DOI: 10.2478/auom-2025-0013
An. Şt. Univ. Ovidius Constanţa

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Vol. 33(1),2025, 261–280

Weak convergence theorems for inertial Krasnoselskii-Mann iterations in the class of enriched nonexpansive operators in Hilbert spaces

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Abstract

In this paper, we present some results about the aproximation of fixed points of nonexpansive and enriched nonexpansive operators. In order to approximate the fixed points of enriched nonexpansive mappings, we use the Krasnoselskii-Mann iteration for which we prove weak convergence theorem and the theorem which offers the convergence rate analysis.

Our results in this paper extend some classical convergence theorems from the literature from the case of nonexpansive mappings to that of enriched nonexpansive mappings. One of our contributions is that the convergence analysis and rate of convergence results are obtained using conditions which appear not complicated and restrictive as assumed in other previous related results in the literature.

1 Introduction and Preliminaries

Our study is based on some results about the aproximation of fixed points of nonexpansive and enriched nonexpansive operators. There are numerous works in this regard (for example [6], [7], [9], [10], [14], [16], [35] and references

Key Words: weak convergence, inertial iteration, nonexpansive operator, enriched nonexpansive operator, Hilbert space.
2010 Mathematics Subject Classification: Primary 46G05, 46L05; Secondary 47A30, 47B47.
Received: 18.11.2023
Accepted: 07.02.2024

to them). Of course, the bibliografical references are extensive and they are mentioned at the end of this paper. At the beginning of this research, we remind some basical notions which will be used in our study.

Definition 1.1. Let K be a nonempty subset of a real normed linear space X. A mapping $T : K \to K$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \;\; \forall x, y \in K. \tag{1}$$

An element $x \in K$ is said to be a *fixed point* of T is $Tx = x$ and the set of fixed points of T is denoted by $F(T)$.

Definition 1.2. [9] Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an *enriched nonexpansive mapping* if there exists $b \in [0, \infty)$ such that

$$\|b(x - y) + Tx - Ty\| \leq (b + 1)\|x - y\|, \forall x, y \in X. \tag{2}$$

To indicate the constant involved in (2) we shall also call T as a b-enriched nonexpansive mapping.

Remark 1.3. 1) It is easy to see that any nonexpansive mapping T is a 0-enriched mapping, i.e., it satisfies (2) with $b = 0$.

2) We note that, according to Theorem 12.1 in [25], in a Hilbert space any enriched nonexpansive mapping which is also firmly nonexpansive is nonexpansive. T is said to be *firmly nonexpansive* if

$$\|T(x) - T(y)\|^2 + \|(Id - T)(x) - (Id - T)(y)\|^2 \leq \|x - y\|^2$$

$(x, y \in X)$.

3) It is very important to note that, similar to the case of nonexpansive mappings, any enriched nonexpansive mapping is continuous.

Example 1.4. 1) $T : [0, 4] \to [0, 4]$, $Tx = 4 - x$, for all $x \in [0, 4]$ is nonexpansive and T has a unique fixed point, $F(T) = \{2\}$.

2) If $T : [0, 10] \to [0, 10]$, $Tx = 2x - 10$, then T is not nonexpansive, because, for $x = 5$ and $y = 4$, then $\|Tx - Ty\| \leq \|x - y\| \Leftrightarrow 2 \leq 1$, which is false.

3) [9] Let $X = \left[\frac{1}{2}, 2\right]$ be endowed with usual norm and $T : X \to X$ be defined by $Tx = \frac{1}{x}$, for all $x \in \left[\frac{1}{2}, 2\right]$. Then T is a $\frac{3}{2}$ - enriched nonexpansive mapping and $F(T) = \{1\}$.

Throughout this paper, we take H as a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$.

To approximate fixed point of a nonexpansive mapping T, Krasnoselski-iMann iteration [27], [33], [38] is often used: $x_1 \in H$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n \quad \forall n = 1, 2, ... \tag{3}$$

where $\alpha_n \in (0, 1)$.

