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On the metrizability of suprametric space

Erdal Karapınar and Marija Cvetković

Abstract

The question of metrizability of suprametric space is answered positively. The observed metric coincides with a suprametric in a way that convergence and continuity are preserved between suprametric space and associated metric space along with the propery of a Cauchy sequence. Consequently, a suprametric space is complete if and only if associated metric space is complete. Fixed point theorems in suprametric space are obtained as a corollary of well-known fixed point results.

1 Introduction and Preliminaries

Suprametric space presents a generalization of a metric space obtained with the modified triangle inequality. This idea was introduced by M. Berzig in [5] where the topological properties of this abstract space were discussed along with some fixed point results. The impact of suprametric space is seen through application in solving equations with specific examples on solving matrix and integral equations. The work of Berzig was further continued in [6, 12, 13, 14] with various aspects of applications.

In the sequel, we collect essential definitions and fixed point theorems for both metric and suprametric space.

Definition 1. If X is a non-empty set, then a function $d : X \times X \to [0, \infty)$ fulfilling:

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(d_1) $d(x,y) = d(y,x) = 0 \Longleftrightarrow x = y$;

(d_2) $d(x,y) = d(y,x)$;

(d_3) $d(x,z) \leq d(x,y) + d(y,z)$;

for all $x, y, z \in X$, is a metric on X and (X,d) is a metric space.

Definition 2. Let (X,d) be a metric space and (x_n) be a sequence in X. The sequence (x_n) converges to $x \in X$ if for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_0) < \varepsilon$ for all $n \geq n_0$.

Definition 3. Let (X,d) be a metric space and (x_n) be a sequence in X. If for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$, $d(x_n, x_m) < \varepsilon$, then (x_n) is a Cauchy sequence in X.

A convergent sequence in a metric space (X,d) is a Cauchy sequence in (X,d), while converse do not hold in general.

Definition 4. If any Cauchy sequence is convergent in a metric space (X,d), then (X,d) is a complete metric space.

Evidently, the following result holds.

Lemma 1. Let (X,d) be a metric space and $(x_n) \subseteq X$ a sequence.

(1) A sequence (x_n) converges to x if and only if $\lim_{n\to\,\infty} d(x_n, x) = 0$.

(2) A sequence $(x_n) \subseteq X$ is Cauchy sequence in a metric space (X,d) if and only if $\lim_{m,n\to\infty} d(x_m, x_n) = 0$.

Definition 5. Let (X,d) be a metric space and $T : X \mapsto X$ a mapping. If $\lim_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} Tx_n = Tx$ for any sequence $(x_n) \subseteq X$, then T is a continuous mapping.

A self-mapping T on a metric space (X,d) is called a Lipschitz mapping if there exists $q \geq 0$ such that inequality

$$d(Tx, Ty) \leq qd(x,y) \tag{1.1}$$

holds for all $x, y \in X$.
If the inequality (1.1) is fulfilled for $q < 1$, then the mapping T is a contraction. Famous Banach results states existence and uniqueness of a fixed point for a contraction on a complete metric space and further convergence of a sequence of successive approximations to the observed fixed point.

Theorem 2. [4] Let (X,d) be a non-empty complete metric space and $T : X \mapsto X$ a contraction. Then there exists a unique fixed point $x^* \in X$ and for any $x \in X$ the iterative sequence $(T^n x)$ converges to the fixed point of a mapping T.

The notion of suprametric originated in [5].

Definition 6. If X is a nonempty set and $d : X \times X \to [0, \infty)$ is a function fulfilling (d_1), (d_2) and

(d_3^*) $d(x, y) \leq d(x, z) + d(z, y) + \rho d(x, z)d(z, y)$

for any $x, y, z \in X$, then a mapping d is a suprametric on X. The pair (X, d) is a suprametric space.

