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Some aspects of statistical causality

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Abstract

Causal thinking is deeply embedded in scientific understanding of the problems of applied statistics. This can not always be done by experiments and the researcher is restricted to observing the system he wants to describe. This is the case in many fields, for example, in economics, demography, neuroscience, et cetera.

In this paper we give different concepts of causality between σ-algebas and between Hilbert spaces, using conditional independence and conditional orthogonality, respectively, that can be applied on both stochastic processes and events. These definitions are based on Granger's definition of causality which has great applications in economics (see Florens, Mouchart, 1982; Florens, Fougère, 1996; McCrorie, Chambers, 2006) and also in some other disciplines; for example, see a recent application in neuroscience (see Valdes-Sosa, Roebroeck, Daunizeau, Friston, 2011). The study of Granger's causality has been mainly preoccupied with discrete time processes (i.e. time series). We shall instead concentrate on continuous-time processes. Many of systems to which it is natural to apply tests of causality, take place in continuous time. For example, this is generally the case within economy, demography, finance. The given definitions of causality extend the ones already given in the case of discrete-time processes.

This paper represents a comprehensive survey of causality concepts between flows of information represented by filtrations and by Hilbert spaces. Also, there are given some new results in Section 4 and Section 5.

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1 Introduction

One of the exciting new developments in the field of probability and statistics is a renewal of interest in the causality concept. This has led to several new approaches to defining and studying causality in stochastics and statistical terms. Causality is based on a notion of the past influencing the present and the future. This has very natural links to the type of stochastic processes.

Many scientific studies focus on finding causal relationships between observed processes. From observations it is possible to infer statistical relations, but these cannot automatically be considered as causal. A criterion of causal dependence remains to be found. A good entry point to the statistical literature on how to detect and measure causal effects could be found in Holland (1986). One solution is to equate it to stochastic dependence, see Suppes (1970) and Good (1961/62). A more sophisticated approach is adapted by Granger (1969) where causality is studied in framework of time series. Pursuing the idea, we shall see that conditional independence can serve as a basis for a general probabilistic theory of causality for both processes and single events as it is shown in Florens, Fourege (1996), Florens, Mouchart, Rolin (1990), Gill, Petrović (1987), Granger (1969), Mykland (1986), Petrović (1989, 1996), Petrović, Dimitrijević (2011), Petrović, Dimitrijević, Valjarević (2016), Valjarević, Dimitrijević, Petrović (2023).

Attempts at incorporating time when discovering causal effects are further found in the theories of predictive causality. This is related to probabilistic causality as defined by Suppes (1970). A development can be collected under the heading predictive causality. One direction is the concept of Granger causality (Granger, 1969), which is well known in economics, and the closely related idea of local dependence (Schweder, 1970). The predictive causal ideas are based on stochastic processes. Granger causality is focused on measurements taken over time and how they may influence one another. The idea is that causality is about the present and past influencing future occurrences. The idea behind Granger causality was also formulated by Schweder (1970) in a Markov chain setting, using the name local dependence, and this concept was later extended to more general stochastic processes by Mykland (1986).

The object of this paper is to consider the models of causality which are based on Granger's model of causality. Granger's causality relies on the concept of optimal predictor and this is taken to mean optimal predictor in the last square sense. Granger and Newbold (1972) considered criterion of optimality based on conditional independence. This definition is compared to some concepts in Chamberlain (1982) and Florens and Mouchart (1982). In this paper we show how conditional independence can serve as a basis for a general probability theory of causality that can be applied on both processes and single events.

After the famous paper of Granger (1969) many authors considered different ways of defining causality. These results mainly belong to predicting theory. Namely, the question of interest is: is it possible to reduce the available information in order to predict a given stochastic process? Granger's causality is one of the most popular measure to reveal causality influence of time series, widely applied in economics.

The Granger's causality is focused on discrete time stochastic processes (time series), but, many of systems to which it is natural to apply tests of causality, take place in continuous time. For example, this is generally the case within economy, demography, neurocsience. Modern finance theory uses extensively diffusion processes. In this case, it may be difficult to use a discrete-time model. Also, the observed "causality" in a discrete-time model may depend on the length of the interval between each of two successive samplings, as in the case of Granger's causality shown in McCrorie and Chambers (2006).

So, in this paper we consider the continuous time processes. We consider the different causality concepts in continuous time models and analyze their properties. The given definitions of causality extend the ones already given in the case of discrete-time processes.

The paper is organized as follows. After Introduction, in Section 2 we, first, consider concept of causality between events and between σ-algebras. In Section 3 we develop concept of causality between flows of information that are represented by filtrations, we work in σ-algebraic framework. Also, in this section, we extend the given causality concept from fixed times to stopping times. This concept of causality is shown to be closely related with the notion of extremality of measures (Petrović and Valjarević (2018a)), stable subspaces (Petrović and Valjarević (2013)), separable processes (Valjarević and Petrović (2020)) and measurable separability of σ-algebras (Valjarević and Merkle, 2021). In (Valjarević and Merkle, 2021) some results are applied on Bayesian experiment. Also, weak solutions of stochastic differential equations, as well as solutions of martingale problem can be expressed using the given concept of causality (Petrović and Valjarević, 2015, 2018b).

In the Section 4 we present different concepts of causality between flows of information that are represented by families of Hilbert spaces. In the Section 5 we consider a problem (that follows directly from realization problem): how to find Markovian representations for a given family of Hilbert spaces (understood as outputs of a stochastic dynamic system S_1) provided it is in a certain causality relationship with another family of Hilbert spaces (i. e. with some information about states of a stochastic dynamic system S_2).

2 Causality between events and between σ-algebras

A probabilistic model for a time–dependent system is described by $(\Omega, \mathcal{A}, \mathcal{F}_t, P)$ where $(\Omega, \mathcal{A}, P)$ is a probability space with filtration $\{\mathcal{F}_t,\ t \in I\}$ where $\mathcal{F}_t$ is a set of all events in the model up to and including time t and $\mathcal{F}_t$ is a subset of $\mathcal{A}$. $\mathcal{F}_\infty$ is the smallest $\sigma-$algebra containing all the $\mathcal{F}_t$ (even if $\sup I < +\infty$), $\mathcal{F}_\infty = \bigvee_{t \in I} \mathcal{F}_t$.

The definition of causality uses the conditional independence of σ-algebras.

Let us recall that two σ-algebras $\mathcal{M}_1$ and $\mathcal{M}_2$ are called independent if,

$$E[x_1 x_2] = E[x_1]E[x_2],$$

where the random variable $x_1 : \Omega \to R_+$ is $\mathcal{M}_1$-measurable and $x_2 : \Omega \to R_+$ is $\mathcal{M}_2$-measurable.

Conditional independence is like independence but formulated in terms of conditional expectations.