Cominetti et al. [20] showed that the fixed point residual

$$\|x_n - Tx_n\| = O\left(\frac{1}{\sqrt{\sigma_n}}\right)$$

in (3), where

$$\sigma_n := \sum_{k=1}^{n} \alpha_k (1 - \alpha_k), \quad n \in \mathbb{N}.$$

In [11], Bot et al. studied the inertial KrasnoselskiiMann algorithm of the form: $x_0, x_1 \in H$,

$$\begin{cases} y_n = x_n + \theta_n (x_n - x_{n-1}) \\ x_{n+1} = y_n + \alpha_n (T y_n - y_n). \end{cases} \tag{4}$$

Weak convergence results are obtained in real Hilbert spaces for the class of nonexpansive operators under the conditions that $\{\theta_n\}$ is a non-decreasing sequence with $0 \leq \theta_n \leq \theta < 1$, $\forall n \geq 1$ and $\theta, \sigma, \delta > 0$ such that

(a) $\delta > \dfrac{\theta^2(1+\theta) + \theta\sigma}{1 - \theta^2}$; and

(b) $0 < \alpha \leq \alpha_n \leq \mu := \dfrac{\delta - \theta[\theta(1+\theta) + \theta\delta + \sigma]}{\delta[1 + \theta(1+\theta) + \theta\delta + \sigma]}$.

Inspired by the works of Bot et al. [11] and Liang et al. [28], in this paper, we first give weak convergence analysis and the nonasymptotic $O(1/n)$ convergence rate result in terms of fixed point residual of inertial KrasnoselskiiMann iteration (1.2) under different conditions assumed by Bot et al. [11].
The following tools will be needed in proving our convergence result.

Lemma 1.5. *[35] Let X be a real inner product space. Then*

$$\|tx + sy\|^2 = t(t+s)\|x\|^2 + s(t+s)\|y\|^2 - st\|x - y\|^2,$$

$\forall x, y \in X, \quad \forall s, t \in \mathbb{R}$.

Lemma 1.6. *[32] Assume $\varphi_n \in [0, \infty)$ and $\delta_n \in [0, \infty)$ satisfy:*

(1). $\varphi_{n+1} - \varphi_n \leq \theta_n(\varphi_n - \varphi_{n-1}) + \delta_n$,

(2). $\sum_{n=1}^{\infty} \delta_n < \infty$,

(3). $\{\theta_n\} \subset [0, \theta]$, *where* $\theta \in (0, 1)$.

Then the sequence $\{\varphi_n\}$ *is convergent with*

$$\sum_{n=1}^{\infty} [\varphi_{n+1} - \varphi_n]_+ < \infty,$$

where $[t]_+ := max\{t, 0\}$, *for any* $t \in \mathbb{R}$.

2 Convergence Analysis

In this section, we consider the convergence analysis of (4) for the class of enriched nonexpansive operators and under a seemingly weaker condition, different from the conditions imposed in [11].

Theorem 2.1. *Let* C *be a bounded closed convex subset of a Hilbert space* H *and* $T : C \to C$ *be an enriched nonexpansive operator and* $F(T) \neq \emptyset$. *Let the sequence* $\{x_n\}$ *in* C *be generated by:* $x_0 = x_1 \in C$,

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = y_n + \alpha_n(Ty_n - y_n), \end{cases} \tag{5}$$

where we assume that $\{\alpha_n\} \subset (0, 1)$ *and* $\{\theta_n\} \subset [0, 1)$ *such that the following conditions hold:*

(a) $0 \leq \theta_n \leq \theta_{n+1} < \theta$, *where* $\theta < \dfrac{\sqrt{1 + 8\epsilon} - 1 - 2\epsilon}{2(1 - \epsilon)}$, *for some* $\epsilon \in (0, 1)$, *and*

(b) $0 < \alpha \leq \alpha_n \leq \dfrac{1}{1 + \epsilon}$.

Then $\{x_n\}$ *converges weakly to a fixed point of* T.

Proof. Step 1: We first prove that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Let $x^* \in F(T)$. From (5), we get

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= (1 - \alpha_n)\|y_n - x^*\|^2 + \alpha_n\|Ty_n - x^*\|^2 \\ &\quad -\alpha_n(1 - \alpha_n)\|y_n - Ty_n\|^2 \\ &\leq \|y_n - x^*\|^2 - \alpha_n(1 - \alpha_n)\|y_n - Ty_n\|^2. \end{aligned} \tag{6}$$

Now,

$$\begin{aligned} \|y_n - x^*\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - x^*\|^2 \\ &= \|(1+\theta_n)(x_n - x^*) - \theta_n(x_{n-1} - x^*)\|^2 \\ &= (1+\theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{7}$$

Observe that

$$\|x_{n+1} - y_n\|^2 = \alpha_n^2\|y_n - Ty_n\|^2$$

and so

$$\|y_n - Ty_n\|^2 = \frac{1}{\alpha_n^2}\|x_{n+1} - y_n\|^2. \tag{8}$$

Putting (8) into (6), we get

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \|y_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|y_n - Ty_n\|^2 \\ &= \|y_n - x^*\|^2 - \frac{(1-\alpha_n)}{\alpha_n}\|x_{n+1} - y_n\|^2. \end{aligned} \tag{9}$$

