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Some results in the level sets of some bipolar fuzzy relations

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Abstract

In this paper, some properties of a BPFR were studied. Also, level sets of BPFRs were studied with their properties. Some results were given and, in some cases, examples and counter-examples were constructed. In particular, we have shown that if a BPFR ϱ is a subset of another BPFR ρ, (s,t)-level (or strong level) subsets of ϱ are classical subsets of the (s,t)-level (or strong level) subsets of ρ and that the strong level subsets of a BPFR is a subset of the ordinary level subset of the same among others. By these, we have improved on some of the results in the previous work and have laid the right foundation for further works on these concepts.

1 INTRODUCTION

In 1965, Zadeh [18] started fuzzy set. Since then, fuzzy sets and their applications have been vigorously studied in various disciplines. In 1971, Zadeh [19] also defined the notion of fuzzy relations and fuzzy orderings as the generalizations of usual relations and orderings respectively. For more on fuzzy relations, please refer to [6, 7, 9, 10, 13]. Meanwhile, fuzzy matrix theory was first introduced in 1977 by Thomason [17] as a branch of fuzzy set.

Atanassaov [3, 4], introduced intuitionistic fuzzy set (IFS) which is an extension of fuzzy sets. Rosenfeld [16] introduced fuzzy subgroup of a group,

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which was redefined by Anthony and Sherwood [2], and was later generalized as intuitionistic fuzzy group (IFG) by Biswas [8]. This has motivated the interest of many researchers towards the generalization of IFG. In [20], Zhan and Tan noted that intuitionistic fuzzy subgroup is a generalization of fuzzy subgroup [16]. Deschrijver and Kerre [11] worked on the composition of intuitionistic fuzzy relations.

Zhang [21] introduced the concept of bipolar fuzzy sets (BPFSs) as a generalisation of fuzzy sets. Lee [15] extended fuzzy sets to bipolar valued fuzzy sets with membership values in $[-1, 1]$, and also, in [14], compared it with other fuzzy sets. Akran and Dudek [1] have extended the notion to graph theory. Also, Azhagappan and Kamaraj [5] have extended the notion to topological spaces. For more applications of bipolar fuzzy sets refer to [15].

Although Dudziak and Pekala [12] reported that they have studied bipolar fuzzy relations (BPFRs) but they have actually studied intuitionistic fuzzy set of [3, 4]. It was Lee and Hur [13], that introduced BPFRs. In this paper, we are going to give some additional results of BPFRs.

The main motivation for this article is to improve on some of the results in the work of Hur [13], as hard as it may be to say, so that further works on these concepts can be laid on the right foundation. The remaining part of this paper will be as follow: Section 2 gives some preliminary definitions and results that would be used in the main work; Section 3 is where the main results (relating to level subsets of BPFRs and some examples and counter examples) are discussed and Section 4 is the conclusion.

2 PRELIMINARIES

In this section, we will provide important results that are necessary and useful in understanding and proving the main results presented in this paper.

Definition 2.1 ([18])**.** A fuzzy subset of χ *(the universe)* is a class of objects in χ whose membership degree can be determined by $\mu : \chi \longrightarrow [0, 1]$.

Definition 2.2 ([13])**.** Let χ be a non empty set. The pair $U = (U^+, U^-)$ is a bipolar fuzzy subset of χ if $U^+ : \chi \longrightarrow [0, 1]$ and $U^- : \chi \longrightarrow [-1, 0]$ are mappings.

Remark 2.3 ([13])**.** $U^+(x)$ is the degree to which x satisfies the property of the BPFS and $U^-(x)$ is the degree to which it satisfies its counter property. The empty set is $0_{bp} = (0^+_{bp}, 0^-_{bp})$ and the whole set is $1_{bp} = (1^+_{bp}, 1^-_{bp})$. The set of all BPFSs in χ is denoted as $BPF(\chi)$. For the union, intersection, equality and complement of BPFSs, refer to [18].

