\left(d_1-\cot ud_2\right)}{\left(-\csc u+\cot ud_1+d_2\right)^2} \end{cases}$$

and

$$\begin{cases} H_t=\dfrac{\cos u{d_1}'+\sin u{d_2}'}{2(-1+\cos ud_1+\sin ud_2)^2}, \\ H_u=\dfrac{-\sin ud_1+\cos ud_2}{2(-1+\cos ud_1+\sin ud_2)^2}. \end{cases}$$

Thus, we find $K_tH_u-K_uH_t=0$. So, we can say that the sweeping surface is a Weingarten surface. □

Theorem 3.7. *Let* $\Psi(t,u)$ *be a sweeping surface constructed by regular polynomial spine curve with the Flc frame, then the* $t-$*parameter curves of the sweeping are*

i. geodesic curves if and only if

$$\nu'=\frac{\nu^2\left(-\sin ud_1+\cos ud_2\right)d_3+\nu\left(\cos u{d_1}'+\sin u{d_2}'\right)}{\left(1-\cos ud_1-\sin ud_2\right)}$$

and

$$(d_3\nu)' = \nu^2\left(\sin u d_1 - \cos u d_2 + \cos 2u d_2 d_1 - \frac{\sin 2u}{2}\right),$$

ii. asymptotic curves if and only if

$$u = \arctan\left(-\frac{d_1}{d_2}\right) \text{ and } d_3 = 0.$$

Theorem 3.8. *Let $\Psi(t,u)$ be a sweeping surface constructed by a regular polynomial spine curve with the Flc frame, then the $u-$parameter curves of the sweeping are*

i. geodesic curves,

ii. not asymptotic curves.

Theorem 3.9. *Let $\Psi(t,u)$ be a sweeping surface constructed by a regular polynomial spine curve with the Flc frame, then the t and $u-$parameter curves of the sweeping surface are principal curvature lines if and only if $d_3 = 0$.*

4 Modeling Examples of Sweeping Surfaces with Regular Polynomial Spine Curves

Example 4.1. *Let us consider a 4^{rd} degree polynomial curve $\sigma(t)$ represented by*

$$\sigma(t) = \left(t, t, t^4\right),$$

see Figure 1. It is obvious that it is impossible to construct the Frenet frame at $t = 0$. Thus, applying to the Frenet-like curve apparatus $\{T, D_2, D_1, d_1, d_2, d_3\}$ of this polynomial curve is a necessity. They are found as follows:

$$T = \left(\frac{1}{\sqrt{2+16t^6}}, \frac{1}{\sqrt{2+16t^6}}, \frac{4t^3}{\sqrt{2+16t^6}}\right),$$

$$D_2 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0\right),$$

$$D_1 = \left(-\frac{2\sqrt{2}t^3}{\sqrt{2+16t^6}}, -\frac{2\sqrt{2}t^3}{\sqrt{2+16t^6}}, \frac{\sqrt{2}}{\sqrt{2+16t^6}}\right)$$

and

$$d_1 = 1,\ d_2 = 0,\ d_3 = 0.$$

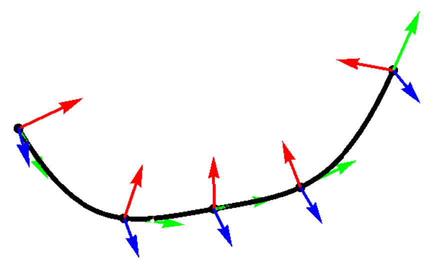

Figure 1: The polynomial spine curve $\sigma(t)$, and its tangent vectors T (green), normal-like vectors D_2 (red), and binormal-like vectors D_1 (blue) for $t \in (-1, 1)$.

Considering the planar profile curve $r(u) = (0, \cos u, \sin u)$, the equation of the sweeping surface with respect to the Flc frame defined by Eq. (3.4) takes the form:

$$\Psi(t,u) = \left(t - \frac{2t^3 \cos u}{\sqrt{1+8t^6}} + \frac{\sin u}{\sqrt{2}}, t - \frac{2t^3 \cos u}{\sqrt{1+8t^6}} - \frac{\sin u}{\sqrt{2}}, t^4 + \frac{\cos u}{\sqrt{1+8t^6}} \right).$$

see Figure 2. The partial derivatives of $\Psi(t,u)$ in terms of t and u are

$$\begin{cases} \Psi_t = \left(1 - \dfrac{6t^2 \cos u}{(1+8t^6)^{3/2}}, 1 - \dfrac{6t^2 \cos u}{(1+8t^6)^{3/2}}, 4t^3 - \dfrac{24t^5 \cos u}{(1+8t^6)^{3/2}} \right), \\ \Psi_u = \left(\dfrac{\cos u}{\sqrt{2}} + \dfrac{2t^3 \sin u}{\sqrt{1+8t^6}}, -\dfrac{\cos u}{\sqrt{2}} + \dfrac{2t^3 \sin u}{\sqrt{1+8t^6}}, -\dfrac{\sin u}{\sqrt{1+8t^6}} \right). \end{cases}$$

