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A digital 3D Jordan-Brouwer separation theorem

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Abstract

We introduce a connectedness in the digital space $\mathbb{Z}^3$ induced by a quaternary relation. Using this connectedness, we prove a digital 3D Jordan-Brouwer separation theorem for boundary surfaces of the digital polyhedra that may be face-to-face tiled with certain digital tetrahedra in $\mathbb{Z}^3$. An advantage of the digital Jordan surfaces obtained over those given by the Khalimsky topology is that the former may bend at the acute dihedral angle $\frac{\pi}{4}$.

1 Introduction

Digital images represent an important source of data and the role of computer imagery is to extract and process these data to make them convenient as input for computer programs to get the desired information. For example, digital image data may be used for diagnosing the objects investigated in medicine, material science, etc. And 3D imaging has a major role as modern efficient imaging technologies are being developed providing high quality 3D digital image data (e.g., 3D tomograph data in medicine).

In 3D imagery, a significant role is played by digital Jordan surfaces, i.e., the surfaces satisfying a digital analog of the 3D Jordan-Brouwer separation theorem – cf. [2]. Such surfaces separate the digital space $\mathbb{Z}^3$ into two connected components, hence represent boundaries of objects in digital pictures.

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Therefore, when studying and processing 3D digital images, it is desirable to equip the digital space $\mathbb{Z}^3$ with a connectedness allowing to define digital Jordan surfaces. In the classical approach, adjacency relations (6-, 18- and 26-adjacencies) are employed to obtain connectedness on $\mathbb{Z}^3$ – see [6, 11]. Such an approach was used, for instance, in [1, 5, 9, 10] to introduce and study digital Jordan surfaces. A disadvantage of the classical approach is that two kinds of connectedness have to be similtaneously employed, one for the surface and the other for its complement. Therefore, a new approach was proposed in [4] using just one connectedness, namely that one provided by the Kahlimsky topology. Digital Jordan surfaces with respect to the Khalimsky topology were studied in [7] and [8].

In this note, we propose another approach to defining digital Jordan surfaces, namely an approach based on employing a quaternary relation to obtain a connectedness in $\mathbb{Z}^3$. More precisely, we introduce a quaternary relation R on $\mathbb{Z}$ and study the connectedness in $\mathbb{Z}^3$ induced by the relation R^3 obtained as a special product of three copies of the relation R. We prove a digital analog of the 3D Jordan-Brouwer separation theorem for the digital space $\mathbb{Z}^3$ equipped with the (connectedness induced by the) relation R^3. More precisely, we show that the boundary surfaces of a polyhedron that may be face-to-face tiled with certain digital tetrahedra separates $\mathbb{Z}^3$ into exactly two connected components. This may be considered to be a 3D extension of the digital Jordan curve theorem proved in [13] that is based on employing a connectedness in $\mathbb{Z}^2$ induced by a ternary relation. But, first of all, it improves the result in [15] where a similar statement was proved for the boundary surfaces of the polyhedra obtained by face-to-face tiling with certain digital prisms. We also show an advantage of the digital Jordan surfaces with respect to the connectedness induced by the relation R^3 over those with respect to the Khalimsky topology on $\mathbb{Z}^3$.

2 Preliminaries

As usual, by a *quaternary relation* R on X we understand a subset $R \subseteq X^4$, i.e., a set of ordered quadruples $(x_0, x_1, x_2, x_3) = (x_i \mid i < 4)$ with $x_0, x_1, x_2, x_3 \in X$. The pair (X, R) is then called a *quaternary relational system*. We denote by Δ_X the quaternary *diagonal* on X, i.e., the quaternary relation $\Delta_X = \{(x_i \mid i < 4) \in X^n;\ x_0 = x_1 = x_2 = x_3\}$. For every quaternary relation R on X, we put $\bar{R} = R \cup \Delta_X$. $\bar{R}$ is called the *reflexive hull* of R.