Definition 2.1. (compare with Rozanov, 1977 and Dellacherie and Meyer, 1980a)

Let$(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}$ arbitrary sub-σ-algebras from $\mathcal{A}$. It is said that $\mathcal{M}$ is splitting for $\mathcal{M}_1$ and $\mathcal{M}_2$ or that $\mathcal{M}_1$ and $\mathcal{M}_2$ are conditionally independent given $\mathcal{M}$ (and written as $\mathcal{M}_1 \perp \mathcal{M}_2 | \mathcal{M}$) if

$$E[x_1 x_2 | \mathcal{M}] = E[x_1 | \mathcal{M}]E[x_2 | \mathcal{M}],$$

where x_1, x_2 denote positive random variables measurable with respect to the corresponding σ-algebras $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. Also, the basic properties are given in Florens at al. (1990).

Some equivalent conditions for conditional independence are presented below.

Proposition 2.1. (compare with Florens, Mouchart, Rolin, 1990)

Let $\mathcal{M}_1, \mathcal{M}_2$ be σ-algebras from $\mathcal{A}$. The following statements are equivalent:

a) $\mathcal{M}_1 \perp \mathcal{M}_2 | \mathcal{M}$;

b) $\mathcal{M}_2 \perp \mathcal{M}_1 | \mathcal{M}$;

c) $E[x_1 | \mathcal{M}_2 \vee \mathcal{M}] = E[x_1 | \mathcal{M}]$ for all $\mathcal{M}_1$-measurable random variables x_1.

d) $E[x_1 | \mathcal{M}_2 \vee \mathcal{M}]$ is $\mathcal{M}$-measurable for all $\mathcal{M}_2$-measurable random variables x_1.

Now, we give a definition of causality using conditional indenpendence.

Definition 2.2. (see Mykland, 1986)

Let

$$\mathcal{M}_1 \subset \mathcal{F}_\infty.$$

be a σ-algebra. A σ-algebra $\mathcal{M}_2$ is a cause of $\mathcal{M}_1$ at time t (relative to $(\Omega, \mathcal{A}, \mathcal{F}_t, P)$) if and only if

$$\mathcal{M}_2 \subset \mathcal{F}_t \tag{1}$$

and

$$\mathcal{M}_1 \perp \mathcal{F}_t | \mathcal{M}_2. \tag{2}$$

This way we can describe the causes of single events ($\mathcal{M}_1 = \{\emptyset, \Omega, A, A^c\}$) and sets of events. Note that if

$$\mathcal{M}_1 \subset \mathcal{F}_t,$$

then $\mathcal{M}_1$ is a cause of $\mathcal{M}_1$ at time t.

This model of causality is closely related to the Bayesian definition of sufficiency. In Bayesian terminology, (1) and (2) would define $\mathcal{M}_2$ as a sufficient σ-algebra if $\mathcal{F}_t$ and $\mathcal{M}_1$ had represented observations and parameters, respectively. For more details see Florens, Mouchart (1979) and Mouchart, Rolin (1979).

For a probability space $(\Omega, \mathcal{A}, P)$ define the set of all sub-σ-algebras,

$$\mathcal{F} = \{\mathcal{M} \in \mathcal{A} | \mathcal{M} \text{ a } \sigma\text{-algebra containing all null sets of } \mathcal{A}\}.$$

For a sub-σ-algebra $\mathcal{M} \in \mathcal{F}$ define the set of all positive random variables $x : \Omega \to R_+$ which are measurable with respect to $\mathcal{M}$ and denote it by

$$L_+(\Omega, R_+, \mathcal{M}) = L_+(\mathcal{M}) = \{x : \Omega \to R_+ |\ x \text{ is } \mathcal{M} - \text{measurable}\}.$$

The motivation to introduce the projection of σ-algebras is to construct the "smallest" sub-σ-algebra of a σ-algebra $\mathcal{M}_2$ conditionally on which $\mathcal{M}_2$ becomes independent of another given σ-algebra $\mathcal{M}_1$.

The projection of a σ-algebra $\mathcal{M}_1$ on a σ-algebra $\mathcal{M}_2$ is the smallest sub-σ-algebra of $\mathcal{M}_2$ with respect to which conditional expectations of $\mathcal{M}_1$-measurable functions given $\mathcal{M}_2$ are measurable. More precisely, we have the following definition.

Definition 2.3.
Let $\mathcal{M}_1, \mathcal{M}_2$ be σ-algebras from $\mathcal{A}$. The projection of the σ-algebra $\mathcal{M}_1$ on $\mathcal{M}_2$ is the σ-algebra

$$\sigma(\mathcal{M}_1 | \mathcal{M}_2) = \sigma\{E[x_1 | \mathcal{M}_2] : x_1 \in L_+(\mathcal{M}_1)\}.$$

In words, the projection is the smallest σ-algebra with respect to which the indicated conditional expectation is measurable. Note that the projection crucially depends on the probability measure.

Now, it is easy to prove the following result which gives an alternative way of defining causality.

Proposition 2.2.
Let the probability system $(\Omega, \mathcal{A}, \mathcal{F}_t, P)$ be given and let $\mathcal{M}_1$ be sub-σ-algebra from $\mathcal{F}_\infty$. $\mathcal{M}_2$ is a cause of $\mathcal{M}_1$ at time t if and only if

$$\mathcal{M}_2 \subset \mathcal{F}_t$$

and

$$\sigma(\mathcal{M}_1 | \mathcal{F}_t) \subset \mathcal{M}_2.$$

It follows directly from Florens, Mouchart (1982).

The Proposition 2.2, says that up-to null sets, $\mathcal{M}_2$ is cause at time t iff it contains the cause at time t and has itself to be occured by t.

Proposition 2.2, also, implies invariance of causality under stochastic equivalence (see Mykland 1986): If $\mathcal{M}_2$ is a cause of $\mathcal{M}_1$ at time t (relative to $(\Omega, \mathcal{A}, \mathcal{F}_t, P)$) and if

$$\mathcal{M}_1 = \mathcal{M}_1', \ \mathcal{M}_2 = \mathcal{M}_2', \ \{\mathcal{F}_t\} = \{\mathcal{F}_t'\} \quad (P - a.s.)$$

and

$$\mathcal{M}_1' \subset \mathcal{F}_\infty', \quad \mathcal{M}_2' \subset \mathcal{F}_t',$$

then $\mathcal{M}_2'$ is a cause of $\mathcal{M}_1'$ at time t relative to (relative to $(\Omega, \mathcal{A}, \mathcal{F}_t', P)$).

3 Causality between filtrations

In this part we develop a concept of causality between flows of information that are represented by filtrations, we work in σ-algebraic framework. The benefit of this approach is to obtain a theory invariant not only to linear transformation of the variables, but also to any change of coordinates and theory which easily can deal with nonlinear transformations.