Definition 2.4 ([13]). $\varrho = (\varrho^-, \varrho^+)$ is a BPFR from χ to Υ with membership functions $\varrho^- : \chi \times \Upsilon \longrightarrow [-1, 0]$ and $\varrho^+ : \chi \times \Upsilon \longrightarrow [1, 0]$. If $\chi = \Upsilon$, ϱ is called BPFR on χ. The empty BPFR on χ, denoted by $\varrho_0 = (\varrho_0^-, \varrho_0^+)$ is defined as $\varrho_0^+(m, n) = 0 = \varrho_0^-(m, n)$ and the whole BPFR on χ denoted by $\varrho_1 = (\varrho_1^-, \varrho_1^+)$ is defined as $\varrho_1^+(m, n) = 1$ and $\varrho_1^-(m, n) = -1$ for each $(m, n) \in \chi \times \chi$.

Remark 2.5. The set of all BPFRs from χ to Υ is denoted by $BPFR(\chi \times \Upsilon)$.

Definition 2.6 ([13]). Let $\varrho \in BPFR(\chi \times \Upsilon)$. Then, the complement of ϱ, denoted by

$$\varrho^{\complement} = ((\varrho^{\complement})^-, (\varrho^{\complement})^+),$$

is a BPFR from χ to Υ and is defined by

$$(\varrho^{\complement})^+(m, n) = 1 - \varrho^+(m, n)$$

and

$$(\varrho^{\complement})^-(m, n) = -1 - \varrho^-(m, n)$$

for each $(m, n) \in \chi \times \Upsilon$.

Definition 2.7 ([13]).

$$\varrho^{-1} = ((\varrho^{-1})^-, (\varrho^{-1})^+),$$

a BPFR from Υ to χ defined as

$$\varrho^{-1}(n, m) = \varrho(m, n),$$

for each $(n, m) \in \Upsilon \times \chi$, is called the inverse of ϱ.

Definition 2.8 ([13]). The BPFR I_x on χ defined for each $(m, n) \in \chi \times \chi$ as

$$I_m^+(m, n) = \begin{cases} 1 & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n \end{cases}$$

and

$$I_m^-(m, n) = \begin{cases} -1 & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n \end{cases}$$

is the identity BPFR.

Definition 2.9 ([18]). For $(p, q), (r, z) \in [-1, 0] \times [0, 1]$, the order " $\leq$ " and the equality " $=$ " are defined as follows:

1. $(p, q) \leq (r, z)$ iff $p \geq r$ and $q \leq z$;

2. $(p, q) = (r, z)$ iff $p = r$ and $q = z$.

Definition 2.10 ([13]). Let $\varrho \in BPFR(\chi \times \Upsilon)$ and $(s,t) \in [-1, 0] \times [0, 1]$, the set

$$[\varrho]^*_{(s,t)} = \{(m,n) \in \chi \times \Upsilon : \varrho^-(m,n) < s, \varrho^+(m,n) > t\}$$

is called the strong (s,t)-level subset of ϱ, and the set

$$[\varrho]_{(s,t)} = \{(m,n) \in \chi \times \Upsilon : \varrho^-(m,n) \leq s, \varrho^+(m,n) \geq t\}$$

is called the (s,t)-level subset of ϱ, both of which are classical relations.

Remark 2.11. $[\varrho]^*_{(s,t)}$ and $[\varrho]_{(s,t)}$ can actually be respectively referred to as strong bipolar level subset and bipolar level subset of a BPFR ϱ.

3 LEVEL SETS OF BPFRs

In this section, we state some generalizations of some results in the work of Hur [13]. In particular, **Proposition 3.1** is the generalization of the **Proposition 22** of [13]. Also, with reference to **Proposition 23** of [13], (1) and (2) are straightforward from the definitions. But (3) and (4) are not correct by some counter examples in this section. Besides, we give some more properties of the level sets of BPFRs.