Let R_j be a quaternary relation on a set X_j for every $j = 1, 2, ..., m$ ($m > 1$ an integer). Recall that the *Cartesian product* of the relations R_j, $j = 1, 2, ..., m$, is the quaternary relation $\prod_{j=1}^{m} R_j$ on the Cartesian product $\prod_{j=1}^{m} X_j$ of the sets X_j, $j = 1, 2, ..., m$, given by $\prod_{j=1}^{m} R_j = \{((x_i^1, x_i^2, ..., x_i^m) \mid$

$i < 4)$; $(x_i^j \mid i < 4) \in R_j$ for every $j \in J\}$. We put $\bigotimes_{j=1}^m R_j = \prod_{j=1}^m \bar{R}_j - \Delta_Y$ where $Y = \prod_{j=1}^m X_j$ and call the n-ary relation $\bigotimes_{j=1}^m R_j$ on $\prod_{j=1}^m X_j$ the *strong product* of R_j, $j = 1, 2, ..., m$.

If $X_j = X$ and $R_j = R$ for every $j = 1, 2, ..., m$, we write R^m instead of $\bigotimes_{j=1}^m R_j$.

Given a quaternary relation R on a set X, we put
$R^* = \{(x_i \mid i \leq m) \in X^{m+1};\ 0 < m < 4$ and there exists $(y_i \mid i < 4) \in R$ such that $x_i = y_i$ for every $i \leq m$ or $x_i = y_{m-i}$ for every $i \leq m\}$.
The elements of R^* will be called *R-initial segments*. Thus, the R-initial segments are the ordered sequences of at least two and at most four members that are the initial parts of the quadruples belonging to R ordered according to the quadruples or conversely.

Definition 2.1. Let R be a quaternary relation on a set X. A sequence $C = (x_i \mid i \leq r)$, $r > 0$ an integer, of elements of X is called an *R-walk* if there is an increasing sequence $(i_k \mid k \leq p)$ of non-negative integers with $i_0 = 0$ and $i_p = r$ such that $i_k - i_{k-1} < 4$ and $(x_i \mid i_{k-1} \leq i \leq i_k) \in R^*$ for every k with $0 < k \leq p$.

Evidently, every R-initial segment is an R-walk. Observe also that, if $(x_i \mid i \leq r)$ is an R-walk, then its reverse, i.e., the sequence $(x_{r-i} \mid i \leq r)$, is an R-walk, too. And, if $(x_i \mid i \leq r)$ and $(y_i \mid i \leq s)$ are R-walks such that $x_r = y_0$, then their union, i.e., the sequence $(z_i \mid i \leq r+s)$ where $z_i = x_i$ for all $i \leq r$ and $z_i = y_{i-r}$ for all i with $r \leq i \leq r+s$, is an R-walk, too.

Definition 2.2. Let R be a quaternary relation on a set X. A set $Y \subseteq X$ is said to be *R-connected* if any two different elements $x, y \in Y$ can be joined by an *R-walk* contained in Y (i.e., there is an R-walk $(x_i \mid i \leq r)$ with $\{x_i \mid i \leq r\} \subseteq Y$ such that $x_0 = x$ and $x_r = y$). A maximal (with respect to set inclusion) R-connected set is called an *R-component* of X.

Note that, given a quaternary relation R on a set X, every R-initial segment is R-connected. Of course, the union of a finite sequence of nonempty R-connected sets is R-connected if the intersection of every consecutive pair of sets in the sequence is nonempty. In particular, every R-walk is R-connected.

If R is a quaternary relation on a set X and $Y \subseteq X$, then there is a quaternary relation on Y induced by R, namely $R \cap Y^4$. The relational system $(Y, R \cap Y^4)$ is then called a *relational subsystem* of (X, R) and is denoted by Y for short. If a subset $A \subseteq Y$ is $R \cap Y^4$-connected or is an $R \cap Y^4$-component of Y, then we briefly say that it is R-connected or is an R-component of Y, respectively. And we say that Y *separates* X into exactly two R-components if the subset $X - Y$ of X has exactly two R-components.