Example 1. Let $X = \{1, 2, 3\}$ and $d : X \times X \mapsto [0, \infty)$ fulfilling (d_1) and (d_2) such that $d(1, 2) = 1$, $d(1, 3) = 3$ and $d(2, 3) = 1$. Then (X, d) is not a metric space since $d(1, 3) > d(1, 2) + d(2, 3)$, but is a suprametric space for $\rho = 1$.

Evidently, any metric is a suprametric, but there are several approaches on creating a suprametric from metric and omitting triangle inequality in general.

Example 2. If (X, d) is a metric space, then the functions

$$d_\alpha(x, y) = d(x, y)(d(x, y) + \alpha),$$
$$d_\beta(x, y) = \beta(e^{d(x,y)} - 1)$$

for any $x, y \in X$ are suprametrics on X with $\rho = \frac{2}{\alpha}$ and $\rho = \frac{1}{\beta}$, respectively. While the function

$$d_\gamma(x, y) = e^{-\gamma d(x,y)^2} - 1$$

for any $x, y \in X$ is a suprametric with a constant $\rho = 1$.

Note that if (d_3^*) is fulfilled for some $\rho > 0$ then it also holds for any greater value of constant ρ.

The terms of convergent and Cauchy sequence along with the continuity of mappings are introduced analogously to equivalent terms in metric space.

Definition 7. Let (X, d) be a suprametric space and (x_n) be a sequence in X. The sequence (x_n) converges to $x \in X$ if for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_0) < \varepsilon$ for all $n \geq n_0$.

Definition 8. Let (X, d) be a suprametric space and (x_n) be a sequence in X. If for all $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n \geq n_0$, then (x_n) is a Cauchy sequence in X.

Definition 9. If any Cauchy sequence is convergent in a suprametric space (X, d), then (X, d) is a complete suprametric space.

Definition 10. Let (X, d) be a suprametric space. A mapping $T : X \mapsto X$ is continuous at a point $x \in X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(Tx, Ty) < \varepsilon$ whenever $d(x, y) < \delta$. If T is continuous at every point of X, then it is a continuous mapping on a suprametric space (X, d).

In order to obtain some fixed point results in suprametric space, we intend to use local approach to the contraction and for that purpose we list several theorems concerning this topic. Meyers in [11] introduced a term of a local contraction.

Definition 11. A mapping $T : X \mapsto X$ on a metric space (X, d) is a local contraction if there exist functions $\lambda, \mu : X \mapsto [0, \infty)$ for which a mapping T is a contraction on a $B[x, \mu(x)]$ with a contractive constant $\lambda(x)$, i.e.,

$$(\forall x \in X)\,(\forall y \in X)\ d(x, y) \leq \mu(x) \Longrightarrow d(Tx, Ty) \leq \lambda(x)d(x, y).$$

If both functions λ and μ are constant, then a local contraction is an uniform local contraction. In order to emphasize with respect to which functions, we will use the notion of a μ, λ-uniform local contraction.
The existence of a fixed point of a local contraction is associated to the question of chainability of a metric space (X, d).

Definition 12. A metric space (X, d) is ε-chainable for some $\varepsilon > 0$ if for any $x, y \in X$ there exists a path $x_0 = x, x_1, \ldots, x_{n-1}, x_n = y \in X$ such that $d(x_i, x_{i+1}) < \varepsilon$ for any $i = \overline{0, n-1}$.

Among several results of [11] concerning existence and uniqueness of a fixed point of a local contraction, we present the following:

Theorem 3. [11] Let $T : X \mapsto X$ be a μ, λ-uniform local contraction on the μ-chainable metric space (X, d). There exists a metric d^* topologically equivalent to d under which T is a contraction. Moreover, (X, d) is complete whenever (X, d^*) is.

Corollary 4. If $T : X \mapsto X$ is a μ, λ-uniform local contraction on the μ-chainable complete metric space (X, d) then it possesses a unique fixed point in X.