Proposition 3.1. *Let $\varrho, \varsigma \in BPFR(\chi \times \Upsilon)$, and $(k,l), (s,t), (u,v) \in [-1, 0] \times [0, 1]$.*

1. *If $\varrho \subseteq \varsigma$, then $[\varrho]_{(k,l)} \subseteq [\varsigma]_{(k,l)}$ and $[\varrho]^*_{(k,l)} \subseteq [\varsigma]^*_{(k,l)}$.*
2. *If $(s,t) \leq (u,v)$, then $[\varrho]_{(u,v)} \subseteq [\varrho]_{(s,t)}$ and $[\varrho]^*_{(u,v)} \subseteq [\varrho]^*_{(s,t)}$.*

Proof. 1. Let $\varrho, \varsigma \in BPFR(\chi, \Upsilon)$ and $k, l \in [-1, 0] \times [0, 1]$. Suppose $\varrho \subseteq \varsigma$ and let $(m,n) \in [\varrho]_{(k,l)}$, then $\varrho^+(m,n) \leq \varsigma^+(m,n)$ and $\varrho^-(m,n) \geq \varsigma^-(m,n)$. But, $[\varrho]_{(k,l)} = \{(m,n)|\varrho^+(m,n) \geq l, \varrho^-(m,n) \leq k\}$. Hence, $\varsigma^+(m,n) \geq \varrho^+(m,n) \geq l$ and $\varsigma^-(m,n) \leq \varrho^-(m,n) \leq k$. Therefore, $(m,n) \in [\varsigma]_{(k,l)}$. Thus, $[\varrho]_{(k,l)} \subseteq [\varsigma]_{(k,l)}$.

Furthermore, $[\varrho]^*_{(k,l)} = \{(m,n)|\varrho^+(m,n) > l, \varrho^-(m,n) < k\}$. Hence,

$$\varsigma^+(m,n) \geq \varrho^+(m,n) > l$$

and

$$\varsigma^-(m,n) \leq \varrho^-(m,n) < k.$$

Therefore, $(m,n) \in [\varsigma]^*_{(k,l)}$. Thus, $[\varrho]^*_{(k,l)} \subseteq [\varsigma]^*_{(k,l)}$.

2. Let $\varrho \in BPFR(\chi \times \Upsilon)$, and let (s,t), $(u,v) \in [-1,0] \times [0,1]$. If $(s,t) \leq (u,v)$, then $s \geq u$ and $t \leq v$. Let $(m,n) \in [\varrho]_{(u,v)}$, then $\varrho^-(m,n) \leq u$ and $\varrho^+(m,n) \geq v$. Since $s \geq u$ and $t \leq v$, then $\varrho^-(m,n) \leq u \leq s$ and $\varrho^+(m,n) \geq v \geq t$. Hence, $(m,n) \in [\varrho]_{(s,t)}$, thus $[\varrho]_{(u,v)} \subseteq [\varrho]_{(s,t)}$.

Furthermore, for $(m,n) \in [\varrho]^*_{(u,v)}$, we have that $\varrho^-(m,n) < u, \varrho^+(m,n) > v$. Since $s \geq u$ and $t \leq v$, we have that $\varrho^-(m,n) < u \leq s$ and $\varrho^+(m,n) > v \geq t$. Hence, $[\varrho]^*_{(u,v)} \subseteq [\varrho]^*_{(s,t)}$. □

The following example is to illustrate the result of **Proposition 3.1**.