We will need the following statement (Theorem 3.5) proved in [14]:

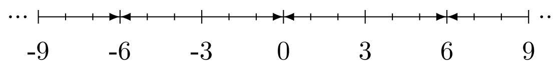

Figure 1: A portion of R.

Proposition 2.3. *Let R_j be a quaternary relation on a set X_j and $Y_j \subseteq X_j$ be a subset for every $j = 1, 2, ..., m$ ($m > 0$ an integer). If Y_j is R_j-connected for every $j = 1, 2, ..., m$, then $\bigotimes_{j=1}^{m} Y_j$ is $\prod_{j=1}^{m} R_j$-connected.*

As usual, two quaternary relational systems (X, R) and (Y, S) are said to be *isomorphic* if there exists a bijection $f : X \to Y$ such that, for every $(x_i \mid i < 4) \in X^4$, $(x_i \mid i < 4) \in R \Leftrightarrow (f(x_i) \mid i < 4) \in S$. Clearly, in this case, a subset $A \subseteq X$ is R-connected if and only if $f(A)$ is S-connected.

In the sequel, we will work with digital polygons in $\mathbb{Z}^2$ and digital polyhedra in $\mathbb{Z}^3$, namely with digital squares, digital triangles, digital cubes, digital prisms and digital tetrahedra. They are obtained by digitizing the squares and triangles in $\mathbb{R}^2$ and the cubes, prisms, and tetrahedra in $\mathbb{R}^3$ having vertices with integer coordinates, i.e., by their intersections with $\mathbb{Z}^2$ and $\mathbb{Z}^3$, respectively. We will also employ the concepts such as sides, faces and diagonals of the digital polygons or digital polyhedra. These are obtained by digitizing the corresponding concepts concerning the polygons or polyhedra (in $\mathbb{R}^2$ or $\mathbb{R}^3$) from which the digital polygons or digital polyhedra under consideration have been digitized.

3 A connectedness in $\mathbb{Z}^3$ associated with a quaternary relation

From now on, R will denote the quaternary relation on $\mathbb{Z}$ given as follows:
$R = \{(x_i \mid i < 4)$; there exists an integer k such that $x_i = (3+6k)+i$ for all $i < 4$ or $x_i = (3+6k)-i$ for all $i < 4\}$.
The quaternary relation R is demonstrated in Figure 1 where the quadruples belonging to R are represented by line segments directed from the first to the last members of the quadruples.

Since $\mathbb{Z}$ is evidently R-connected, Proposition 2.3 implies:

Theorem 3.1. *$\mathbb{Z}^m$ is R^m-connected for every positive integer m.*

Remark 3.2. *For every positive integer m, the relation R^m induces a graph G with the vertex set $\mathbb{Z}^m$ and the set of edges $\{\{p, q\};\ p, q \in \mathbb{Z}^m$ and there are $(r_i \mid i < 4) \in R^m$ and $i_0, 0 \le i_0 < 4$, such that $p = r_{i_0}$ and $q = r_{i_0+1}\}$. The elements (quadruples) of R^m may be considered to be paths of length 4 in the graph G. Then G is a graph with 4-path partition according to the terminology*

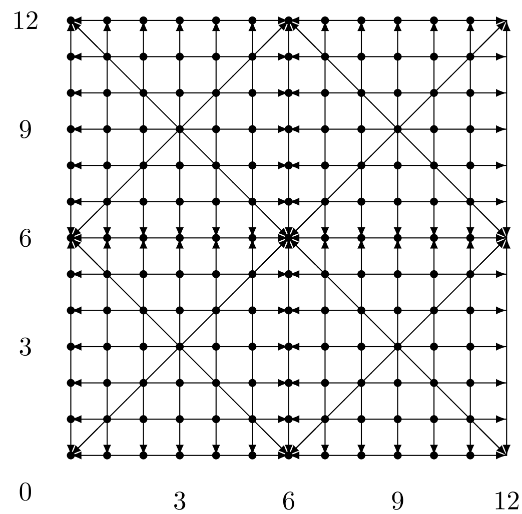

Figure 2: A portion of R^2.

introduced in [12] where graphs with n-path partitions ($n > 1$ an integer) were used for structuring the digital spaces.