Substituting the above equations Ψ_t and Ψ_u into the equation (2.1), the normal vector field of $\Psi(t,u)$ is obtained as follows:

$$U(t,u) = \rho \left(2\sqrt{2}t^3 \cos u - \sqrt{1+8t^6} \sin u, 2\sqrt{2}t^3 \cos u - \sqrt{1+8t^6} \sin u, -\cos u \right)$$

where

$$\rho = \frac{\left(1+8t^6\right)^{3/2} - 6t^2 \cos u}{\sqrt{(1+8t^6)\left((1+8t^6)^3 - 12t^2(1+8t^6)^{3/2} \cos u + 36t^4 \cos^2 u\right)}}$$

and

$$\left(1+8t^6\right)^3 - 12t^2\left(1+8t^6\right)^{3/2} \cos u + 36t^4 \cos^2 u \neq 0.$$

The first fundamental quadratic form I *of* $\Psi(t,u)$ *is defined by*

$$\mathrm{I} = Edt^2 + 2Fdtdu + Gdu^2$$

such that

$$E = \frac{2\left(\left(1+8t^6\right)^3 - 12t^2\left(1+8t^6\right)^{3/2}\cos u + 36t^4\cot^2 u\right)}{\left(1+8t^6\right)^2},$$
$$F = 0,\ G = 1.$$

On the other hand, the second fundamental quadratic form II *of* $\Psi(t,u)$ *is defined by*

$$\mathrm{II} = kdt^2 + 2ldtdu + mdu^2$$

where

$$k = -\frac{12t^2\cos u\sqrt{1+8t^6\left(3+8t^6\left(3+8t^6\right)\right) - 12t^2\cos u\left(\left(1+8t^6\right)^{3/2} - 3t^2\cos u\right)}}{\left(1+8t^6\right)^2},$$
$$l = 0,\ m = \frac{\left(1+8t^6\right)^{3/2} - 6t^2\cos u}{\sqrt{\left(1+8t^6\right)^3 - 12t^2\left(1+8t^6\right)^{3/2}\cos u + 36t^4\cot^2 u}}.$$

The Gaussian and mean curvatures of $\Psi(t,u)$ *are*

$$K(t,u) = -\frac{6t^2\cos u}{\left(1+8t^6\right)^{3/2} - 6t^2\cos u}$$

and

$$H(t,u) = \frac{\left(1+8t^6\right)^{3/2} - 12t^2\cos u}{2\sqrt{\left(1+8t^6\right)^3 - 12t^2\left(1+8t^6\right)^{3/2}\cos u + 36t^4\cot^2 u}},$$

where $\left(1+8t^6\right)^3 - 12t^2\left(1+8t^6\right)^{3/2}\cos u + 36t^4\cos u^2 \neq 0$ *and* $\left(1+8t^6\right)^{3/2} - 6t^2\cos u \neq 0$.

The necessary and sufficient condition for this sweeping surface to be a developable surface is $\cos u = 0$, *i.e.,* $u = \mp\frac{\pi}{2}$. *However, in that case the profile curve degenerates to* $(0,0,\mp 1)$, *and then* $\Psi(t,u)$ *degenerates to the curve* $\left\{t+\frac{1}{\sqrt{2}}, t-\frac{1}{\sqrt{2}}, t^4\right\}$.

Secondly, the necessary and sufficient condition for this sweeping surface $\Psi(t,u)$ *to be a minimal surface is* $\left(1+8t^6\right)^{3/2} = 12t^2\cos u$. *In other words, the condition to be minimal is*

$$u = \mp\arccos\left(\frac{\sqrt{1+8t^6} + 8t^6\sqrt{1+8t^6}}{12t^2}\right) \quad \text{and} \quad t \neq 0.$$

The partial derivatives of the Gaussian and mean curvatures of this surface in terms of t and u are

$$\begin{cases} K_t = \dfrac{12t\sqrt{1+8t^6}\left(-1+28t^6\right)\cos u}{\left((1+8t^6)^{3/2}-6t^2\cos u\right)^2}, \\ K_u = \dfrac{6t^2\left(1+8t^6\right)^{3/2}\sin u}{\left((1+8t^6)^{3/2}-6t^2\cos u\right)^2} \end{cases}$$

and

$$\begin{cases} H_t = \dfrac{6t\left(-1+28t^6\right)\cos u\left(\left(1+8t^6\right)^2-6t^2\sqrt{1+8t^6}\cos u\right)}{\left((1+8t^6)^3-12t^2(1+8t^6)^{3/2}\cos u+36t^4\cot^2 u\right)^{3/2}}, \\ H_u = \dfrac{3t^2\left(1+8t^6\right)\left(\left(1+8t^6\right)^2-6t^2\sqrt{1+8t^6}\cos u\right)\sin u}{\left((1+8t^6)^3-12t^2(1+8t^6)^{3/2}\cos u+36t^4\cot^2 u\right)^{3/2}}. \end{cases}$$