Proof. Theorem 3 claims existence of a complete metric d^* on X in dependence to which T is a contraction. Having a contraction on a complete metric spaces, conclusion follows from Theorem 2, □

2 Main results

Theorem 5. If (X, d) is a suprametric space for some $\rho > 0$ and $D : X \times X \mapsto [0, \infty)$ a mapping defined by

$$D(x, y) = \ln(\rho d(x, y) + 1) \tag{2.1}$$

for any $x, y \in X$, then (X, D) is a metric space.

Proof. Assume that (X, d) is a suprametric space with some $\rho > 0$ and that $D : X \times X \mapsto [0, \infty)$ is defined as in (2.1). Evidently, a mapping D is well-defined since for arbitrary $x, y \in X$,

$$d(x, y) \geq 0 \implies \rho d(x, y) + 1 \geq 1 \implies D(x, y) \geq 0.$$

It remains to prove the validity of $(d_1) - (d_3)$ for a mapping D.

(d_1) If $x, y \in X$ are such that $D(x, y) = 0$, then

$$D(x, y) = 0 \iff d(x, y) = 0 \iff x = y$$

follows from (d_1^*).

(d_2) For any $x, y \in X$, the symmetry of suprametric implies

$$D(x, y) = \ln(\rho d(x, y) + 1) = \ln(\rho d(y, x) + 1) = D(y, x).$$

(d_3) If $x, y, z \in X$ are arbitrary, then

$$\begin{aligned} D(x, y) &= \ln(\rho d(x, y) + 1) \\ &\leq \ln\left(\rho(d(x, z) + d(z, y) + \rho d(x, z)d(z, y)) + 1\right) \\ &= \ln\left(\rho d(x, z) + 1\right)\left(\rho d(z, y) + 1\right) \\ &= \ln\left(\rho d(x, z) + 1\right) + \ln\left(\rho d(z, y) + 1\right) \\ &= D(x, z) + D(z, y). \end{aligned}$$

Thus, triangle inequality holds for any $x, y, z \in X$.

According to previous considerations, (X, D) is a metric space. □

Considering the topological properties of induced metric space, we may note the relation between the open balls in both settings and recall that their collection in both cases presents a base of a observed Hausdorff topology. To make a distinction, the open ball with center x and radius $r > 0$ in suprametric space (X, d) will be denoted with $B_d(x, r)$, while for the metric space (X, D) we intend to use $B_D(x, r)$.

Theorem 6. Let (X, d) be a suprametric space and D a metric on X defined by (2.1). If $x \in X$ and $r > 0$ are arbitrary, then

$$B_d(x, r) = B_D(x, \ln(1 + \rho r))$$

Proof. If $x \in X$ is arbitrary point in suprametric space (X, d) with some $\rho > 0$ and $r > 0$, then

$$y \in B_d(x, r) \iff d(x, y) < r \iff D(x, y) < \ln(1+\rho r) \iff y \in B_D(x, \ln(1+\rho r)).$$

Evidently, $r > 0$ is equivalent to $\ln(1 + \rho r) > 0$, and the assertion hold. □

Theorem 7. Let (X,d) be a suprametric space and D a metric on X defined by (2.1). If $x \in X$ and $r > 0$ are arbitrary, then

$$B_d[x,r] = B_D[x, \ln(1+\rho r)].$$

Proof. Proof is analogous to the proof of Theorem 6. For some $x \in X$ and $r > 0$, we have

$$d(x,y) \leq r \iff, D(x,y) \leq \ln(1+\rho r).$$

Further, $B_d[x,r] = B_D[x, \ln(1+\rho r)]$. □

Since we intend to discuss on fixed point results in suprametric space, it is important to consider the question of completeness in the case of newly obtained metric space.

Theorem 8. A suprametric space (X,d) is complete if and only if the induced metric space (X,D) is complete where $D : X \times X \mapsto [0,\infty)$ is defined by (2.1).