The relation R^2 is demonstrated in Figure 2 where, as in Figure 1, the quadruples belonging to R^2 are represented by line segments directed from the first to the last members of the quadruples.

Definition 3.3. Each of the following subsets of $\mathbb{Z}^2$ will be called an R^2-*triangle*:

(1) $\{(x, y) \in \mathbb{Z}^2;\ 6k \le x \le 6k + 6,\ 6l \le y \le x + 6l - 6k\}$, $k, l \in \mathbb{Z}$,

(2) $\{(x, y) \in \mathbb{Z}^2;\ 6k \le x \le 6k + 6,\ x + 6l - 6k \le y \le 6l + 6\}$, $k, l \in \mathbb{Z}$,

(3) $\{(x, y) \in \mathbb{Z}^2;\ 6k \le x \le 6k + 6,\ 6l \le y \le 6k + 6l + 6 - x\}$, $k, l \in \mathbb{Z}$,

(4) $\{(x, y) \in \mathbb{Z}^2;\ 6k \le x \le 6k + 6,\ 6k + 6l + 6 - x \le y \le 6l + 6\}$, $k, l \in \mathbb{Z}$.

Every R^2-triangle is a digital rectangular triangle having 28 points. If we define R^2-*squares* to be the sets $\{(x, y) \in \mathbb{Z}^2;\ 6k \le x \le 6k+6,\ 6l \le y \le 6l+6\}$, $k, l \in \mathbb{Z}$, then every R^2-triangle is one of the two (digital) triangles obtained by splitting an R^2-square along one of its two diagonals - cf. Figure 2.

The following statement follows from (the proof of) Theorem 4.7 in [14]:

Lemma 3.4. *Every R^2-triangle is R^2-connected and so is every set obtained from an R^2-triangle by deleting some of its sides.*

Since every R^2-square is the union of a pair of R^2-triangles having a common hypotenuse (which is a diagonal of the square), Lemma 1 is valid for R^2-squares as well.

Now, we will proceed from $\mathbb{Z}^2$ to $\mathbb{Z}^3$. Clearly, the coordinate planes xy, xz, and zy, when regarded as the relational subsystems of $(\mathbb{Z}^3, R^3)$, are isomorphic to $(\mathbb{Z}^2, R^2)$. The same is true for the digital planes $\{(x, y, 6k);\ (x, y) \in \mathbb{Z}^2\}$, $\{(x, 6k, z);\ (x, y) \in \mathbb{Z}^2\}$, and $\{(6k, y, z);\ (x, y) \in \mathbb{Z}^2\}$ where $k \in \mathbb{Z}$ (these planes may be obtained by shifting the coordinate planes along the coordinate axes perpendicular to them).

Definition 3.5. A subsets $P \subseteq \mathbb{Z}^3$ will be called an R^3-*prism* if:

(1) There are an R^2-triangle A in the coordinate plane xy and an integer $k \in \mathbb{Z}$ such that $P = \{(x, y, z) \in \mathbb{Z}^3;\ (x, y) \in A \text{ and } 6k \le z \le (6k + 6)\}$ or

(2) there are an R^2-triangle A in the coordinate plane xz and an integer $k \in \mathbb{Z}$ such that $P = \{(x, y, z) \in \mathbb{Z}^3;\ (x, z) \in A \text{ and } 6k \le y \le (6k + 6)\}$ or

(3) there are an R^2-triangle A in the coordinate plane yz and an integer $k \in \mathbb{Z}$ such that $P = \{(x, y, z) \in \mathbb{Z}^3;\ (y, z) \in A \text{ and } 6k \le x \le (6k + 6)\}$.