Let the system $(\Omega, \mathcal{A}, \mathcal{F}_t, P)$ be given, where $(\Omega, \mathcal{A}, P)$ is a probability space and $\{\mathcal{F}_t, \ t \in I\}$ is a filtration, $\mathcal{F}_t$ is a σ-algebra of all events in the model up to and including time t and $\mathcal{F}_t$ is a subset of $\mathcal{A}$. We suppose that the filtration $\{\mathcal{F}_t\}$ satisfies the "usual conditions", which means that each $\mathcal{F}_t$ is right continuous and complete. Analogous notation will be used for filtrations $\mathbf{E} = \{\mathcal{E}_t, \ t \in I\}$ and $\mathbf{G} = \{\mathcal{G}_t, \ t \in I\}$.

We now give a definition of causality formulated in terms of σ-algebras (filtrations).

Definition 3.1. (compare with Gill, Petrović, 1987 and Valjarević, Petrović, 2020)

Let $\mathbf{F} = \{\mathcal{F}_t\}$, $\mathbf{G} = \{\mathcal{G}_t\}$ and $\mathbf{E} = \{\mathcal{E}_t,\}$, $t \in I$ be filtrations on the same probability space. It is said that **G is a cause of E** within **F** relative to P (and written as $\mathbf{E} \mid< \mathbf{G}; \mathbf{F}$;P) if $E_\infty \subset F_\infty$, $\mathcal{G}_t \subseteq \mathcal{F}_t$ and if $\mathcal{E}_\infty$ is conditionally independent of $\mathcal{F}_t$ given $\mathcal{G}_t$ for each t,

$$\mathcal{E}_\infty \perp \mathcal{F}_t | \mathcal{G}_t$$

i.e.

$$(\forall t \in I)(\forall A \in \mathcal{E}_\infty)\ P(A|\mathcal{F}_t) = P(A|\mathcal{G}_t).$$

If there is no doubt about P, we omit "relative to P".

Intuitively, $\mathbf{E} \mid< \mathbf{G}; \mathbf{F}$ means that, for arbitrary t, information about $\mathcal{E}_\infty$ provided by $\mathcal{F}_t$ is not "bigger" than that provided by $\mathcal{G}_t$.

A definition, similar to Definition 3.1 was first given in (Mykland, 1986): "It is said that **G** is a cause of **E** within **F** relative to P (and written as $\mathbf{E} \mid< \mathbf{G}; \mathbf{F}$;P) if $\mathcal{E}_t \subseteq \mathcal{F}_t$, $\mathcal{G}_t \subseteq \mathcal{F}_t$ and if $\mathcal{E}_\infty \perp \mathcal{F}_t|\mathcal{G}_t$ for each t". However, the definition from (Mykland, 1986) contains also the condition $\mathcal{E}_t \subseteq \mathcal{F}_t$ for each t, (instead of $\mathcal{E}_\infty \subseteq \mathcal{F}_\infty$ in Definition 3.1) which does not have intuitive justification. Since Definition 3.1 is more general than the definition given in (Mykland, 1986), all results related to causality in the sense of Definition 3.1 will be true and in the sense of the definition from (Mykland, 1986), when we add the condition $\mathcal{E}_t \subseteq \mathcal{F}_t$ for each t to them.

If **G** and **F** are such that $\mathbf{G} \mid< \mathbf{G}; \mathbf{F}$, we shall say that **G** is its own cause within **F** or that **G** is self-caused within **F** (compare with Mykland, 1986). It should be mentioned that the notion of subordination (as introduced by Rozanov, 1977) is equivalent to the notion of being one's own cause, as defined here.

The following result shows that the relationship "being one's own cause" for the σ-fields is the transitive relationship.

Proposition 3.1. (see Petrović, Valjarević, Dimitrijević, 2016)

Let $\mathbf{E} = \{\mathcal{E}_t\}$, $\mathbf{G} = \{\mathcal{G}_t\}$ and $\mathbf{F} = \{\mathcal{F}_t\}$, $t \in I \subset R$, be filtrations on the same probability space $(\Omega, \mathcal{A}, P)$. Then from $\mathbf{E} \mid< \mathbf{E}; \mathbf{G}$ and $\mathbf{G} \mid< \mathbf{G}; \mathbf{F}$ it follows that $\mathbf{E} \mid< \mathbf{E}; \mathbf{F}$ holds.

If **G** and **F** are such that $\mathbf{G} \mid< \mathbf{G}; \mathbf{G} \bigvee \mathbf{F}$ (where $\mathbf{G} \bigvee \mathbf{F}$ is a family determined by $(G \bigvee F)_t = G_t \bigvee F_t$), we shall say that **F** does not cause **G**. It is clear that the interpretation of Granger's causality is now that $\mathbf{F}$ does not cause $\mathbf{G}$ if $\mathbf{G} \models \mathbf{G}; \mathbf{G} \bigvee \mathbf{F}$ (see Mykland, 1986). Without difficulty, it can be shown that this term and the term "$\mathbf{F}$ does not anticipate $\mathbf{G}$" (as introduced by Rozanov, 1982) are identical.

It should be mentioned that the definition of causality in Mykland (1986) is equivalent to the definition of strong global noncausality, given by Florens and Fougères (1996). So, Definition 3.2 is a generalization of the notion of strong global noncausality.

A family of σ-algebras induced by a stochastic process $\mathbf{X} = \{X_t, t \in I\}$ is given by $\mathbf{F}^X = \{\mathcal{F}_t^X, t \in I\}$, where

$$\mathcal{F}_t^X = \sigma\{X_u, u \in I, u \leq t\},$$

being the smallest σ-algebra with respect to which the random variables $X_u, u \leq t$ are measurable.

The process $\{X_t\}$ is $(\mathcal{F}_t)$-adapted if $\mathcal{F}_t^X \subseteq \mathcal{F}_t$ for each t.

A family of σ-algebras may be induced by several processes, e.g. $\mathbf{F}^{X,Y} = \{\mathcal{F}_t^{X,Y}, t \in I\}$, where

$$\mathcal{F}_t^{X,Y} = \mathcal{F}_t^X \bigvee \mathcal{F}_t^Y, t \in I.$$

Definition 3.1 can be applied to stochastic processes. It will be said that stochastic processes are in a certain relationship if and only if the corresponding induced filtrations are in that relationship. Specially, $(\mathcal{F}_t)-$adapted stochastic process $X = \{X_t\}$ is its own cause if $\mathbf{F}^{\mathbf{X}} = (\mathcal{F}_{\mathbf{t}}^{\mathbf{X}})$ is its own cause within $\mathbf{F} = (\mathcal{F}_{\mathbf{t}})$ if $\mathbf{F}^{\mathbf{X}} \models \mathbf{F}^{\mathbf{X}}; \mathbf{F}; \mathbf{P}$.