Example 3.2. Let the BPFRs ϱ and ς be listed as

$\varrho =$

	s	t	u
s	$(-0.3, 0.5)$	$(-0.6, 0.4)$	$(-0.8, 0.7)$
t	$(-0.2, 0.7)$	$(-0.5, 0.2)$	$(-0.1, 0.6)$
u	$(-0.4, 0.6)$	$(-0.7, 0.2)$	$(-0.5, 0.3)$

and

$\varsigma =$

	s	t	u
s	$(-0.5, 0.6)$	$(-0.8, 0.6)$	$(-1.0, 0.9)$
t	$(-0.4, 0.8)$	$(-0.7, 0.3)$	$(-0.3, 0.8)$
u	$(-0.6, 0.7)$	$(-0.9, 0.3)$	$(-0.7, 0.4)$

1. If $(a,b) = (-0.4, 0.6)$, then,

$$\begin{aligned} [\varrho]_{(a,b)} &= [\varrho]_{(-0.4,0.6)} \\ &= \left\{(m,n) \in \chi \times \Upsilon : \varrho^+(m,n) \geq 0.6, \varrho^-(m,n) \leq -0.4\right\} \\ &= \{(s,u),(u,s)\} \end{aligned}$$

and

$$\begin{aligned} [\varsigma]_{(a,b)} &= [\varsigma]_{(-0.4,0.6)} \\ &= \left\{(m,n) \in \chi \times \Upsilon : \varsigma^+(m,n) \geq 0.6, \varsigma^-(m,n) \leq -0.4\right\} \\ &= \{(s,s),(s,u),(s,t),(t,s),(u,s)\}. \end{aligned}$$

Hence, $[\varrho]_{(a,b)} \subseteq [\varsigma]_{(a,b)}$.
Furthermore,

$$\begin{aligned} [\varrho]^*_{(-0.4,0.6)} &= \left\{(m,n) \in \chi \times \Upsilon : \varrho^+(m,n) > 0.6, \varrho^-(m,n) < -0.4\right\} \\ &= \{(s,u)\} \end{aligned}$$

and

$$\begin{aligned} [\varsigma]^*_{(a,b)} &= [\varsigma]^*_{(-0.4,0.6)} \\ &= \left\{(m,n) \in \chi \times \Upsilon : \varrho^+(m,n)>0.6, \varrho^-(m,n)<-0.4\right\} \\ &= \{(s,u),(u,s)\}. \end{aligned}$$

Hence, $[\varrho]^*_{(a,b)} \subseteq [\varsigma]^*_{(a,b)}$.

2. For $(p,q) \leq (r,z)$, let $p=-0.4$, $q=0.7$, $r=-0.9$ and $z=0.8$, then,

$$\begin{aligned} [\varrho]_{(r,z)} &= [\varrho]_{(-0.9,0.8)} \\ &= \left\{(m,n) \in \chi \times \Upsilon : \varrho^+(m,n) \geq 0.8, \varrho^-(m,n) \leq -0.9\right\} \\ &= \emptyset \end{aligned}$$

and

$$\begin{aligned} [\varrho]_{(p,q)} &= [\varrho]_{(-0.4,0.6)} \\ &= \left\{(m,n) \in \chi \times \Upsilon : \varrho^+(m,n) \geq 0.6, \varrho^-(m,n) \leq -0.4\right\} \\ &= \{(s,u),(u,s)\}. \end{aligned}$$

Thus, $[\varrho]_{(r,z)} \subseteq [\varrho]_{(p,q)}$.

Similarly,

$$\begin{aligned} [\varrho]^*_{(r,z)} &= [\varrho]^*_{(-0.9,0.8)} \\ &= \left\{(m,n) \in \chi \times \Upsilon : \varrho^+(m,n)>0.8, \varrho^-(m,n)<-0.9\right\} \\ &= \emptyset \end{aligned}$$

and

$$\begin{aligned} [\varrho]^*_{(p,q)} &= [\varrho]^*_{(-0.4,0.6)} \\ &= \left\{(m,n) \in \chi \times \Upsilon : \varrho^+(m,n)>0.6, \varrho^-(m,n)<-0.4\right\} \\ &= \{(s,u)\}. \end{aligned}$$

Hence, $[\varrho]^*_{(r,z)} \subseteq [\varrho]^*_{(p,q)}$.

Proposition 3.3. *For* $(a,b) \in [-1,0] \times [0,1]$, *then,* $[\varrho]^*_{(a,b)} \subseteq [\varrho]_{(a,b)}$.