Thus, we get $K_tH_u - K_uH_t = 0$. So, we can say that $\Psi(t,u)$ is a Weingarten surface.

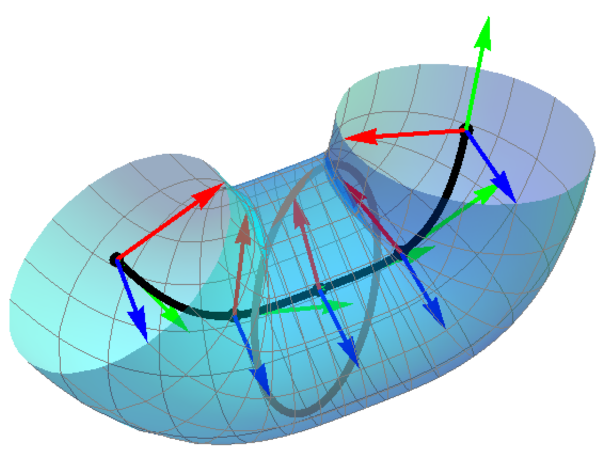

Figure 2: The sweeping surface $\Psi(t,u)$ with polynomial spine curve (black), and the planar profile curve (gray) for $t \in (-1,1)$ and $u \in (-\pi,-\pi)$.

Example 4.2. *Let us consider a* 3^{rd} *degree polynomial curve* $\sigma(t)$ *represented by*

$$\sigma(t) = \left(t, \frac{t^2}{2}, \frac{t^3}{6}\right),$$

see Figure 3 and the Frenet-like curve apparatus $\{T, D_2, D_1, d_1, d_2, d_3\}$ *of this polynomial curve is found as follows:*

$$\begin{aligned} T &= \left(\frac{2}{2+t^2}, \frac{2t}{2+t^2}, \frac{t^2}{2+t^2}\right), \\ D_2 &= \left(-\frac{t^2}{\sqrt{1+t^2}\,(2+t^2)}, -\frac{t^3}{\sqrt{1+t^2}\,(2+t^2)}, \frac{2\sqrt{1+t^2}}{2+t^2}\right), \\ D_1 &= \left(\frac{s}{\sqrt{1+t^2}}, -\frac{1}{\sqrt{1+t^2}}, 0\right) \end{aligned}$$

and

$$d_1 = \frac{4s}{\sqrt{1+t^2}(2+t^2)^2},\ d_2 = -\frac{4}{\sqrt{1+t^2}(2+t^2)^2},\ d_3 = \frac{2t^2}{(1+t^2)\,(2+t^2)^2}.$$

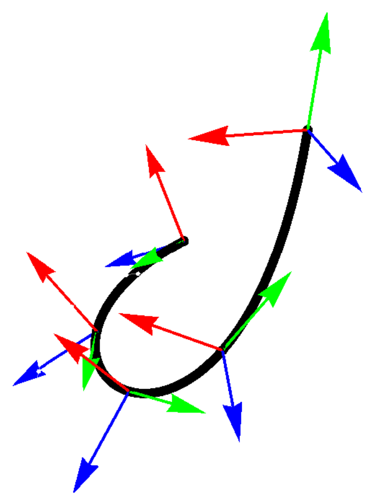

Figure 3: The polynomial spine curve $\sigma(t)$, and its tangent vectors T (green), normal-like vectors D_2 (red), and binormal-like vectors D_1 (blue) for $t \in (-2, 2)$.