Now,

$$\begin{aligned} \|x_{n+1} - y_n\|^2 &= \|x_{n+1} - x_n - \theta_n(x_n - x_{n-1})\|^2 \\ &= \|x_{n+1} - x_n\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 - 2\theta_n\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle \\ &\geq \|x_{n+1} - x_n\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 - 2\theta_n\|x_{n+1} - x_n\|\|x_n - x_{n-1}\| \\ &\geq (1-\theta_n)\|x_{n+1} - x_n\|^2 + (\theta_n^2 - \theta_n)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{10}$$

Putting (7) and (10) into (9), we get

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1+\theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 + \\ &\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2 - \frac{(1-\alpha_n)}{\alpha_n}[(1-\theta_n)\|x_{n+1} - x_n\|^2 \\ &+(\theta_n^2 - \theta_n)\|x_n - x_{n-1}\|^2] = (1+\theta_n)\|x_n - x^*\|^2 \\ &-\theta_n\|x_{n-1} - x^*\|^2 - \frac{(1-\alpha_n)}{\alpha_n}(1-\theta_n)\|x_{n+1} - x_n\|^2 \\ &+[\theta_n(1+\theta_n) - \frac{(1-\alpha_n)}{\alpha_n}(\theta_n^2 - \theta_n)]\|x_n - x_{n-1}\|^2 \\ &= (1+\theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 - \rho_n\|x_{n+1} - x_n\|^2 \end{aligned} \tag{11}$$

$+\sigma_n\|x_n - x_{n-1}\|^2$, where $\rho_n := \frac{(1-\alpha_n)}{\alpha_n}(1-\theta_n)$ and $\sigma_n := \theta_n(1+\theta_n) - \frac{(1-\alpha_n)}{\alpha_n}(\theta_n^2 - \theta_n)$.

Let $\Gamma_n := \|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 + \sigma_n\|x_n - x_{n-1}\|^2$. Then we obtain from (11) that

$$\Gamma_{n+1} - \Gamma_n = \|x_{n+1} - x^*\|^2 - (1+\theta_{n+1})\|x_n - x^*\|^2$$
$$+\theta_n\|x_{n-1} - x^*\|^2 + \sigma_{n+1}\|x_{n+1} - x_n\|^2 - \sigma_n\|x_n - x_{n-1}\|^2$$
$$\leq \|x_{n+1} - x^*\|^2 - (1+\theta_n)\|x_n - x^*\|^2$$
$$+\theta_n\|x_{n-1} - x^*\|^2 + \sigma_{n+1}\|x_{n+1} - x_n\|^2 - \sigma_n\|x_n - x_{n-1}\|^2$$
$$\leq -(\rho_n - \sigma_{n+1})\|x_{n+1} - x_n\|^2. \tag{12}$$

Since $0 \leq \theta_n \leq \theta_{n+1} < \theta$, we have

$$\rho_n - \sigma_{n+1} = \frac{(1-\alpha_n)}{\alpha_n}(1-\theta_n) - \theta_{n+1}(1+\theta_{n+1})$$
$$+\frac{(1-\alpha_{n+1})}{\alpha_{n+1}}(\theta_{n+1}^2 - \theta_{n+1}) \geq \frac{(1-\alpha_n)}{\alpha_n}(1-\theta_{n+1})$$
$$-\theta_{n+1}(1+\theta_{n+1}) + \frac{(1-\alpha_{n+1})}{\alpha_{n+1}}(\theta_{n+1}^2 - \theta_{n+1})$$
$$\geq e(1-\theta) - \theta(1+\theta) + e(\theta^2 - \theta) = e - 2e\theta - \theta - \theta^2 + e\theta^2 = \tag{13}$$

$-(1-e)\theta^2 - (1+2e)\theta + e$. Combining (12) and (13), we get

$$\Gamma_{n+1} - \Gamma_n \leq -\delta\|x_{n+1} - x_n\|^2, \tag{14}$$

where $\delta := -(1-e)\theta^2 - (1+2e)\theta + e$. Therefore, $\Gamma_{n+1} \leq \Gamma_n$. Hence $\{\Gamma_n\}$ is nonincreasing. Furthermore,

$$\Gamma_n = \|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 + \sigma_n\|x_n - x_{n-1}\|^2$$
$$\|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2.$$