Proof. Suppose that (X,d) is a suprametric space with a constant $\rho > 0$ and $D : X \times X \mapsto [0,\infty)$ is defined by (2.1).
If (X,d) is a complete suprametric space, observe a Cauchy sequence $(x_n) \subseteq X$ in a metric space (X,D). For arbitrary $\varepsilon > 0$ let $\delta > 0$ be such that $e^{\delta} - 1 < \rho\varepsilon$. Due to the presumption that it is a Cauchy sequence, there exists some $n_0 \in \mathbb{N}$ such that $D(x_n, x_m) < \delta$ for any $n, m \geq n_0$. The inequality further implies $d(x_n, x_m) < \frac{1}{\rho}\left(e^{\delta} - 1\right) < \varepsilon$ and the sequence (x_n) is a Cauchy sequence in a suprametric space (X,d). Since it is a complete suprametric space, there exists the limit of the sequence $x^* \in X$ satisfying $\lim_{n\to\infty} d(x_n, x^*) = 0$. Furthermore,

$$\lim_{n\to\infty} D(x_n, x^*) = \lim_{n\to\infty} \ln\left(\rho d(x_n, x^*) + 1\right) = 0.$$

Therefore, (x_n) is convergent in a metric space (X,D) with the same limit point.
Otherwise, assume that induced metric space (X,D) is complete and observe a Cauchy sequence (x_n) in a suprametric space (X,d). For arbitrary $\varepsilon > 0$ choose $\delta > 0$ such that $\ln(\rho\delta + 1) < \varepsilon$. For a chosen δ let $m_0 \in \mathbb{N}$ be such that $d(x_n, x_m) < \delta$ for any $n, m \geq m_0$. Then,

$$D(x_n, x_m) = \ln(\rho d(x_n, x_m) + 1) < \ln(\rho\delta + 1) < \varepsilon,$$

for any $n, m \geq m_0$. Accordingly, (x_n) is a Cauchy sequence in (X,D) and thus converges to some $x^* \in X$ with respect do D, meaning that $\lim_{n\to\infty} D(x_n, x^*) = 0$.
Moreover,

$$\lim_{n\to\infty} d(x_n, x^*) = \lim_{n\to\infty} \frac{1}{\rho}\left(e^{D(x_n,x^*)} - 1\right) = 0.$$

Consequently, (x_n) is a Cauchy sequence in (X, d) with the identical limit x^*. It is proven that completeness of a suprametric space implies completeness of induced metric space from Theorem 5 and vice versa. □

Remark 1. The inequality $x - \frac{x^2}{2} < \ln(1+x) < x$ (or $x < e^x - 1$) for any $x > 0$ may be used in the proof of Theorem 8 instead of observed estimations.

Theorem 9. A mapping $T : X \mapsto X$ is continuous with respect to suprametric d with a constant $\rho > 0$ if and only if it is continuous with respect to the induced metric D defined in (2.1).

Proof. This proof is easily deduced from the proof of Theorem 8.
Since $\lim_{n\to\infty} d(Tx_n, Tx) = 0$ if and only if $\lim_{n\to\infty} D(Tx_n, Tx) = 0$ as mentioned in the previous remark for arbitrary point $x \in X$ and a sequence $(x_n) \subseteq X$ converging towards it in either (X, d) or (X, D). □

3 Fixed point results

Theorem 10. If (X, d) is a complete suprametric space with $\rho > 0$ and $T : X \mapsto X$ a contraction, then T is a local contraction in an associated metric space (X, D).

Proof. Let (X, d) be a complete suprametric space for some $\rho > 0$ and $T : X \mapsto X$ a contraction on (X, d) with a contractive constant $q \in [0, 1)$, meaning that

$$d(Tx, Ty) \leq qd(x, y), \tag{3.1}$$

for all $x, y \in X$.
The inequality

$$(\rho t + 1)^{1-\frac{1}{n}} > \rho qt + 1 \tag{3.2}$$

has a solution n_0 (and any $n \geq n_0$) in the set of integers for $t \in (0, \varepsilon(n_0))$. For arbitrary $x \in X$ let n_0 be a large enough solution of the inequality (3.2) such that $\frac{2}{1-q} d(x, Tx) \in (0, \varepsilon(n_0))$. Let $r = \frac{1}{1-q} d(x, Tx)$ and $X^* = B_d[x, r]$ presents a complete subspace of a metric space (X, D) due to the Theorem 7. We will discuss on the restriction of a mapping T on a set X^* and its codomain. If $y \in X^*$, then