Proof. Let

$$[\varrho]^*_{(a,b)} = \left\{(m,n) \in \chi \times \Upsilon : \varrho^+(m,n)>b, \varrho^-(m,n)<a\right\}.$$

Let $a = \min\{a_i\}$ and $b = \max\{b_i\}$, where i is an integer, then we have

$$[\varrho]^*_{(a_i,b_i)} = \left\{(m,n) \in \chi \times \Upsilon : \varrho^+(m,n) > b_i, \varrho^-(m,n) < a_i\right\}.$$

And, for some integer j, we have b_j and a_j such that

$$\begin{aligned} [\varrho]_{(a,b)} &= \left\{(m,n) \in \chi \times \Upsilon : \varrho^+(m,n) \geq b, \varrho^-(m,n) \leq a\right\} \\ &= \cap\left\{(m,n) \in \chi \times \Upsilon : \varrho^+(m,n) \geq b, \varrho^-(m,n) \leq a\right\} \\ &= \cap\left\{(m,n) \in \chi \times \Upsilon : \varrho^+(m,n) \geq b > b_j, \varrho^-(m,n) \leq a < a_j\right\} \\ &= \cap\left\{(m,n) \in \chi \times \Upsilon : \varrho^+(m,n) > b_j, \varrho^-(m,n) < a_j\right\} \\ &= \cap[\varrho]^*_{(a_j,b_j)}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \cap[\varrho]^*_{(a_i,b_i)} &= \cap\left\{(m,n) \in \chi \times \Upsilon : \varrho^+(m,n) > b_i, \varrho^-(m,n) < a_i\right\} \\ &= \cap\left\{(m,n) \in \chi \times \Upsilon : \varrho^+(m,n) > b_i \geq b_j, \varrho^-(m,n) < a_i \leq a_j\right\} \\ &= \cap\left\{(m,n) \in \chi \times \Upsilon : \varrho^+(m,n) > b_j, \varrho^-(m,n) < a_j\right\} \\ &= \cap[\varrho]^*_{(a_j,b_j)} \\ &= [\varrho]_{(a,b)}. \end{aligned}$$

Since $(a_i, b_i) \leq (a, b)$ and by Proposition 3.1, we have

$$[\varrho]^*_{(a,b)} \subseteq [\varrho]^*_{(a_i,b_i)}.$$

Thus,

$$[\varrho]^*_{(a,b)} \subseteq \cap[\varrho]^*_{(a_i,b_i)} = [\varrho]_{(a,b)}.$$

□

The following are counter examples to **Proposition 23** (3) and (4) of [13].

Example 3.4. Consider the BPFR ϱ defined by

$\varrho =$

	s	t	u
s	$(-0.4, 0.7)$	$(-0.7, 0.6)$	$(-1.0, 0.7)$
t	$(-0.2, 0.9)$	$(-0.6, 0.4)$	$(-0.3, 0.6)$
u	$(-0.4, 0.6)$	$(-0.7, 0.4)$	$(-0.4, 0.5)$

1. From $(r, z) < (p, q)$, we get $r > p$ and $z < q$. Let $(p, q) = (-0.2, 0.6)$, where $(p, q) \in [-1, 0] \times [0, 1]$, then the possible values of (r, z) are listed as follows: $(0, 0)$, $(0, 0.1)$, $(0, 0.2)$, $(0, 0.3)$, $(0, 0.4)$, $(0, 0.5)$, $(-0.1, 0)$, $(-0.1, 0.1)$, $(-0.1, 0.2)$, $(-0.1, 0.3)$, $(-0.1, 0.4)$ and $(-0.1, 0.5)$. Thus

$$[\varrho]_{(0,0)} = \{(s,s), (s,t), (s,u), (t,s), (t,t), (t,u), (u,s), (u,t), (u,u)\}.$$