Now we give some properties of causality concept from Definition 3.1.

Lemma 3.1.
$\mathbf{E} \models \mathbf{G}; \mathbf{F}$ if and only if $\mathcal{E}_\infty \subset \mathcal{F}_\infty$, $\mathcal{G}_t \subseteq \mathcal{F}_t$ and $\sigma(\mathcal{E}_\infty|\mathcal{F}_t) = \sigma(\mathcal{E}_\infty|\mathcal{G}_t)$.
It follows directly from Florens, Mouchart (1982).

The following result gives the invariance under convergence for causality concept from Definition 3.1.

Proposition 3.2 (see Mykland, 1986 and Petrović, Dimitrijević, 2011)
Let $\mathbf{E} = \{\mathcal{E}_t\}$, $\mathbf{G} = \{\mathcal{G}_t\}$ and $\mathbf{F} = \{\mathcal{F}_t\}$, $t \in I \subset R$, be filtrations on the same probability space $(\Omega, \mathcal{A}, P)$. If $\{X_t^n\}$ is a sequence of stochstic processes satisfying X_t^n converges to X_t in probability when $n \to \infty$, for each $t \in I$ and $\mathbf{F}^{\mathbf{X^n}} \models \mathbf{G}; \mathbf{F}$ for each n, then for the process $X = \{X_t, t \in I\}$ holds $\mathbf{F}^{\mathbf{X}} \models \mathbf{G}; \mathbf{F}$.

Some of the properties of causality relationship from Definition 3.1 could be find in Mykland (1986), Gill, Petrović (1987), Petrović, Dimitrijević, Valjarević, (2016), Petrović, Valjarević, (2018), Valjarević,, Petrović (2020).

Gégout-Petit and Commenges (2009) use conditional independence of filtrations to establish some causality relations, too. In their terminology causality relationship $\mathbf{F}^X \mid< \mathbf{F}^X; \mathbf{F}^{X,Z}$ would be interpreted as $(\mathcal{F}_t^X)$ is filtration-based strong local independent of filtration $(\mathcal{F}_t^Z)$.

Remark.

The condition of Granger's causality is actually a condition of transitivity largely used in sequential analysis (in statistics), see (Bahadur, 1954) and (Hall, Wijsman, Gosh, 1965).

We now extend Definition 3.1 from fixed times to stopping times, i.e. we give characterization of causality using σ-field associated to stopping times. This generalization involves stopping times – a class of random variables that plays the essential role in the Theory of Martingales (for details see Elliot, 1982 and Cohen, Elliot, 2015).

Let us briefly recall some basics about stopping times and σ-algebras.

Suppose that $(\Omega, \mathcal{A}, P)$ is a probability space and $\mathbf{F} = \{\mathcal{F}_t,\ t \in I\}$ is a given filtration ($\mathcal{F}_t \subseteq \mathcal{A}$ for each $t \in I$).

- A random variable $T : \Omega \to R \cup \{\infty\}$ is a stopping time with respect to filtration $\mathbf{F} = \{\mathcal{F}_t\}$, provided that $\{\omega \mid T(\omega) \le t\} \in \mathcal{F}_t$, for all t.
- $\mathcal{F}_T = \{A \in \mathcal{F} \mid A \cap \{T \le t\} \in \mathcal{F}_t, \text{for all } t\}$ is σ-field and intuitively $\mathcal{F}_T$ is the information available at time T.
- If S and T are stopping times with respect to the filtration $\mathbf{F}$, then $S \wedge T$ is a stopping time with respect to the filtration $\mathbf{F}$, too. Specially, if T is a stopping time and t some real number, then $t \wedge T$ defined by

$$t \wedge T(\omega) = \min(t, T(\omega)) = \begin{cases} T(\omega), & T(\omega) < t \\ t, & T(\omega) \ge t \end{cases}$$

  is a stopping time.
- If S and T are stopping times such that $S \le T$ then $\mathcal{F}_S \subseteq \mathcal{F}_T$, and as a consequence we get that $\mathcal{F}_{s\wedge T} \subseteq \mathcal{F}_{t\wedge T}$ for all $s < t$.

In many situations we observe some systems up to some random time, for example till the time when something happens for the first time. For a process X, we set $X_T(\omega) = X_{T(\omega)}(\omega)$, whenever $T(\omega) < +\infty$. We define the stopped process $X^T = \{X_{t\wedge T}, t \in I\}$ with

$$X_t^T(\omega) = X_{t\wedge T(\omega)}(\omega) = X_t 1_{\{t<T\}} + X_T 1_{\{t\ge T\}}.$$

Now, we define a concept of causality for the stopped processes as a generalization of the concept given by the Definition 3.1. More precisely, we define the concept of causality for the stopped (progressively measurable) process X^T using the stopped filtration $\mathbf{F}^T = \{\mathcal{F}_{t\wedge T}\}$, i.e. using the σ-algebras associated to stopping times.

The following definition gives causality between filtrations $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{E}$ up to stopping time T.

Definition 3.2. (see Petrović, Dimitrijević, Valjarević,, 2016)
Let $\mathbf{F} = \{\mathcal{F}_t\}$, $\mathbf{G} = \{\mathcal{G}_t\}$, $\mathbf{E} = \{\mathcal{E}_t\}$, $t \in I$, be given filtrations on the probability space $(\Omega, \mathcal{F}, P)$ and let T be a stopping time relative to filtration $\mathbf{E}$. It is said that filtration $\mathbf{G}$ is a cause of $\mathbf{E}$ within $\mathbf{F}$ relative to P up to stopping time T or that filtration $\mathbf{G}^T$ is a cause of $\mathbf{E}^T$ within $\mathbf{F}^T$ relative to P (and written as $\mathbf{E}^T \mid\!< \mathbf{G}^T; \mathbf{F}^T; P$) if $\mathbf{E}^T \subseteq \mathbf{F}^T$, $\mathbf{G}^T \subseteq \mathbf{F}^T$ and if $\mathcal{E}_T$ is conditionally independent of $\mathcal{F}_{t\wedge T}$ given $\mathcal{G}_{t\wedge T}$ for each t, i.e.

$$\mathcal{E}_T \perp \mathcal{F}_{t\wedge T} | \mathcal{G}_{t\wedge T},$$

or, equivalently,

$$(\forall t \in I)(\forall A \in \mathcal{E}_T) \qquad P(A|\mathcal{F}_{t\wedge T}) = P(A|\mathcal{G}_{t\wedge T}).$$

The causality between stopped processes and stopped filtrations are considered in papers Petrović, Dimitrijević, Valjarević, (2016) and Valjarević, Petrović (2021).

The following results give some properties of the causality concept up to some stopping time.