Similarly to subsets of the Euclidean space $\mathbb{R}^3$, a pair of subsets of $\mathbb{Z}^3$ is said to be *congruent* if one may be transformed into the other by a combination of a translation, a rotation and a reflection (i.e., mirroring). We will also use the concepts of a 3D *tiling* or, equivalently, *tessellation* (see [3]) restricted from $\mathbb{R}^3$ to $\mathbb{Z}^3$. More precisely, we will work with a face-to-face tiling (tessellation) of a digital polyhedron in $\mathbb{Z}^3$ with certain digital polyhedra (so that any two polyhedra of such a tiling are disjoint or only share one vertex or one full edge or one full face).

Clearly, all R^3-prisms are congruent to each other and each of them is a digital triangular prism with 196 points. If we define R^3-*cubes* to be the sets $\{(x, y, z) \in \mathbb{Z}^3;\ 6k \le x \le 6k + 6,\ 6l \le y \le (6l + 6),\ 6m \le z \le (6k + 6)\}$, $k, l, m \in \mathbb{Z}$, then each R^3-prism is one of the two (digital) prisms obtained by splitting an R^3-cube along a plane perpendicular to a face of the cube and containing a diagonal of the face. In other words, every R^3-cube may be tiled (in six ways) with a pair of R^3-prisms. A splitting of an R^3-cube into a pair of R^3-prisms is demonstrated in Figure 3 where only the edges of the pair of R^3-prisms are visualized. Thus, every R^3-cube gives rise to 12 different

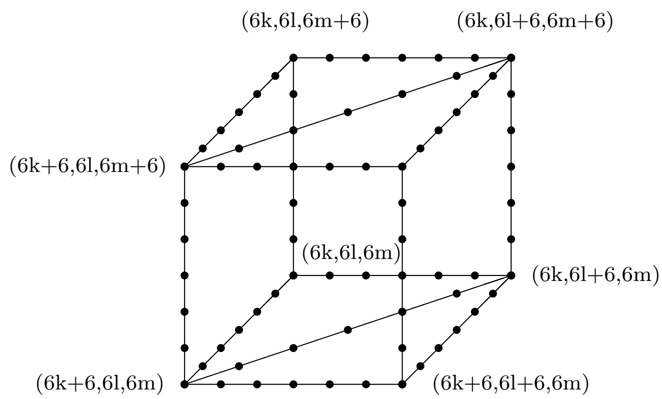

Figure 3: Splitting an R^3-cube into a pair of R^3-prisms

R^3-prisms and the cube is the union of any pair of them having a common face.

The following digital 3D Jordan-Brouwer separation theorem follows from [15], Theorem 2:

Theorem 3.6. *Let S be the boundary surface of such a digital polyhedron that may be tiled with R-prisms. Then S separates $\mathbb{Z}^3$ into exactly two R^3-components and the union of S with each of them is R^3-connected.*

The goal of this note is to improve Theorem 3.6 by proving its statement for boundary surfaces of the polyhedra that may be face-to-face tiled with certain tetrahedra finer than the R^3-prisms. Of course, if a triangular prism is one of the two prisms obtained by cutting a cube (in $\mathbb{R}^3$) by a plane that is perpendicular to a face of the cube and contains a diagonal of the face, then the prism can be tessellated with three tetrahedra congruent to each other and such a tessellation may be done in two different ways. Each of the two tessellations is called *canonical* and this concept applies also to the digital case. Thus, there are two canonical tessellations of every R^3-prism, each with three digital tetrahedra congruent to each other.

Definition 3.7. Each digital tetrahedron of a canonical tessellation of an R^3-prism will be called an R^3-*tetrahedron*.

A canonical tessellation of an R^3-prism is demonstrated in Figure 4 where only the edges of the three R^3-tetrahedra of the tessellation are visualized. Clearly, each of the three R^3-tetrahedra is a digital tetrahedron with 84 points.