It is also established that

$$\begin{aligned}[\varrho]_{(0,0)} &= [\varrho]_{(0,0.1)} = [\varrho]_{(0,0.2)} = [\varrho]_{(0,0.3)} \\ &= [\varrho]_{(0,0.4)} = [\varrho]_{(-0.1,0)} = [\varrho]_{(-0.1,0.1)} \\ &= [\varrho]_{(-0.1,0.2)} = [\varrho]_{(-0.1,0.3)} = [\varrho]_{(-0.1,0.4)}.\end{aligned}$$

Hence,

$$[\varrho]_{(0,0.5)} = \{(s,s),(s,t),(s,u),(t,s),(t,u),(u,s),(u,u)\} = [\varrho]_{(-0.1,0.5)}.$$

Thus,

$$\bigcap_{(r,z)<(p,q)} [\varrho]_{(r,z)} = \{(s,s),(s,t),(s,u),(t,s),(t,u),(u,s),(u,u)\}.$$

Also,

$$[\varrho]_{(-0.2,0.6)} = \{(s,s),(s,u),(t,s),(t,u),(s,u),(u,s)\}.$$

Hence,

$$[\varrho]_{(p,q)} \neq \bigcap_{(r,z)<(p,q)} [\varrho]_{(r,z)}.$$

2. From $(r,z)>(p,q)$ we obtain $r<p$ and $z>q$. Let $(p,q) = (-0.2, 0.6)$, where $(p,q) \in [-1,0] \times [0,1]$.

Then the possible values of (r,z) are $(-0.3,0.7)$, $(-0.3,0.8)$, $(-0.3,0.9)$, $(-0.3,1)$, $(-0.4,0.7)$, $(-0.4,0.8)$, $(-0.4,0.9)$, $(-0.4,1)$, $(-0.5,0.7)$, $(-0.5,0.8)$, $(-0.5,0.9)$, $(-0.5,1)$, $(-0.6,0.7)$, $(-0.6,0.8)$, $(-0.6,0.9)$, $(-0.6,1)$, $(-0.7,0.7)$, $(-0.7,0.8)$, $(-0.7,0.9)$, $(-0.7,1)$, $(-0.8,0.7)$, $(-0.8,0.8)$, $(-0.8,0.9)$, $(-0.8,1)$, $(-0.9,0.7)$, $(-0.9,0.8)$, $(-0.9,0.9)$, $(-0.9,1)$, $(-1,0.7)$, $(-1,0.8)$, $(-1,0.9)$, and $(-1,1)$.

$$\begin{aligned}[\varrho]^*_{(-0.3,0.7)} &= [\varrho]^*_{(-0.3,0.8)} = [\varrho]^*_{(-0.3,0.9)} = [\varrho]^*_{(-0.3,1)} = [\varrho]^*_{(-0.4,0.7)} \\ &= [\varrho]^*_{(-0.4,0.8)} = [\varrho]^*_{(-0.4,0.9)} = [\varrho]^*_{(-0.4,1)} = [\varrho]^*_{(-0.5,0.7)} \\ &= [\varrho]^*_{(-0.5,0.8)} = [\varrho]^*_{(-0.5,0.9)} = [\varrho]^*_{(-0.5,1)} = [\varrho]^*_{(-0.6,0.7)} \\ &= [\varrho]^*_{(-0.6,0.8)} = [\varrho]^*_{(-0.6,0.9)} = [\varrho]^*_{(-0.6,1)} = [\varrho]^*_{(-0.7,0.7)} \\ &= [\varrho]^*_{(-0.7,0.8)} = [\varrho]^*_{(-0.7,0.9)} = [\varrho]^*_{(-0.7,1)} = [\varrho]^*_{(-0.8,0.7)} \\ &= [\varrho]^*_{(-0.8,0.8)} = [\varrho]^*_{(-0.8,0.9)} = [\varrho]^*_{(-0.8,1)} = [\varrho]^*_{(-0.9,0.7)} \\ &= [\varrho]^*_{(-0.9,0.8)} = [\varrho]^*_{(-0.9,0.9)} = [\varrho]^*_{(-0.9,1)} = [\varrho]^*_{(-1,0.7)} \\ &= [\varrho]^*_{(-1,0.8)} = [\varrho]^*_{(-1,0.9)} = [\varrho]^*_{(-1,1)} = \emptyset.\end{aligned}$$

Thus, $\bigcup_{(r,z)>(p,q)} [\varrho]^*_{(r,z)} = \emptyset$.