Theorem 3.1. (see Petrović, Dimitrijević, Valjarević, 2016)
On the probability space $(\Omega, \mathcal{F}, P)$, let the filtrations $\mathbf{F} = \{\mathcal{F}_t\}$, $\mathbf{G} = \{\mathcal{G}_t\}$, $\mathbf{H} = \{\mathcal{H}_t\}$ and $\mathbf{J} = \{\mathcal{J}_t\}$ be given and let T be a stopping time relative to $\mathbf{J}$ and $\mathbf{H}$. Then the following statements are equivalent

(i) $\mathbf{J}^T \mid\!< \mathbf{H}^T; \mathbf{G}^T; P$ and $\mathbf{J}^T \mid\!< \mathbf{G}^T; \mathbf{F}^T; P$,

(ii) $\mathbf{J}^T \mid\!< \mathbf{H}^T; \mathbf{F}^T; P$ and $\mathbf{H}^T \subseteq \mathbf{G}^T \subseteq \mathbf{F}^T; P$.

From the following result it follows that the relationship "being **its** own cause" for filtrations associated to stopping times is transitive relationship.

Theorem 3.2. (see Petrović, Dimitrijević, Valjarević, 2016)
Let $\mathbf{F} = \{\mathcal{F}_t\}$, $\mathbf{G} = \{\mathcal{G}_t\}$, $\mathbf{H} = \{\mathcal{H}_t\}$ be filtrations on the probability space $(\Omega, \mathcal{F}, P)$. If T is a stopping time relative to $\mathbf{H}$, then from

$$\mathbf{H}^T \mid\!< \mathbf{H}^T; \mathbf{G}^T; P \quad \text{and} \quad \mathbf{G}^T \mid\!< \mathbf{G}^T; \mathbf{F}^T; P,$$

it follows that

$$\mathbf{H}^T \mid\!< \mathbf{H}^T; \mathbf{F}^T; P.$$

If relationship "being one's own cause" holds up to stopping time T and if S is another stopping time such that $S \leq T$, it is natural to expect that the same relationship will hold up to stopping time S, as is shown in the next theorem.

Theorem 3.3. (see Petrović, Dimitrijević, Valjarević, 2016)
Let $\mathbf{F} = \{\mathcal{F}_t\}$ and $\mathbf{G} = \{\mathcal{G}_t\}$ be filtrations on the probability space $(\Omega, \mathcal{F}, P)$ such that $\mathbf{G} \subseteq \mathbf{F}$ and let T and S be a two stopping times relative to $\mathbf{G}$, such that $S \leq T$. Then, from

$$\mathbf{G}^T \mid\!< \mathbf{G}^T; \mathbf{F}^T; P \quad \text{it follows} \quad \mathbf{G}^S \mid\!< \mathbf{G}^S; \mathbf{F}^S; P.$$

Now, we consider the connection between the preservation martingale property and given causality concept. If filtration $\mathbf{G}$ is its own cause within filtration $\mathbf{F}$, then every $\mathbf{G}$-martingale is $\mathbf{F}$-martingale, too (see Bremaund, Yor, 1978 and Yor, 1979). Notice that there is a strong connectionn between the preservation of the martingale property and the causality concept. It is well known that the martingale property remains valid if the filtration decrease, but if the filtration increases the preservation of martingale property is directly connected to the concept of self causality. Namely, in the theory of martingales the concept of selfcausality is equivalent to the hypothesis (H): if $\mathbf{G} \subseteq \mathbf{F}$, every $\mathbf{G}$-martingale is a $\mathbf{F}$-martingale, that is, $\mathbf{G}$ is immersed in $\mathbf{F}$.

Now, we give the similar result for the concept of causality up to some stopping time T and the stopped martingales.

Theorem 3.4. (see Petrović, Dimitrijević, Valjarević, 2016)
Let the process M be a uniformly integrable right continuous martingale with respect to the filtration $\mathbf{G} = \{\mathcal{G}_t\}$, T is a $(\mathcal{G}_t)$-stopping time and $\mathbf{G} \subset \mathbf{F}$. Then the stopped process M^T is martingale with respect to the filtration $\mathbf{F}^T = \{\mathcal{F}_{t \wedge T}\}$ if and only if $\mathbf{G}^T \mid\!< \mathbf{G}^T; \mathbf{F}^T; P$.

4 Causality between families of Hilbert spaces

Causality concepts between families of Hilbert spaces were studied by Hosoya (1977), Florens and Mouchart (1985). In the papers of Florens and Mouchart (1982), Mykland (1986), Gill and Petrović (1987), Petrović (1989, 1996, 2013) it is shown how conditional orthogonality can serve as a basis for a general probabilistic theory of causality for both processes and single events.

Let $\mathcal{F}$ be a Hilbert space whose inner product is defined by $(\cdot,\cdot)$. For arbitrary subspaces F_1 and F_2 of $\mathcal{F}$ (all subspaces are taken to be closed) we have that:

- $F_1 \perp F_2$ means that F_1 and F_2 are orthogonal,
- the orthogonal projection of $x \in F_1$ onto F_2 is denoted by $P(x|F_2)$,
- $P(F_1|F_2)$ will denote the orthogonal projection of F_1 onto F_2,
- $F_1 \ominus F_2$ will denote a Hilbert space generated by all elements $x - P(x|F_2)$, where $x \in F_1$.

  If $F_2 \subseteq F_1$, then $F_1 \ominus F_2$ coincides with $F_1 \cap F_2^{\perp}$, where $F_2^{\perp}$ is the orthogonal complement of F_2 in $\mathcal{F}$; i.e. $F_2^{\perp} = \mathcal{F} \ominus F_2$.

Definition 4.1. (see Gill, Petrović, 1987)
If F_1 and F_2 are arbitrary subspaces of Hilbert space $\mathcal{F}$, then it is said that X is splitting for F_1 and F_2 or that F_1 and F_2 are conditionally orthogonal given X (and written as $F_1 \perp F_2|X$) if

$$F_1 \ominus X \perp F_2 \ominus X, \tag{1}$$

or, equivalently,

$$(x_1, x_2) = (P(x_1|X), P(x_2|X)) \text{ for all } x_1 \in F_1, x_2 \in F_2.$$

When X is trivial, i.e. $X = \{0\}$, this reduces to the usual orthogonality $F_1 \perp F_2$.

The notion of splitting was first given by M. P. Kean 1963.

The following results gives an alternative way of defining splitting.

Lemma 4.1. (see Gill, Petrović, 1987 and Putten and Schuppen, 1979)
$F_1 \perp F_2|X$ if and only if $P(F_i|F_j \vee X) \subseteq X$, for $i,j = 1,2,\ i \neq j$.
Corollary 4.1.1.
$F_1 \perp F_2|F$ if and only if $F_1' \perp F_2'|F$ for all $F_i' \subseteq F_i \bigvee F,\ \ i = 1,2.$

The following result will be needed later.