Therefore,

$$\|x_n - x^*\|^2 \leq \theta_n\|x_{n-1} - x^*\|^2 + \Gamma_n$$
$$\leq \theta\|x_{n-1} - x^*\|^2 + \Gamma_1$$
$$\vdots$$
$$\leq \theta^n\|x_0 - x^*\|^2 + \Gamma_1(\theta^{n-1} + \theta^{n-2} + ... + 1)$$
$$\leq \theta^n\|x_0 - x^*\|^2 + \frac{\Gamma_1}{1-\theta} \tag{15}$$

and it can also be seen that

$$-\theta\|x_{n-1}-x^*\|^2 \le \|x_n-x^*\|^2-\theta\|x_{n-1}-x^*\|^2 \le \Gamma_n \le \Gamma_1. \tag{16}$$

Note that

$$\Gamma_{n+1} = \|x_{n+1}-x^*\|^2-\theta_{n+1}\|x_n-x^*\|^2+\sigma_{n+1}\|x_{n+1}-x_n\|^2$$
$$\ge -\theta_{n+1}\|x_n-x^*\|^2. \tag{17}$$

Using (15) and (17), we get

$$-\Gamma_{n+1} \le \theta_{n+1}\|x_n-x^*\|^2 \le \theta\|x_n-x^*\|^2$$
$$\le \theta^{n+1}\|x_0-x^*\|^2+\frac{\theta\Gamma_1}{1-\theta}.$$

By (14), we have

$$\delta\|x_{n+1}-x_n\|^2 \le \Gamma_n-\Gamma_{n+1},$$

and so by (15) and (16), we get

$$\delta\sum_{j=1}^{n}\|x_{j+1}-x_j\|^2 \le \Gamma_1-\Gamma_{n+1} \le \Gamma_1+\theta\|x_n-x^*\|^2$$
$$\le \Gamma_1+\theta^{n+1}\|x_0-x^*\|^2+\frac{\theta\Gamma_1}{1-\theta} = \theta^{n+1}\|x_0-x^*\|^2+\frac{\Gamma_1}{1-\theta}. \tag{18}$$

This shows that

$$\sum_{n=1}^{\infty}\|x_{n+1}-x_n\|^2 \le \frac{\Gamma_1}{\delta(1-\theta)} < \infty. \tag{19}$$

Therefore,

$$\sum_{n=1}^{\infty}\theta_n\|x_{n+1}-x_n\|^2 < \infty.$$

From above, we deduce that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.
Step 2: We show that $\lim_{n\to\infty}\|x_n-x^*\|$ exists and $\lim_{n\to\infty}\|y_n-Ty_n\| = 0$. Using (7) in (6), we have

$$\|x_{n+1}-x^*\|^2 \le (1+\theta_n)\|x_n-x^*\|^2-\theta_n\|x_{n-1}-x^*\|^2$$
$$+\theta_n(1+\theta_n)\|x_n-x_{n-1}\|^2-\alpha_n(1-\alpha_n)\|y_n-Ty_n\|^2$$
$$\le \|x_n-x^*\|^2-\theta_n(\|x_n-x^*\|^2-\|x_{n-1}-x^*\|^2)$$
$$+2\|x_n-x_{n-1}\|^2-\alpha_n(1-\alpha_n)\|y_n-Ty_n\|^2.$$

Therefore,

$$\alpha_n(1-\alpha_n)\|y_n - Ty_n\|^2 \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2$$
$$-\theta(\|x_{n-1} - x^*\|^2 - \|x_n - x^*\|^2) + 2\|x_n - x_{n-1}\|^2 \tag{20}$$

From (11), we have

$$\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 + \theta_n(\|x_n - x^*\|^2$$
$$-\|x_{n-1} - x^*\|^2) + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2$$
$$\le \|x_n - x^*\|^2 + \theta_n(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2) + 2\|x_n - x_{n-1}\|^2 \tag{21}$$

Using Lemma 1.6, we see that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Since $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we have from (20) that

$$\lim_{n\to\infty}\alpha_n(1-\alpha_n)\|y_n - Ty_n\| = 0. \tag{22}$$

In view of condition (b) in (22), this yields

$$\lim_{n\to\infty}\|y_n - Ty_n\| = 0.$$

Step 3: Finally, we show that $\{x_n\}$ converges weakly to a fixed point of T. From (5), we have that

$$\lim_{n\to\infty}\|y_n - x_n\| = \lim_{n\to\infty}\theta_n\|x_n - x_{n-1}\| = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to some element $p \in H$. Also, $\lim_{n\to\infty}\|y_n - x_n\| = 0$ implies that there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ that converges weakly to $p \in H$. Following the same method of proof given in [6, Teorema 5.14 (iii)] and [11, Teorema 5] we can show that the entire sequence $\{x_n\}$ converges weakly to $p \in F(T)$. This completes the proof. □