For the planar profile (cross-section) curve $r(u) = (0, \cos u, \sin u)$, *the equation of the sweeping surface defined by Eq. (3.4) based on the Flc frame, takes the following form:*

$$\Psi(t,u) = \begin{pmatrix} t\left(1 + \dfrac{-t\cos u + \left(2+t^2\right)\sin u}{\sqrt{1+t^2}\left(2+t^2\right)}\right), \dfrac{t^2}{2} - \dfrac{t^3\cos u + \left(2+t^2\right)\sin u}{\sqrt{1+t^2}\left(2+t^2\right)}, \\ \dfrac{t^3}{6} + \dfrac{2\sqrt{1+t^2}\cos u}{2+t^2} \end{pmatrix},$$

see Figure 4.
By straightforward computations, the normal vector field of the sweeping surface is obtained as follows:

$$U(t,u) = \left(-\frac{t^2\cos u + t\sin u}{\sqrt{1+t^2}\left(2+t^2\right)}, -\frac{t^3\cos u - \sin u}{\sqrt{1+t^2}\left(2+t^2\right)}, \frac{2\sqrt{1+t^2}\cos u}{2+t^2}\right).$$

The Gaussian and mean curvatures of the sweeping surfaces given by Eq. (3.4) are obtained as

$$K(t,u) = \frac{4\left(t\cos u - \sin u\right)}{4t\cos u - 4\sin u - \sqrt{1+t^2}\left(2+t^2\right)^2}$$

and

$$H(t,u) = \frac{8t\cos u - 8\sin u - \sqrt{1+t^2}\left(2+t^2\right)^2}{2\left(4\sin u - 4t\cos u + \sqrt{1+t^2}\left(2+t^2\right)^2\right)},$$

where $4\sin u - 4t\cos u + \sqrt{1+t^2}\left(2+t^2\right)^2 \neq 0$.
So, the $\Psi(t,u)$ *is*

i. a developable surface if and only if $u = \arctan t$,

ii. a minimal surface if and only if $8t\cos u - 8\sin u = \sqrt{1+t^2}\left(2+t^2\right)^2$.

On the other hand, the partial differentiation of the equations of the Gaussian and mean curvatures of this sweeping surface in terms of t *and* u *are found*

$$\begin{cases} K_t = \dfrac{\left(2+t^2\right)\left(4\left(-2+3t^2+4t^4\right)\cos u - 4t\left(6+5t^2\right)\sin u\right)}{\sqrt{1+t^2}\left(4t\cos u - 4\sin u - \sqrt{1+t^2}\left(2+t^2\right)^2\right)^2}, \\ K_u = \dfrac{\sqrt{1+t^2}\left(2+t^2\right)^2\csc u\left(4s + 4\cot u\right)}{\left(4t\cot u - \sqrt{1+t^2}\left(2+t^2\right)^2\csc u - 4\right)^2} \end{cases}$$

and

$$\begin{cases} H_t = \dfrac{\left(2+t^2\right)\left(-4\left(-2+3t^2+4t^4\right)\cos u + 4t\left(6+5t^2\right)\sin u\right)}{2\sqrt{1+t^2}\left(4t\cos u - 4\sin u - \sqrt{1+t^2}(2+t^2)^2\right)^2}, \\ H_u = \dfrac{-\sqrt{1+t^2}\left(2+t^2\right)^2\left(4t\sin u + 4\cos u\right)}{2\left(4t\cos u - 4\sin u - \sqrt{1+t^2}(2+t^2)^2\right)^2}. \end{cases}$$

Thus, we get $K_t H_u - K_u H_t = 0$. *So, it is seen that this sweeping surface is a Weingarten surface.*

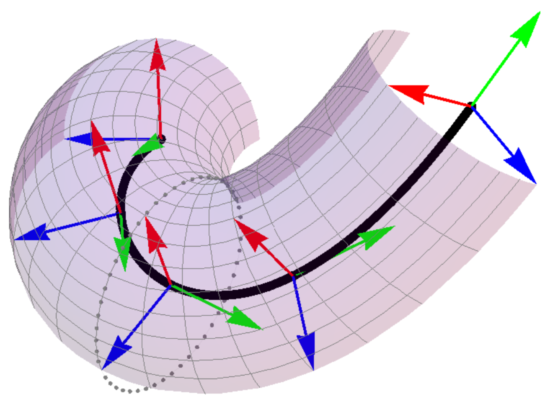

Figure 4: The sweeping surface $\Psi(t,u)$ with polynomial spine curve (black), and the planar profile curve (gray) for $t \in (-2,2)$ and $u \in (-\pi/2, -\pi/2)$.

5 Conclusion

This paper investigates sweeping surfaces in Euclidean 3-space with polynomial spine curves. The novelty is using the Frenet-like curve frame instead of the Frenet frame of the polynomial spine curves. The research includes deriving conditions for sweeping surfaces to be minimal, developable, and Weingarten surface. A comprehensive analysis of the parameter curves of the sweeping surface is conducted to determine conditions for them to be geodesics, asymptotics, and principal curvature lines. Furthermore, the research provides some examples of sweeping surfaces along with illustrated graphics. The study contributes novel insights, as sweeping surfaces have not been previously examined in the context of the Flc Frame elements that make an original contribution to the field. This study stands as a valuable foundation for future research in the intersection of sweeping surfaces and polynomial curves.

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