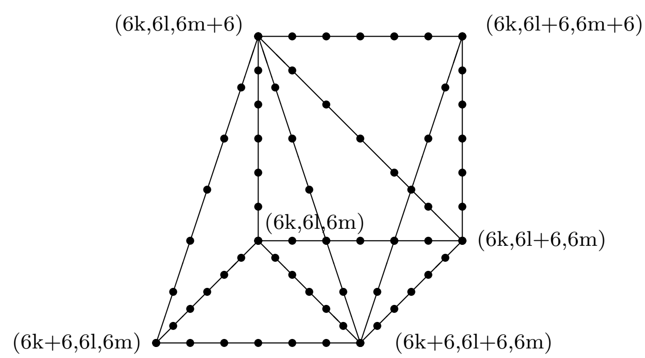

Figure 4: A canonical tessellation of an R^3-prism.

Every R^3-cube may be tessellated, in 4 ways, with 6 different (but congruent) R^3-tetrahedra. Thus, each R^3-cube gives rise to 24 different but congruent R^3-tetrahedra:

Given $k, l, m \in \mathbb{Z}$, put $A = (6k, 6l, 6m)$, $B = (6k + 6, 6l, 6m)$, $C = (6k + 6, 6l+6, 6m)$, $D = (6k, 6l+6, 6m)$, $E = (6k, 6l, 6m+6)$, $F = (6k+6, 6l, 6m+6)$, $G = (6k + 6, 6l + 6, 6m + 6)$, $H = (6k, 6l + 6, 6m + 6)$. The 24 R^3-tetrahedra are $ABCE$, $ACDE$, $CDEH$, $ABDH$, $BCDH$, $ABEH$, $BEFH$, $BFGH$, $BCGH$, $CEFG$, $CEGH$, $BCEF$, $BCFG$, $BFGH$, $BCDF$, $ABDF$, $ADEF$, $DEFH$, $ACDG$, $ADGH$, $AEGH$, $ABFG$, $AEFG$, and $ABCG$.

Since the digital space $\mathbb{Z}^3$ may be tiled with the (congruent) R^3-cubes, it may be tiled with (congruent) R^3-tetrahedra.

Proposition 3.8. *Every R^3-tetrahedron is R^3-connected and so is every set obtained from an R^3-tetrahedron by removing some of its faces.*

Proof. Let $k, l, m \in \mathbb{Z}$ and let T be the R^3-tetrahedron with vertices $(6k, 6l, 6m)$, $(6k + 6, 6l, 6m)$, $(6k + 6, 6l + 6, 6m)$, $(6k, 6l, 6m + 6)$ (see Figure 4). Put $T_1 = \{(x, y, z) \in T;\ 6k \le x \le 6k + 3,\ 6l \le y \le x + 6l - 6k\}$, $T_2 = \{(x, y, z) \in T;\ 6k + 3 \le x \le 6k + 6,\ 6l \le y \le 6l + 3)\}$, $T_3 = \{(x, y, z) \in T;\ 6k + 3 \le x \le 6k + 6,\ 6l + 3 \le y \le x + 6l - 6k\}$. Then $T = T_1 \cup T_2 \cup T_3$. Put $S = \{(x, y, z) \in T;\ z = 6m\}$, $S_1 = T_1 \cap T_2$ and $S_2 = T_2 \cap T_3$.
Clearly, every point of T_1 can be joined by an R^3-walk (consisting of at most two R^3-initial segments) with a point of S, namely with the orthogonal projection of the point on S. Since S is R^3-connected (it is isomorphic to an R^2-triangle), every pair of points of S can be joined by an R^3-walk in S. Therefore, every pair of points of T_1 can be joined by an R^3-walk in T. Further, every point of T_2 can be joined by an R^3-walk (an R^3-initial segment)