Remark 2.2. We can see from the conditions (a) and (b) placed on the inertial factor θ_n and parameter α_n that

$$\lim_{\epsilon\to 0}\frac{\sqrt{1+8\epsilon} - 1 - 2\epsilon}{2(1-\epsilon)} = 0 \text{ and } \lim_{\epsilon\to 1}\frac{\sqrt{1+8\epsilon} - 1 - 2\epsilon}{2(1-\epsilon)} = \frac{1}{3};$$

and

$$\lim_{\epsilon\to 0}\frac{1}{1+\epsilon} = 1 \text{ and } \lim_{\epsilon\to 1}\frac{1}{1+\epsilon} = \frac{1}{2}.$$

This implies that (5) is reduced to the KrasnoselskiiMann iteration (3) when ϵ is chosen very close to zero and in this case, there is no much advantage of (5) over (3). On the other hand, there is significant advantage of (5) over (3) when ϵ is chosen close to 1.

3 Rate of Convergence

In this section, we investigate the convergence rate analysis of inertial Krasnoselskii-Mann algorithm proposed in (5) under the conditions (a) and (b) of Theorem 2.1. The arguments of proof of the result in this section are similar to the ones used in [39].

Theorem 3.1. *Let C be a bounded closed convex subset of a Hilbert space H and $T : C \to C$ be an enriched nonexpansive operator and $F(T) \neq \emptyset$. Suppose $\{x_n\}$ is generated by (5), with $\{\alpha_n\}$ and $\{\theta_n\}$ in $[0,1]$ satisfying conditions (a) and (b) in Theorem 2.1 above. Then, for any $x^* \in F(T)$ and $n > 0$, it holds that*

$$\min_{1\leq i\leq n} \|x_i - Tx_i\| = O(\frac{1}{\sqrt{n}}).$$

Proof. Let $x^* \in F(T)$. By Lemma 1.5, we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= (1-\alpha_n)\|y_n - x^*\|^2 + \alpha_n\|Ty_n - x^*\|^2 \\ &\quad -\alpha_n(1-\alpha_n)\|Ty_n - y_n\|^2 \\ &\leq \|y_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|Ty_n - y_n\|^2. \end{aligned} \tag{23}$$

Using Lemma 1.5 in (5), we obtain

$$\|y_n - x^*\|^2 = \|(1+\theta_n)(x_n - x^*) - \theta_n(x_{n-1} - x^*)\|^2$$
$$= (1+\theta_n)\|x_n - x^*\|^2 - \theta_n\|x_{n-1} - x^*\|^2 + \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2.$$

Using the last equality in (23) and (5), we obtain

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 - (1+\theta_n)\|x_n - x^*\|^2 + \theta_n\|x_{n-1} - x^*\|^2 \\ &\leq -\alpha_n(1-\alpha_n)\|Ty_n - y_n\|^2 + \theta_n(1-\theta_n)\|x_n - x_{n-1}\|^2 \\ &= \frac{-\alpha_n(1-\alpha_n)}{\alpha_n^2}\|x_{n+1} - y_n\|^2 + \theta_n(1-\theta_n)\|x_n - x_{n-1}\|^2 \\ &= \frac{-(1-\alpha_n)}{\alpha_n}\|x_{n+1} - y_n\|^2 + \theta_n(1-\theta_n)\|x_n - x_{n-1}\|^2 \\ &\leq -\frac{e}{\alpha(1+e)}\|x_{n+1} - y_n\|^2 + \theta_n(1-\theta_n)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{24}$$

This implies from (24) that

$$\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 - \theta_n(\|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2)$$

$$\leq -\frac{e}{\alpha(1+e)}\|x_{n+1}-y_n\|^2 + \theta_n(1-\theta_n)\|x_n - x_{n-1}\|^2. \tag{25}$$

Let

$$\delta_n := \theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2;$$
$$\varphi_n := \|x_n - x^*\|^2;$$
$$V_n := \varphi_n - \varphi_{n-1}.$$

and

$$[V_n]_+ := \max\{V_n, 0\}, \ \ \forall n \geq 1.$$

Then, we obtain from (25) that

$$\frac{e}{\alpha(1+e)}\|x_{n+1}-y_n\|^2 \leq \varphi_n - \varphi_{n+1} + \theta_n(\varphi_n - \varphi_{n-1}) + \delta_n$$

$$\varphi_n - \varphi_{n+1} + \theta[V_n]_+ + \delta_n. \tag{26}$$

From (19), we have

$$\sum_{n=1}^{\infty}\|x_{n+1}-x_n\|^2 \leq \frac{\Gamma_1}{1-\theta} \leq \frac{\|x_0 - x^*\|^2}{1-\theta}.$$