$$\begin{aligned} d(Ty, x) &\leq d(Ty, Tx) + d(Tx, x) \\ &\leq qd(x, y) + d(x, Tx) \\ &\leq qr + d(x, Tx) \\ &= \left(\frac{q}{1-q} + 1\right) d(x, Tx) \\ &= r, \end{aligned}$$

leads to the conclusion that $Ty \in X^*$ and, as mentioned, we may observe a restriction $T^* : X^* \mapsto X^*$ of a mapping T.
For any $x_1, x_2 \in X^*$, due to (3.1), we get

$$\begin{aligned} D(T^*x_1, T^*x_2) &= D(Tx_1, Tx_2) \\ &= \ln(\rho d(Tx_1, Tx_2) + 1) \\ &\le \ln(\rho q d(x_1, x_2) + 1) \\ &< \ln\left((\rho d(x_1, x_2) + 1)^{\left(1-\frac{1}{n_0}\right)}\right) \\ &= \left(1 - \frac{1}{n_0}\right) \ln\left((\rho d(x_1, x_2) + 1)\right) \\ &= \left(1 - \frac{1}{n_0}\right) D(x_1, x_2) \end{aligned}$$

since $d(x_1, x_2) \le d(x_1, x) + d(x, x_2) \le \frac{2}{1-q} d(x, Tx)$ implies that $d(x_1, x_2) \in (0, \varepsilon(n_0))$ and (3.2) is fulfilled.
Accordingly, T^* is a contraction on a complete metric space X^* and by Banach fixed point theorem it possesses a unique fixed point in X^*.
The perceived fixed point in X^* is unique in whole space X because of (3.1).
□

Theorem 11. If (X, d) is a complete suprametric space with a constant $\rho > 0$ and $T : X \mapsto X$ a contraction on (X, d) with a contractive constant $q > 0$ such that $\mu = \sup_{x \in X} d(x, Tx) < \infty$, then T is a $\rho\mu$,q-uniform local contraction on X.

Proof. In a complete suprametric space with a contraction $T : X \mapsto X$ determined by a constant $q \in [0, 1)$ observe arbitrary points $x, y \in X$. The question of chainability will be answered for a complete metric space (X, D) associated to suprametric space as in Theorems 5 and 8.
If $\mu = \sup_{x \in X} d(x, Tx) < \infty$, then

$$d(T^n x, T^n y) \le q^n d(x, y),$$

implies that there exists some $n_0 \in \mathbb{N}$ such that $d(T^n x, T^n y) < \mu$ for any $n \ge n_0$.
Additionally, $d(T^i x, T^{i+1} x) \le d(x, Tx)$ for any $i \in \mathbb{N}$ due to the contractive condition.
Consequently, $x_0 = x$, $x_1 = Tx, \ldots$, $x_{n_0} = T^{n_0} x$, $x_{n_0+1} = T^{n_0} y$, $x_{n_0+2} = T^{n_0-1} y, \ldots$, $x_{2n_0} = Ty$ and $x_{2n_0+1} = y$ is a $\rho\mu$-chain between x and y as $d(x_i, x_{i+1}) < \mu$ gives

$$D(x_i, x_{i+1}) = \ln(\rho d(x_i, x_{i+1}) + 1) \le \rho\mu$$

for any $i = \overline{0, 2n_0}$ due to the previous considerations. Thus, the space (X, D) is $\rho\mu$-chainable.
There exists some $\lambda \in [0, 1)$ such that $q\rho t + 1 \leq (\rho t + 1)^\lambda$ for any $t \in [0, 2\rho\mu]$. A mapping T is a contraction with a contractive constant λ on a closed ball $B[x, \rho\mu]$ for each $x \in X$ as the inequalities