Lemma 4.2.
A minimal space $F^i \subseteq F_i$ such that $F_1 \perp F_2|F^i$ is defined by $F^i = P(F_j|F_i)$, where $i,j = 1,2,\ i \neq j$.
Proof.
From $F_1 \ominus F_2 = F_1 \ominus P(F_1|F_2) \perp F_2$ and $P(F_1|F_2) \subseteq F_2$, it follows that $F_1 \ominus P(F_1|F_2) \perp F_2 \ominus P(F_1|F_2)$, i.e. $F_1 \perp F_2|P(F_1|F_2)$. To prove the

minimality of $F^2 = P(F_1|F_2)$, let us suppose that $F \subseteq F_2$ is such that $F_1 \perp F_2|F$. According to Lemma 4.1, $F_1 \perp F_2|F$ is equivalent to $P(F_1|F_2 \bigvee F) \subseteq F$. However, since $F \subseteq F_2$, the last inclusion becomes $F^2 \subseteq F$, as we wanted to prove.

Let $\mathbf{F} = (F_t), t \in \mathbf{R}$ be a family of Hilbert spaces. We shall think about F_t as an information available at time t, or as a current information. Total information F_∞ carried by $\mathbf{F}$ is defined by $F_\infty = \vee_{t \in R} F_t$, while past and future information of $\mathbf{F}$ at t is defined as $F_{\leq t} = \vee_{s \leq t} F_s$ and $F_{\geq t} = \vee_{s \geq t} F_s$, respectively.

Analogous notation will be used for families of Hilbert spaces $\mathbf{G} = (G_t)$ and $\mathbf{E} = (E_t)$.

We use the following intuitively plausible notion of causality between families of Hilbert spaces.

Definition 4.2. (Gill, Petrović, 1987; Petrović, 1996)
Let $\mathbf{E}$, $\mathbf{G}$ and $\mathbf{F}$ be arbitrary families of Hilbert spaces. It is said that $\mathbf{G}$ is a cause of $\mathbf{E}$ within $\mathbf{F}$ (and written as $\mathbf{E} \mid\!< \mathbf{G}; \mathbf{F}$) if $E_\infty \subseteq F_\infty$, $G_{\leq t} \subseteq F_{\leq t}$ and

$$E_\infty \perp F_{\leq t} | G_{\leq t} \tag{3}$$

for each t.

The essence of (3) is that all information about E_∞ that gives $F_{\leq t}$ comes via $G_{\leq t}$ for arbitrary t; equivalently, $G_{\leq t}$ contains all the information from the $F_{\leq t}$ needed for predicting E_∞.

Intuitively, $\mathbf{E} \mid\!< \mathbf{G}; \mathbf{F}$ means that, for arbitrary t, information about E_∞ provided by $F_{\leq t}$ is not "bigger" than that provided by $G_{\leq t}$.

If $\mathbf{G}$ and $\mathbf{F}$ are such that $\mathbf{G} \mid\!< \mathbf{G}; \mathbf{F}$, we shall say that $\mathbf{G}$ is its own cause within $\mathbf{F}$., or, equivalently, that $\mathbf{G}$ is self caused within $\mathbf{F}$.

If $\mathbf{G}$ and $\mathbf{F}$ are such that $\mathbf{G} \mid\!< \mathbf{G}; \mathbf{G} \bigvee \mathbf{F}$ (where $\mathbf{G} \bigvee \mathbf{F}$ is a family determined by $(G \bigvee F)_t = G_t \bigvee F_t)$, we shall say that $\mathbf{F}$ does not cause $\mathbf{G}$. It is clear that the interpretation of Granger's causality is now that $\mathbf{F}$ does not cause $\mathbf{G}$ if $\mathbf{G} \mid\!< \mathbf{G}; \mathbf{G} \bigvee \mathbf{F}$ (see Mykland, 1986). Without difficulty, it can be shown that this term and the term "$\mathbf{F}$ does not anticipate $\mathbf{G}$" (as introduced by Rozanov, 1982) are identical.

The Definition 4.2 can be applied to stochastic processes.

Definition 4.3.
It will be said that second order stochastic processes are in a certain relationship if and only if the Hilbert spaces they generate are in that relationship.

So, from the Definition 4.3 it follows that:
stochastic process $\mathbf{Y}$ is a cause of a process $\mathbf{X}$ within process $\mathbf{Z}$ relative to P (i.e. that $\mathbf{F^X} \mid\!< \mathbf{F^Y};\mathbf{F^Z}$ holds) if $F^X_\infty \subseteq F^Z_\infty$, $F^Y_{\leq t} \subseteq F^Z_{\leq t}$ and if F^X_∞ and $F^Z_{\leq t}$ are conditionally orthogonal of given $F^Y_{\leq t}$ for each t, i.e.

$$F^X_\infty \perp F^Z_{\leq t} | F^Y_{\leq t} \text{ for each } t.$$

Now we give one example to illustrate the notions from this part.

Example 4.1.
Let $X(t) = \sum\limits_{n=1}^{N} \int_{-\infty}^{t} g_n(t,u)\, dZ_n(u)$, $t \in [0,1]$ be a proper canonical (or Hida–Cramer) representation of the stochastic process $X(t)$, $t \in [0,1]$. Any process $Z_n(t)$, $n = \overline{1,N}$, is its own cause within $X(t)$, i.e. $\mathbf{F^{Z_n}} \mid\!< \mathbf{F^{Z_n}};\mathbf{F^X}$ holds for any $n = \overline{1,N}$. If we define the process $Y(t)$ as a non–anticipative transformation of $Z_n(t)$, i.e.

$$Y(t) = \int_0^t h(t,u) Z_n(u)\, du\,, \quad t \in [0,1],$$

it is easy to see that Z_n is a cause of Y within X, i.e. that $\mathbf{F^Y} \mid\!< \mathbf{F^{Z_n}};\mathbf{F^X}$ holds.

Example 4.2.
Let $Z(t)$, $0 \leq t \leq T$ be a (t,ω)–measurable signal process such that $\int_0^T E|Z(t)|^2\, dt < \infty$ and let

$$Y(t) = \int_0^t Z(s)\, ds + W(t)\,, \quad 0 \leq t \leq T\,,$$

be the observation process where $W(t)$ is a Wiener process such that $W(t) - W(s)$ is orthogonal on $F^W_{\leq s} \bigvee F^Z_{\leq s}$ for $0 \leq s \leq t \leq T$. Then $Z(t)$ does not cause $W(t)$, i.e. $\mathbf{F^W} \mid\!< \mathbf{F^W};\mathbf{F^{W\vee Z}}$ holds.
If $\mathbf{F^W} \subseteq \mathbf{F^Y}$, then $\mathbf{F^W} \mid\!< \mathbf{F^W};\mathbf{F^Y}$ and $\mathbf{F^W} \mid\!< \mathbf{F^Y};\mathbf{F^{W\vee Z}}$ hold.