Furthermore, we get

$$\sum_{n=1}^{\infty}\theta_n(1+\theta_n)\|x_n - x_{n-1}\|^2$$
$$\leq \sum_{n=1}^{\infty}\theta(1+\theta)\|x_n - x_{n-1}\|^2$$
$$= \theta(1+\theta)\sum_{n=1}^{\infty}\|x_n - x_{n-1}\|^2$$
$$\leq \frac{\theta(1+\theta)\|x_0 - x^*\|^2}{1-\theta} := C_1.$$

From (25,) we obtain

$$V_{n+1} \leq \theta_n V_n + \delta_n \leq \theta[V_n]_+ + \delta_n.$$

Therefore,

$$[V_{n+1}]_+ \leq \theta[V_n]_+ + \delta_n$$
$$\leq \theta^n[V_1]_+ + \sum_{j=1}^{n}\theta^{j-1}\delta_{n+1-j}. \tag{27}$$

Since $x_0 = x_1$, we get

$$V_1 = [V_1]_+ = 0, \delta_1 = 0.$$

From (27), we get

$$\sum_{n=2}^{\infty}[V_n]_+ \leq \frac{1}{1-\theta}\sum_{n=1}^{\infty}\delta_n = \frac{1}{1-\theta}\sum_{n=2}^{\infty}\delta_n. \quad (28)$$

From (26), we get

$$\frac{e}{\alpha(1+e)}\sum_{i=1}^{n}\|x_{i+1}-y_i\|^2 \leq \varphi_1 - \varphi_n + \theta\sum_{i=1}^{n}[V_i]_+ \quad (29)$$

$$+\sum_{i=2}^{n}\delta_i \leq \varphi_1 + \theta C_2 + C_1, \quad (30)$$

where

$$C_2 := \frac{C_1}{1-\theta} \geq \frac{1}{1-\theta}\sum_{i=2}^{\infty}\delta_i \geq \sum_{i=1}^{\infty}[V_i]_+$$

by (28). Now, since $\varphi_1 = \varphi_0$, we get

$$\varphi_1 + \theta C_2 + C_1 = \varphi_1 + \frac{\theta C_1}{1-\theta} + \frac{\theta(1+\theta)\|x_0 - x^*\|^2}{1-\theta}$$

$$= \varphi_1 + \frac{\theta}{1-\theta}\left[\frac{\theta(1+\theta)\|x_0-x^*\|^2}{1-\theta}\right] + \frac{\theta(1+\theta)\|x_0-x^*\|^2}{1-\theta}$$

$$= \left[1 + \frac{\theta^2(1+\theta)}{(1-\theta)^2} + \frac{\theta(1+\theta)}{1-\theta}\right]\|x_0-x^*\|^2. \quad (31)$$

From (29) and (31), we obtain

$$\sum_{i=1}^{n}\|x_{i+1}-y_i\|^2 \leq \frac{\alpha(1+e)}{e}\left[1 + \frac{\theta^2(1+\theta)}{(1-\theta)^2} + \frac{\theta(1+\theta)}{1-\theta}\right]\|x_0-x^*\|^2.$$

Thus,

$$\min_{1\leq i\leq n}\|x_{i+1}-y_i\|^2 \leq \frac{\alpha}{1-\theta}\left[1 + \frac{\theta^2(1+\theta)}{(1-\theta)^2} + \frac{\theta(1+\theta)}{1-\theta}\right]\frac{\|x_0-x^*\|^2}{n}. \quad (32)$$

By (5), we obtain from (32) that

$$\min_{1\le i\le n}\|y_i - Ty_i\|^2 \le \frac{\alpha(1+e)}{\alpha^2 e}\left[1+\frac{\theta^2(1+\theta)}{(1-\theta)^2}+\frac{\theta(1+\theta)}{1-\theta}\right]\frac{\|x_0 - x^*\|^2}{n}.$$

In other words,

$$\min_{1\le i\le n}\|y_i - Ty_i\|^2 = O\left(\frac{1}{n}\right). \tag{33}$$

Consequently, from (18), one can show that

$$\min_{1\le i\le n}\|x_{i+1} - x_i\|^2 = O\left(\frac{1}{n}\right).$$

By (5), we have, for all $i = 1, 2, ..., n$, that

$$\|y_i - x_i\| = \theta_i\|x_i - x_{i-1}\| \le \|x_i - x_{i-1}\|.$$

Also,

$$\begin{aligned}\|x_i - Tx_i\| &\le \|Ty_i - x_i\| + \|Tx_i - Ty_i\|\\ &\le \|y_i - Ty_i\| + \|y_i - x_i\| + \|x_i - y_i\|\\ &= \|y_i - Ty_i\| + 2\|x_i - y_i\|.\end{aligned}$$