$$\begin{aligned} D(Ty, Tz) &= \ln(\rho d(Ty, Tz) + 1) \\ &\leq \ln(q\rho d(y, z) + 1) \\ &\leq \ln\left(\rho d(y, z) + 1\right)^\lambda \\ &= \lambda D(y, z) \end{aligned}$$

hold for any $y, z \in B[x, \rho\mu]$. □

Recalling Mayer's result presented in Theorem 3, we may state the following

Theorem 12. If (X, d) is a complete suprametric space and $T : X \mapsto X$ a contraction such that $\sup_{x\in X} d(x, Tx) < \infty$, then there exists a complete metric d^* on X such that T is a contraction on a complete metric space (X, d^*). Moreover, d^* is topologically equivalent to the induced metric D on X defined in Theorem 5.

Proof. With respect to Theorem 11 we conclude that a contraction $T : X \mapsto X$ on a complete suprametric space (X, d) with a constant ρ is $\rho\mu$,q-uniform local contraction on a complete metric space (X, D) where q is a contractive constant associated to T and $\mu = \sup_{x\in X} d(x, Tx) < \infty$. Hence, Theorem 3 implies existence of a complete metric d^* on X topologically equivalent to D such that T is a contraction on (X, d^*). □

If X is a topological space, $T : X \mapsto X$ a mapping and $x_0 \in X$, then the ω-limit set is the set

$$\omega_T(x_0) = \bigcap_{n\in\mathbb{N}} \overline{\{T^k x_0 \mid k \geq n\}}.$$

Recall the result of Lipeinš [9] regarding the relation between the limit set and the existence of a fixed point of a continuous mapping.

Theorem 13. [9] Let X be a topological space and $T : X \mapsto X$ a continuous mapping. If there exists a continuous mapping $d : X \times X \mapsto \mathbb{R}$ satisfying

$$|d(Tx, Ty)| < |d(x, y)|$$

for all distinct $x, y \in X$ and there exists some $x_0 \in X$ such that $\omega_T(x_0) \neq \emptyset$, then T has a unique fixed point in X. Moreover, the iterative sequence $(T^n x_0) \subseteq X$ converges to the fixed point for an arbitrary initial point $x_0 \in X$.

The succeeding result was obtained in [5] based on Theorem 13.

Theorem 14. [5] If (X, d) is a suprametric space, a mapping $T : X \mapsto X$ satisfies

$$d(Tx, Ty) < d(x, y)$$

for all mutually distinct $x, y \in X$ and there exists $x_0 \in X$ such that $\omega_T(x_0) \neq \emptyset$, then T has a unique fixed point. Moreover, the iterative sequence $(T^n x_0) \subseteq X$ converges to the fixed point in a suprametric space (X, d) for an arbitrary initial point $x_0 \in X$.

Theorem 13 is equivalent to Edelstein theorem in the case of a metric space.

Theorem 15. [7] Let (X, d) be a metric space, $T : X \mapsto X$ a contractive mapping satisfying such that $\omega_T(x_0) \neq \emptyset$ for some $x_0 \in X$, then the sequence $(T^n x_0)$ is convergent and its limit is a unique fixed point of a mapping T.

Consequently, Theorem 12 yields that Theorem 14 is equivalent to Theorem 15.

4 Supratopology and suprametric

The supratopology was introduced in [10] in 1983. as a collection of subsets of a non-empty set X containing X and closed for arbitrary union. It was further investigated in [1, 2, 3, 8, 15] among many others. Known examples are the families of semiopen subsets or preopen subsets of a topological space. Unfortunately, this notation is of earlier term and the choice of denoting the described metric as suprametric was unfortunate since those two terms do not correlate in general.

Definition 13. If X is a non-empty set and $\tau^* \subseteq \mathcal{P}(X)$ a family of subsets satisfying

(τ_1) $\emptyset, X \in \tau^*$;

(τ_2) $(\forall \mathcal{A} \subseteq \tau^*) \bigcup \mathcal{A} \in \tau^*$;

is a supratopology on X and (X, τ^*) is a supratopological space.