with a point of S_1, namely with the orthogonal projection of the point on S_1. Since $S_1 \subseteq T_1$, every pair of points of S_1 can be joined by an R^3-walk in T. Therefore, every pair of points of T_2 can be joined by an R^3-walk in T and this is true also if one of the points belongs to T_1 and the other one belongs to T_2. Finally, every point of T_3 may be joined by an R^3-walk (an R^3-initial segment) with a point of S_2, namely with the orthogonal projection of the point on S_2. Since $S_2 \subseteq T_2$, every pair of points of S_2 can be joined by an R^3-walk in T. Therefore, every pair of points of T_3 can be joined by an R^3-walk in T and this is true also if one of the points belongs to T_2 and the other one belongs to T_3. Consequently, every pair of points of T can be joined by an R^3-walk in T. Therefore, T is R^3-connected.
If T is the set obtained from the R^3-tetrahedron by removing some of its faces, then the proof of R^3-connectedness of T is much the same (note that, in this case, S is obtained from an R^3-triangle by removing some of its sides, hence still connected). And for the other 23 R^3-tetrahedra the proof is analogous because they are congruent to T. □

Since every R^3-prism may be tiled with R^3-tetrahedra, Proposition 3.8 is valid also for R^3-prisms (as well as for R^3-cubes). The whole digital space $\mathbb{Z}^3$ may evidently be tiled with R^3-cubes, hence also with R^3-tetrahedra.

Theorem 3.9. (Digital 3D Jordan-Brouwer separation theorem) *Let S be the boundary surface of a polyhedron that may be tiled with R^3-tetrahedra. Then S separates $\mathbb{Z}^3$ into exactly two R^3-components and the union of S with each of them is R^3-connected.*

Proof. Let S satisfy the conditions of the statement so that S is the union of all faces of a polyhedron $T_F \subseteq \mathbb{Z}^3$ that can be tiled with R^3-tetrahedra. Then the set $T_I = (\mathbb{Z}^3 - T_F) \cup S$ may also be tiled with R^3-tetrahedra. Since all R^3-tetrahedra are R^3-connected and so are all subsets of $\mathbb{Z}^3$ obtained from the tetrahedra by removing some of their faces, T_F, $T_F - S$, T_I, and $T_I - S$ are R^3-connected, too. It is obvious that every R^3-walk $C = (z_i \mid i \leq k)$, $k > 0$ an integer, joining a point of $T_F - S$ with a point of $T_I - S$ meets S (i.e., meets a face of an R^3-tetrahedron contained in S). Thus, the set $\mathbb{Z}^3 - S = (T_F - S) \cup (T_I - S)$ is not R^3-connected. Hence, $T_F - S$ and $T_I - S$ are R^3-components of $\mathbb{Z}^3 - S$, $T_F - S$ finite and $T_I - S$ infinite, with T_F and T_I R^3-connected. The proof is complete. □

Remark 3.10. If we define R to be a binary or ternary relation on $\mathbb{Z}$, i.e., to be obtained by replacing $i < 4$ with $i < 2$ or $i < 3$ in the definition of R at the beginning of this section, then Theorem 3.9 is not valid. More precisely, if R is binary, then the R^3-connectedness coincide with the connectedness with respect to the Khalimsky topology on $\mathbb{Z}^3$. And, if R is ternary, then the sets obtained from an R^3-tetrahedra by deleting its faces is not R^3-connected. The cases of the relation R with arities greater than 4 will be studied in a forthcoming paper.

4 Conclusion

We have found a connectedness structure for the digital space $\mathbb{Z}^3$, namely the quaternary relation R^3, which can be used to obtain a digital 3D Jordan-Brouwer separation theorem (Theorem 3.9). An advantage of the R^3-Jordan surfaces provided by Theorem 3.9 over the Jordan surfaces with respect to the Khalimsky topology on $\mathbb{Z}^3$ (proposed in [7]) is that the former may bend at an acute dihedral angle $\frac{\pi}{4}$ while the latter may never bend in a dihedral angle less than $\frac{\pi}{2}$. And, since every R^3-prism may be tiled with three R^3-tetrahedra, Theorem 3.9 implies the digital 3D Jordan-Brouwer separation theorem proved in [15]. In other words, the variety of digital Jordan surfaces provided by Theorem 3 is richer and containing finer surfaces than the one provided in [15].

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