Also,

$$\begin{aligned}[\varrho]^*_{(p,q)} &= [\varrho]^*_{(-0.2,0.6)}\\ &= [\varrho^-(m,n)<-0.2, \varrho^+(m,n)>0.6]\\ &= \{(s,u),(t,s),(u,s)\}\end{aligned}$$

Hence, $\bigcup_{(r,z)>(p,q)}[\varrho]^*_{(r,z)} \neq [\varrho]^*_{(p,q)}$.

Proposition 3.5. *For each* $(p,q) \in [-1,0]\times[0,1]$, *then*

1. $[\varrho]_{(p,q)} = \bigcap_{(r,z)<(p,q)}[\varrho]^*_{(r,z)}$;
2. $[\varrho]_{(p,q)} \subseteq \bigcap_{(r,z)<(p,q)}[\varrho]_{(r,z)}$;
3. $[\varrho]^*_{(p,q)} \subseteq \bigcap_{(r,z)<(p,q)}[\varrho]^*_{(r,z)} \subseteq \bigcap_{(r,z)<(p,q)}[\varrho]_{(r,z)}$.

Proof. 1. If $(r,z)<(p,q)$, then $r>p$ and $z<q$. Let $(m,n)\in[\varrho]_{(p,q)}$, then $\varrho^-(m,n)\le p<r$ and $\varrho^+(m,n)\ge q>z$ so that $(m,n)\in[\varrho]^*_{(r,z)}\forall(r,z)<(p,q)$. Hence, $(m,n)\in\bigcap[\varrho]^*_{(r,z)}$, $\forall(r,z)<(p,q)$, which implies that $[\varrho]_{(p,q)}\subseteq\bigcap[\varrho]^*_{(r,z)}$. Also, let $(m,n)\in\bigcap[\varrho]^*_{(r,z)}, \forall(r,z)<(p,q)$, then $\varrho^-(m,n)<r$ and $\varrho^+(m,n)>z$, $\forall\ r>p$ and $\forall\ z<q$.

Since $p<r$ and $q>z$, the two possibilities are: (1) $\varrho^-(m,n)\le p$ and $\varrho^+(m,n)\ge q$; (2) $\varrho^-(m,n)>p$ $\varrho^+(m,n)<q$. The second is not a possibility because it will then imply that $\varrho^-(m,n)\ge r$ and $\varrho^+(m,n)\le z$, which contradicts $\varrho^-(m,n)<r$ and $\varrho^+(m,n)>z$. Hence, $(m,n)\in[\varrho]_{(p,q)}$. Thus, $\bigcap[\varrho]^*_{(r,z)}\subseteq[\varrho]_{(p,q)}$.

By the foregoing, $[\varrho]_{(p,q)}=\bigcap[\varrho]^*_{(r,z)}$.

2. By Proposition 3.3 and 3.5(1), $[\varrho]_{(p,q)} = \bigcap[\varrho]^*_{(r,z)} \subseteq \bigcap[\varrho]_{(r,z)}$, which implies that $[\varrho]_{(p,q)}\subseteq\bigcap[\varrho]_{(r,z)}$.

3. By Proposition 3.3 and 3.5(1), $[\varrho]^*_{(p,q)}\subseteq[\varrho]_{(p,q)}=\bigcap[\varrho]^*_{(r,z)}\subseteq\bigcap[\varrho]_{(r,z)}$. □

Proposition 3.6. *If* $(r,z)\le(p,q)$, *then* $\bigcap[\varrho]^*_{(r,z)}\subseteq\bigcap[\varrho]_{(r,z)}=[\varrho]_{(p,q)}$.