In the next section we give some applications. Especially, we consider a problem (that follows directly from realization problem): how to find Markovian representations (even minimal) for a given family of Hilbert spaces (understood as outputs of a stochastic dynamic system S_1) provided it is in a certain causality relationship with another family of Hilbert spaces (i.e. with some informations about states of a stochastic dynamic system S_2).

5 Causality and Stochastic Dynamic Systems

We first give some definitions that we need later.

The notion of minimality and maximality of families of Hilbert spaces is specified in the following definition.

Definition 5.1.
It will be said that $\mathbf{F}$ is a minimal family having a certain property if there is no family $\mathbf{F}^*$ having the same property which is submitted to $\mathbf{F}$.
It will be said that $\mathbf{F}$ is a maximal family having a certain property if there is no family $\mathbf{F}^*$ having the same property such that family $\mathbf{F}$ is submitted to $\mathbf{F}^*$.

It should be understood that a minimal and maximal family having a certain property are not necessarily unique.

In this paper the following definition of markovian property will be used.

Definition 5.2. (compare with Rozanov, 1977)
Family $\mathbf{G} = (G_t)$ will be called Markovian if $P(G_{\geq t}|G_{\leq t}) = G_t$ for each t.

Now we give a definition of a stochastic dynamic system in terms of Hilbert spaces. The characterizing property is the condition that past informations of outputs and states and future informations of outputs and states are conditionally orthogonal given the current state.

Definition 5.3 (compare with Putten and Schuppen, 1979)
A stochastic dynamic system (s.d.s.) is a set of two families: $\mathbf{H}$ (outputs) and $\mathbf{G}$ (states), that satisfy the condition

$$H_{<t} \vee G_{<t} \perp H_{>t} \vee G_{>t} | G_t. \tag{4}$$

For given family of outputs $\mathbf{H}$, any family $\mathbf{G}$ satisfying (4) is called a realization of a s.d.s. with those outputs.

It is clear that realization of a s.d.s. is Markovian.

Suppose that a stochastic dynamic system S_1 causes, in a certain sense, changes of another stochastic dynamic system S_2. It is natural to assume that outputs $\mathbf{H}$ of system S_1 can be registered and that some information $\mathbf{E}$ about the states (or perhaps states themselves) of system S_2 is given. Results that we shall prove will tell us under which conditions concerning the relationships between $\mathbf{H}$ and $\mathbf{E}$ it is possible to find states $\mathbf{G}$ (i.e. Markovian representations) of system S_1 having certain causality relationship in the sense of Definition 1.4 with $\mathbf{H}$ and $\mathbf{E}$.

More precisely, the following cases can be considered:

1° available information about s.d.s. S_2 are a cause of states of a s.d.s. S_1 within outputs of a s.d.s. S_1, i.e. $\mathbf{G} \,|\!\!< \mathbf{E}; \mathbf{H}$ holds;

2° outputs of a s.d.s. S_1 are cause of states of the same system within available information about s.d.s. S_2, i.e. $\mathbf{G} \,|\!\!< \mathbf{H}; \mathbf{E}$ holds;

3° states of a s.d.s. S_1 are a cause of the available information about s.d.s. S_2 within outputs of a s.d.s. S_1, i.e. $\mathbf{E} \,|\!\!< \mathbf{G}; \mathbf{H}$ holds

4° states of a s.d.s. S_1 are a cause of outputs of the same system within available information about s.d.s. S_2, i.e. $\mathbf{H} \,|\!\!< \mathbf{G}; \mathbf{E}$ holds;

5° the available information about S_2 is a cause of outputs of S_1 within states of S_1, i.e. $\mathbf{H} \,|\!\!< \mathbf{E}; \mathbf{G}$ holds;

6° outputs of a s.d.s. S_1 are cause of the available information about S_2 within states of a s.d.s. S_1, i.e. $\mathbf{E} \,|\!\!< \mathbf{H}; \mathbf{G}$ holds.

We consider different kinds of causality between families $\mathbf{G}$, $\mathbf{H}$ and $\mathbf{E}$, while $\mathbf{G}$ and $\mathbf{H}$ are in the same relationship, that is, $\mathbf{G}$ is a realization of an s.d.s. with outputs $\mathbf{H}$ in all cases.

In all cases 1° - 6° it is of interest to find minimal and maximal realizations that satisfy given conditions. We can see that, in some cases, family of extremal realizations is trivial, so as that extremal realizations is unique.

This paper is continuation of the papers (Gill and Petrović 1987), (Petrović 1996 and 2013). In these papers cases 1°, 2°, 4°, 5° and 6° are considered. In the remaining part of this paper we consider case 3°.

For the case 3°, we define some minimal realizations $\mathbf{G}$ (of a s.d.s. with given outputs $\mathbf{H}$) such that $\mathbf{E} \,|\!\!< \mathbf{G}; \mathbf{H}$ holds. It is easy to see that maximal families $\mathbf{G}$ for which $\mathbf{E} \,|\!\!< \mathbf{G}; \mathbf{H}$ holds are all families such that $G_{\leq t} = H_{\leq t}$ for each t.

In case 3° we want to find Markovian family $\mathbf{G}$ which for arbitrary t contains all the information from $H_{\leq t}$ needed for predicting E_∞. The next results give conditions under which $\mathbf{G}$ is a minimal realization of s. d. s. with outputs $\mathbf{H}$ such that $\mathbf{E} \,|\!\!< \mathbf{G}; \mathbf{H}$ holds.

Theorem 5.1.

Let $\mathbf{E}$ and $\mathbf{H}$ be such that $E_\infty \subseteq H_\infty$, $P(H_t|E_\infty) \subseteq H_{\leq t}$ and $H_{<t} \perp H_{>t}|P(H_t|E_\infty)$ for each t. If $\mathbf{H}$ is Markovian, then the family $\mathbf{G}$, defined by

$$G_t = P(H_t|E_\infty), \quad t \in R, \tag{5}$$

is a minimal realization (of a s. d. s. with outputs $\mathbf{H}$) which is a cause of $\mathbf{E}$ within $\mathbf{H}$.

Proof.

From $G_{\leq t} = P(H_{\leq t}|E_{<\infty})$ and Lemma 4.1 it follows that $E_{<\infty} \perp H_{\leq t}|G_{\leq t}$. Also, the definition of $\mathbf{G}$ and the assumption $P(H_t|E_{<\infty}) \subseteq H_{\leq t}$ imply $G_{\leq t} \subseteq$

$H_{\le t}$, which together with the previous conclusion means that $\mathbf{E}\,|\!<\mathbf{G};\mathbf{H}$. The minimality of $\mathbf{G}$ follows from Lemma 4.2.