Therefore,

$$\min_{1\le i\le n}\|x_i - Tx_i\| \le \min_{1\le i\le n}\|y_i - Ty_i\| + 2\min_{1\le i\le n}\|x_i - y_i\|.$$

This implies that

$$\min_{1\le i\le n}\|x_i - Tx_i\| = O\left(\frac{1}{\sqrt{n}}\right).$$

□

4 Numerical experiments

In this section, we firstly present two examples refering to the analysis of the convergence of the algorithm given by Theorem 2.1.

Example 4.1. We choose $\epsilon = \frac{1}{2}$ in algorithm (5). It means that $\theta_n < \sqrt{5}-2$. We choose $\theta_n = \frac{1}{5}$, $\alpha_n = \frac{1}{4}$, $\forall n$, in algorithm (5) and $x_0 = x_1 = 2$. Let

$Tx = \frac{1}{x}$, $x \in \left[\frac{1}{2}, 2\right]$. Then T is $\frac{3}{2}$ - enriched nonexpansive and $F(T) = \{1\}$. Thus, algorithm (5) reduces to

$$\begin{cases} y_n = x_n + \frac{1}{5}(x_n - x_{n-1}) \\ x_{n+1} = y_n + \frac{1}{4}(Ty_n - y_n) \end{cases}$$

$$\Leftrightarrow \begin{cases} y_n = \frac{6}{5}x_n - \frac{1}{5}x_{n-1} \\ x_{n+1} = y_n + \frac{1}{4}\left(\frac{1}{y_n} - y_n\right). \end{cases}$$

In other words,

$$x_{n+1} = \frac{9}{10}x_n - \frac{3}{20}x_{n-1} + \frac{1}{\frac{24}{5}x_n - \frac{4}{5}x_{n-1}}, x_0 = x_1 = 2.$$

We see that

$x_2 = 1.6250$	$x_9 = 1.0005$
$x_3 = 1.3238$	$x_{10} = 1.0001$
$x_4 = 1.1455$	$x_{11} = 1$
$x_5 = 1.0576$	$x_{12} = 1$
$x_6 = 1.0204$	$x_{13} = 1$
$x_7 = 1.0065$	$x_{14} = 1$
$x_8 = 1.0019$	$\vdots$

We can see that $\{x_n\}$ converge to 1.

We remark here that if the conditions imposed on sequences $\{\alpha_n\}$ and $\{\theta_n\}$ in Theorem 2.1 are not satisfied, there is no convergence of the sequence $\{x_n\}$ generated by algorithm (5) to a fixed point of T as the following example shows.

Example 4.2. Suppose that $\epsilon = 1$, $\theta_n = 1$, $\forall n$, $\alpha_n := \dfrac{4}{5}$ and $x_0 = x_1 = 2$ in algorithm (5). Let $Tx = \dfrac{1}{x}$, $x \in \left[\dfrac{1}{2}, 2\right]$. Then T is $\dfrac{3}{2}$ - enriched nonexpansive and $F(T) = \{1\}$. Moreover, algorithm (5) reduces to

$$\begin{cases} y_n = 2x_n - x_{n-1} \\ x_{n+1} = \dfrac{1}{5}y_n + \dfrac{4}{5y_n}. \end{cases}$$

In other words, $x_{n+1} = \dfrac{2}{5}x_n - \dfrac{1}{5}x_{n-1} + \dfrac{4}{10x_n - 5x_{n-1}}$,
$x_0 = x_1 = 2$.

We see that

$$\begin{aligned} x_2 &= 0.8000 \\ x_3 &= -2.0800 \\ x_4 &= -1.1533 \\ x_5 &= -3.5761 \\ &\vdots \\ x_{13} &= -31.9831 \\ x_{14} &= -12.9693 \\ &\vdots \\ x_{33} &= -3.1732 \\ &\vdots \end{aligned}$$

We can see that $\{x_n\}$ does not converge to 1.

In both Example 4.1 and 4.2, the convergence results in [11] cannot be applied here, mainly because T is enriched nonexpansive but not nonexpansive and, secondary, because the assumptions on parameters in Theorem 2.1 are weaker than the corresponding ones in [11].

Further, we present some numerical experiments intended to illustrate the effectiveness of the Krasnoselskii-Mann iteration (3) and of the inertial Krasnoselskii-Mann algorithm (4) in the class of enriched nonexpansive mappings. We remind that the Krasnoselskii-Mann iteration is obtained from the inertial Krasnoselskii-Mann algorithm when $\theta_n = 0$.