Proof. Let $(m,n)\in[\varrho]_{(p,q)}$. Then, $\varrho^-(m,n)\le p$ and $\varrho^+(m,n)\ge q$. But $r\ge p$ and $z\le q$. Hence, $\varrho^-(m,n)\le p\le r$ and $\varrho^+(m,n)\ge q\ge z$. Thus, $(m,n)\in[\varrho]_{(r,z)}$, whence $(m,n)\in\bigcap[\varrho]_{(r,z)}$, for all $(r,z)\le(p,q)$. So $[\varrho]_{(p,q)}\subseteq\bigcap[\varrho]_{(r,z)}$.

Furthermore, let $(m,n)\in\bigcap[\varrho]_{(r,z)}$, then $\varrho^-(m,n)\le r$ and $\varrho^+(m,n)\ge z$, for all $r\ge p$ and all $z\le q$, which imply that $\varrho^-(m,n)\le p$ and $\varrho^+(m,n)\ge q$. Thus, $(m,n)\in[\varrho]_{(p,q)}$ and $\bigcap[\varrho]_{(r,z)}\subseteq[\varrho]_{(p,q)}$. So, $[\varrho]_{(p,q)}=\bigcap[\varrho]_{(r,z)}$.

Therefore, by Proposition 3.3, $\bigcap[\varrho]^*_{(r,z)}\subseteq\bigcap[\varrho]_{(r,z)}=[\varrho]_{(p,q)}$. □

Proposition 3.7. *For* $(r,z)>(p,q)$, *then* $\bigcup[\varrho]^*_{(r,z)}\subseteq[\varrho]^*_{(p,q)}$.

Proof. By Proposition 3.5(1), for all possible values of (r, z), we have

$$[\varrho]_{(r,z)} = \bigcap [\varrho]^*_{(p,q)} \subseteq [\varrho]^*_{(p,q)}.$$

So,

$$\bigcup [\varrho]_{(r,z)} \subseteq \bigcup [\varrho]^*_{(p,q)}.$$

By Proposition 3.3, we get

$$\bigcup [\varrho]^*_{(r,z)} \subseteq \bigcup [\varrho]_{(r,z)} \subseteq \bigcup [\varrho]^*_{(p,q)},$$

whence $\bigcup [\varrho]^*_{(r,z)} \subseteq \bigcup [\varrho]^*_{(p,q)}$. □

Proposition 3.8 provides the correct form of **Proposition 23** (4) of [13].

Proposition 3.8. *For* $(p, q) < (r, z)$*, then* $[\varrho]^*_{(p,q)} = \bigcup [\varrho]_{(r,z)}$.

Proof. As a matter of fact, by Proposition 3.5(1),

$$[\varrho]_{(r,z)} = \bigcap [\varrho]^*_{(p,q)} = [\varrho]^*_{(p,q)}, \ \forall (r, z) > (p, q),$$

which implies that

$$\bigcup [\varrho]_{(r,z)} = \bigcup [\varrho]^*_{(p,q)} = [\varrho]^*_{(p,q)}, \ \forall (r, z) > (p, q).$$

□

4 CONCLUSION

In this paper, we have shown that if a BPFR ϱ is a subset of another BPFR ρ, (s, t)-level (or strong level) subsets of ϱ are classical subsets of the (s, t)-level (or strong level) subsets of ρ and that the strong level subsets of a BPFR is a subset of the ordinary level subset of the same among others. Some counter examples to some results of level sets of BPFRs given by [13] have been provided. Furthermore, some additional properties of (a, b)-level set of BPFRs were proposed and investigated. This work has provided foundation for future studies in the area of partitioning a bipolar fuzzy group and quotient bipolar fuzzy group which would be a generalization of the classical quotient group and the fuzzy quotient group. To this end, the results in this paper can be used to extend the frontier of knowledge in fuzzy group theory.

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