From $\mathbf{E}\,|\!<\mathbf{G};\mathbf{H}$ and the obvious equality $G_{<\infty} = E_{<\infty}$ it follows that $\mathbf{G}\,|\!<\mathbf{G};\mathbf{H}$. From $G_{\le t} \subseteq H_{\le t}$, the fact that $P(G_{\ge t}|G_{\le t}) = P(H_{\ge t}|G_{\le t})$ and the assumption that $\mathbf{H}$ is Markovian we obtain

$$P(G_{\ge t}|G_{\le t}) = P(P(H_{\ge t}|H_{\le t})|G_{\le t}) = P(H_t|G_{\le t})\,. \tag{6}$$

However, $\mathbf{G}\,|\!<\mathbf{G};\mathbf{H}$ means in particular that $H_t \perp G_{<\infty} \ominus G_{\le t}$ so that (6) becomes

$$P(G_{\ge t}|G_{\le t}) = P(H_t|G_{<\infty}) = P(H_t|E_{<\infty}) = G_t$$

which means that $\mathbf{G}$ is Markovian.

From $\mathbf{G}\,|\!<\mathbf{G};\mathbf{H}$, the fact that $\mathbf{G}$ is Markovian and Corollary 4.1.1 it follows that, for any t, $H_{<t} \perp G_{\ge t}|G_t$, which (together with $H_{<t} \perp H_{>t}|G_t$) gives $H_{<t} \perp H_{>t} \bigvee G_{\ge t}|G_t$. However, since $G_{\le t} \subseteq H_{\le t}$, the last relation implies that $\mathbf{G}$ is a realization of a s. d. s. with outputs $\mathbf{H}$. The proof is completed.

We obtain a simpler version of the Theorem 5.1 if $\mathbf{E}$ is its own cause within $\mathbf{H}$.

Corollary 5.1.1.
Let $\mathbf{E}$ be its own cause within $\mathbf{H}$ and $H_{<t} \perp H_{>t}|P(H_t|E_{\le t})$ for each t. If $\mathbf{H}$ is Markovian, then the family $\mathbf{G}$, defined by

$$G_t = P(H_t|E_{\le t}), \quad t \in R,$$

is a minimal realization (of a s. d. s. with outputs $\mathbf{H}$) which is a cause of $\mathbf{E}$ within $\mathbf{H}$.

The assumption in Theorem 5.1 that $\mathbf{H}$ itself is Markovian is rather strong, simply because $\mathbf{H}$ represents outputs of a s. d. s., and thus the properties of $\mathbf{H}$ could hardly be controlled. The following result does not require $\mathbf{H}$ to be Markovian, but provides a realization whose information at time t is equal to its total information accumulated ut to t.

Theorem 5.2. (see Petrović 1998)
Let $\mathbf{E}$ and $\mathbf{H}$ be such that $E_{<\infty} \subseteq H_{<\infty}$, $P(H_t|E_{<\infty}) \subseteq H_{\le t}$ and $H_{<t} \perp H_{>t}|P(H_{\le t}|E_{<\infty})$ for each t. The family $\mathbf{G}$, defined by

$$G_t = P(H_{\le t}|E_{<\infty}), \quad t \in R\,, \tag{7}$$

is a minimal realization (of a s. d. s. with outputs $\mathbf{H}$) which is a cause of $\mathbf{E}$ within $\mathbf{H}$.

Proof.

Since $G_t = G_{\leq t}$ for all t, it is immediately clear that $\mathbf{G}$ is Markovian. From Lemma 4.1 it follows $E_{<\infty} \perp H_{\leq t} | G_t$; that is $E_{<\infty} \perp H_{\leq t} | G_{\leq t}$, which together with $G_{\leq t} \subseteq H_{\leq t}$, means that $\mathbf{E} \models\!< \mathbf{G}; \mathbf{H}$. From the last relation and $G_{<\infty} = E_{<\infty}$ it follows that $\mathbf{G} \models\!< \mathbf{G}; \mathbf{H}$. Now, on a similar way as in Theorem 5.1, we can prove that $\mathbf{G}$ is a realization (of a s. d. s. with outputs $\mathbf{H}$). The minimality of $\mathbf{G}$ follows from Lemma 4.2.

Remark.

It is of an interest to find conditions for the existence of a realization with certain properties less restrictive than those obtained in this paper.

Also, the problems considered here and in the papers Gill and Petrović (1987), Petrović (1996) and (2005) can be considered in the σ-algebraic approach when stochastic dynamic system is defined as a set of two families of σ-algebras ,see Lindquist, Picci, Ruckebush, (1979).

It is clear that all results from this section can be extended on the σ-algebras generated by finite dimensional Gaussian random variables. But, in the case that σ-algebras are arbitrary, the extensions of the proofs from this paper is nontrivial because one can not take an orthogonal complement with respect to a σ-algebra as one can with respect to subspaces in Hilbert space.

6 Conclusion - Some Applications

The study of Granger's causality has been mainly preoccupied with time series. (see Granger, 1969, 1977; McCrorie, Chambers, 2006). But, many of the systems to which it is natural to apply tests of causality, take place in continuous time (see Mykland, 1986; Gill, Petrović,1987; Florens, Fougères,1996) . So, in this paper we considered continuous time processes.

The given concept of causality can be applied to weak solutions of stochastic differential equations (driven by standard Brownian motion, by fractional Brownian motion and with driving semimartingales). These results are given in Mykland (1986), Petrović, Stanojević (2010), Petrović, Dimitrijević, Valjarević (2016).

In Gill, Petrović (1987) the causality concept is applied to some problems that follows directly from stochastic realization problem.

Also, this concept of causality is closely connected to the notion of extremality of measures and martingale problem (see Petrović and Stanojević (2010); Petrović and Valjarević (2018)). The concept of statistical causality is related to the notion separability of stochastic processes (see Valjarević, Petrović, 2020) and measurable separability of σ-algebras and filtrations (Valjarević and Merkle, 2021). In (Valjarević and Merkle, 2021) some results are applied on Bayesian experiment.

The given concept of causality can be connected to the purely discontinuous martingale and filtrations (see Valjarević, Petrović, 2020).

The given causality concept is connected to the optional and predictable processes important in stochastic integration. More precisely, in (Valjarević, Dimitrijević and Petrović, 2023) was established that the preservation of predictability with respect to larger filtrations is implied by the considered notion of (self-)causality. Also in the same paper were considered the connections between the given causality concept and the optional and predictable projections of a stochastic process, which play an important role in the general theory of stochastic processes, semimartingale theory, and stochastic calculus. Some results show that the (self-)causality implies indistinguishability of the optional (or predictable) projections with respect to considered filtrations from those with respect to larger filtrations.

Finally, it should be noted that the notion "being its own cause" sometimes occurs as a useful assumption in the theory of martingales and stochastic integration (see Yor 1979; Strook and Yor 1980).

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