Topological space is a supratopological space, but converse do not hold in general.

Example 3. If $X = \{1, 2, 3\}$ and $\tau^* = \{\emptyset, \{1, 2\}, \{1, 3\}, X\}$, then τ^* is a supratopology on X, but not a topology.

The family generated by open balls

$$\tau = \{A \subseteq X \mid (\forall x \in A)\,(\exists r > 0)\, B_d(x, r) \subseteq A\}$$

in a suprametric space (X,d) presents a Hausdorff topology on X as marked in [5] and as also follows from our observation that the ball $B_d(x,r)$ coincides with a open ball in a metric space (X,D) and that the metric topology is Hausdorff.

Theorem 16. Intersection of two open balls in a suprametric space (X,d) is open set in a generated topological space (X,τ).

Proof. Let $x,y\in X$ be arbitrary points of a suprametric space (X,d) with a constant $\rho>0$ and $\varepsilon,\delta>0$. Assume that $z\in B(x,\varepsilon)\cap B(y,\delta)$ (it their intersection is empty then it is an open set by definition) and let $r=\min\left\{\frac{\varepsilon-d(z,x)}{1+\rho d(z,x)},\frac{\delta-d(z,y)}{1+\rho d(z,y)}\right\}$, then for arbitrary $w\in B(z,r)$

$$\begin{aligned} d(x,w) &\leq d(x,z)+d(z,w)+\rho d(x,z)d(z,w)\\ &< d(x,z)+\frac{\varepsilon-d(z,x)}{1+\rho d(z,x)}+\rho d(x,z)\frac{\varepsilon-d(z,x)}{1+\rho d(z,x)}\\ &=\varepsilon \end{aligned}$$

and

$$\begin{aligned} d(y,w) &\leq d(y,z)+d(z,w)+\rho d(y,z)d(z,w)\\ &< d(y,z)+\frac{\delta-d(z,y)}{1+\rho d(z,y)}+\rho d(y,z)\frac{\delta-d(z,y)}{1+\rho d(z,y)}\\ &=\delta \end{aligned}$$

imply that $z\in B(z,r)\subseteq B(x,\varepsilon)\cap B(y,\delta)$. □

On the other hand, we may define a supratopology that does not generate a suprametric since not every supratopology is a topology.

Example 4. Let $X=\{1,2,3\}$ and $\tau^*=\{\emptyset,\{1,2\},\{1,3\},X\}$ a supratopology on X. Then τ is not generated by any suprametric $d:X\times X\mapsto[0,\infty)$ since for $d(1,2)=a$, $d(1,3)=b$ and $d(2,3)=c$ define $r<\min\{a,b,c\}$ and $B(1,r)=\{1\}\notin\tau^*$.

It would be expected that a suprametric generates a supratopology instead of a topology which is not the case and that is why we needed to emphasize that there is no relation between mentioned terms (except that any topology is a supratopology).

5 Conclusion

The notion of suprametric was introduced with a presumption of novelty and potential for further applications. Unfortunately, from the proposed metrization of suprametric which preserves a property of completeness we may conclude that that a setting of a complete suprametric space may be observed through a obtained metric space. Moreover, observed fixed point theorems are a direct corollary of some well-known result in metric space. We believe that the same approach may be applied for various generalizations of suprametric and that the expected outcome would be analogous equivalent of a generalization of a metric space.

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Erdal KARAPINAR,
Department of Mathematics, Atılım University, İncek, Ankara
Turkey;
Department of Medical Research, China Medical University Hospital
China Medical University, 40402 Taichung, Taiwan
Emailerdalkarapinar@yahoo.com

Marija CVETKOVIĆ
Faculty of Sciences and Mathematics
University of Niš
Višegradska 33
18000 Niš
Serbia
Email: marija.cvetkovic@pmf.edu.rs