Let $Tx = \dfrac{1}{x}$, $x \in \left[\dfrac{1}{2}, 2\right]$. Then T is $\dfrac{3}{2}$ - enriched nonexpansive mapping and

$F(T) = \{1\}$.
In the Table 1, it is easily seen that the Krasnoselskii-Mann iteration (3) converges to $x^* = 1$, for $\alpha_n \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$. The starting values are $x_0 = x_1 = 2$. The numerical experiments illustrate the convergence of the Krasnoselskii-Mann iteration. N denotes the number of iterations needed to reach the exact solution with four exact digits. Note also the fact that, for small values of α_n, the Krasnoselskii-Mann iteration converges slowly, while for high values of α_n, it converges faster.

Table 1: Results of the numerical experiments of (3) for $\alpha_n \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$

α_n / n	0.1	0.2	0.3	0.4	0.5
0	2	2	2	2	2
1	2	2	2	2	2
2	1.8500	1.7000	1.5500	1.4000	1.2500
3	1.7191	1.4776	1.2785	1.1257	1.0250
4	1.6012	1.3175	1.1296	1.0308	1.0003
5	1.5035	1.2058	1.0663	1.0065	1
6	1.4197	1.1305	1.0234	1.0013	1
7	1.3482	1.0813	1.0095	1.0003	1
8	1.2875	1.0500	1.0038	1.0001	1
9	1.2364	1.0305	1.0015	1	1
10	1.1937	1.0185	1.0006	1	1
11	1.1581	1.0111	1.0002	1	1
12	1.1286	1.0067	1.0001	1	1
13	1.1044	1.0040	1	1	1
N	47	21	12	8	4

In the Table 2, it is easily seen that the inertial Krasnoselskii-Mann algorithm (4) converges to $x^* = 1$, for $(\alpha_n, \theta_n) \in \{(0.1, 0.1), (0.1, 0.5), (0.1, 0.8), (0.5, 0.1), (0.5, 0.5), (0.5, 0.8)\}$. The starting values are $x_0 = x_1 = 2$. The numerical experiments illustrate the convergence of the inertial Krasnoselskii-Mann algorithm. N denotes the number of iterations needed to reach the exact solution with four exact digits. Note also the fact that, for small values of α_n and θ_n, the inertial Krasnoselskii-Mann algorithm converges slowly, while for high values of α_n, it converges faster, regardless of the values of θ_n.

Table 2: Results of the numerical experiments of (4) for $(\alpha_n, \theta_n) \in \{(0.1, 0.1), (0.1, 0.5), (0.1, 0.8), (0.5, 0.1), (0.5, 0.5), (0.5, 0.8)\}$

α_n	0.1	0.1	0.1	0.5	0.5	0.5
θ_n n	0.1	0.5	0.8	0.1	0.5	0.8
0	2	2	2	2	2	2
1	2	2	2	2	2	2
2	1.8500	1.8500	1.8500	1.2500	1.2500	1.2500
3	1.7060	1.6538	1.6148	1.0130	1.0089	1.0942
4	1.5816	1.4645	1.3541	1.0001	1.0070	1.0005
5	1.4759	1.3058	1.1182	1	1	1.0030
6	1.3871	1.1854	0.9442	1	1	1
7	1.3129	1.1015	0.8487	1	1	1
8	1.2516	1.0480	0.8245	1	1	1
9	1.20124	1.0170	0.8489	1	1	1
10	1.1601	1.0012	0.8967	1	1	1
11	1.1269	0.9947	0.9484	1	1	1
12	1.1002	0.9931	0.9918	1	1	1
13	1.0789	0.9939	1.0213	1	1	1
N	42	23	44	4	4	5

From Tables 1 and 2, we see that, for small values of α_n, the inertial Krasnoselskii-Mann algorithm is more efficient than the Krasnoselskii-Mann iteration, in terms of number of iterations, while for high values of α_n, the two algorithms converge almost as fast.

5 Final remarks

In this paper we studied the class of enriched nonexpansive mappings in the setting of a Hilbert space H.

The focus of this paper is centered on weak convergence results (Theorem 2.1) and nonasymptotic $O\left(\frac{1}{n}\right)$ convergence rate analysis (Theorem 3.1) of inertial KrasnoselskiiMann iteration in real Hilbert spaces under seemly easy to implement conditions on the iterative parameters.

Theorem 2.1 is an extension of Theorem 2.1 from [35], by considering enriched nonexpansive mappings instead of nonexpansive mappings.

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