DOI: 10.2478/auom-2024-0032
An. Şt. Univ. Ovidius Constanţa **Vol. 32(3),2024, 125–148**

A note on almost sure exponential stability of θ-Euler-Maruyama approximation for neutral stochastic differential equations with time-dependent delay when $\theta \in (\frac{1}{2}, 1)$

Maja Obradović and Marija Milošević

Abstract

This paper is motivated by the paper [2]. The main aim of this paper is to extend the stability result from [16], related to the θ-Euler-Maruyama method ($\theta \in (\frac{1}{2}, 1)$) for a class of neutral stochastic differential equations with time-dependent delay. The theta method is defined such that, in general case, it is implicit in both drift coefficient and neutral term. Sufficient conditions of the a.s. exponential stability of the θ-Euler-Maruyama method, including the linear growth condition on the drift coefficient of the equation, are revealed. The stability result is established for larger class of neutral terms than that considered in the second cited paper. An example is provided to support the main results of the paper.

1 Introduction

(Neutral) stochastic differential delay equations and (neutral) stochastic functional differential equations are considered by many authors (see, for example

Key Words: θ-Euler-Maruyama method, Neutral stochastic differential equations, Time-dependent delay, Almost sure exponential stability, Khasminskii-type condition.
2010 Mathematics Subject Classification: Primary 60H10; Secondary 65C20.
Received: 12.09.2023
Accepted: 20.12.2023

[5, 21, 1, 9, 10, 4, 3, 6, 7, 14, 11, 12]), since these equations describe many real-life phenomena. As known from the existing literature, a very significant task in numerical analysis of stochastic differential equations is to reveal the conditions under which the exact and approximate solution share the same stability properties. For example, in [20, 23, 8], the authors investigated a.s. exponential stability of the Euler-Maruyama or backward Euler approximate solutions for stochastic differential equations. Also, in [15, 16], the θ-Euler-Maruyama method is considered for neutral stochastic differential equations with time-dependent delay, for $\theta \in \left[0, \frac{1}{2}\right]$ and $\theta \in (\frac{1}{2}, 1)$, respectively.

In this paper we will consider the θ-Euler-Maruyama method for a class of neutral stochastic differential equations with time-dependent delay, under the linear growth condition on the drift coefficient of the equation, among other conditions. The main result of this paper is influenced by the paper [2], where the theta method, for $\theta \in (\frac{1}{2}, 1]$ is considered for a class of neutral stochastic differential equations with constant delay and Markovian switching. It should be noted that in the cited paper certain sufficient conditions, without the linear growth condition on the drift and diffusion coefficients, are applied for obtaining the appropriate stability results. Additionally, in [13] and [17], the backward Euler method is studied for a class of neutral stochastic differential equations with bounded time-dependent delay, as well as for a class of neutral stochastic differential equations with unbounded delay and Markovian switching. On the other hand, the a.s. exponential stability of the θ-Euler-Maruyama method, when $\theta \in (\frac{1}{2}, 1)$, for neutral stochastic differential equations with time-dependent delay, which is considered in the present paper, required the application of the technique different than that from [13, 17]. It should be emphasized that in the paper [16], the a.s. exponential stability result for the same method and the same type of equations is established under certain highly nonlinear conditions, including an additional condition on the drift coefficient, comparing to the conditions from the paper [2]. The reason for that is the fact that in [16] the approximate equation is defined by parameterizing not only the drift coefficient by θ, but also the neutral term, which was not the case in [2]. Comparing to the paper [16], in the present paper, the linear growth condition on the drift coefficient of the equation is added, but without the additional condition on the drift coefficient although the approximate equation is defined by parameterizing the drift coefficient and the neutral term by θ, since the argument of the neutral term in the approximation could be the present state of the system described by our equation. On the other hand, in this paper the condition on the neutral term is weaker than the one in the paper [16].

This paper is organized in following way. After introducing the basic notation and hypotheses which are necessary for proving the main result of this paper in Section 2, we will impose the assumptions under which the discrete θ-Euler-Maruyama approximate solution for neutral stochastic differential equations with time-dependent delay is a.s. asymptotically exponentially stable, for $\theta \in (\frac{1}{2}, 1)$. Moreover, we will present the verification that the assumptions from the present paper, for the certain extent, allow the extension of the corresponding result from the paper [16]. However, these assumptions include the linear growth condition on the drift coefficient. In Section 3 an example and numerical simulations will be presented in order to illustrate our theory.

First, we will introduce some standard notation and definitions which are fundamental for the following consideration. Assume that all random variables and processes considered here are defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $w = \{w(t), t \geq 0\}$ be an m-dimensional standard Brownian motion and $\mathcal{F}_t = \sigma\{w(s), 0 \leq s \leq t\}$. Additionally, let $|x|$ stand for the Euclidean norm of $x \in R^d$ and, for simplicity, $|A|^2 = trace(A^T A)$ for matrix A, where A^T is the transpose of a vector or a matrix.

Let τ be a fixed positive number and let $C([-\tau, 0]; R^d)$ be the family of continuous functions $\varphi : [-\tau, 0] \to R^d$ with the supremum norm $\|\varphi\| = \sup_{-\tau\leq t\leq 0} |\varphi(t)|$. Moreover, let $C^b_{\mathcal{F}_0}([-\tau, 0]; R^d)$ be the family of $\mathcal{F}_0$-measurable, $C([-\tau, 0]; R^d)$-valued bounded random variables.

For the delay function $\delta : R_+ \to [0, \tau]$, which is Borel-measurable, we consider the following neutral stochastic differential equation with time-dependent delay

$$\begin{aligned} &d[x(t) - u(x(t - \delta(t)), t)] \\ &\quad = f(x(t), x(t - \delta(t)), t)dt + g(x(t), x(t - \delta(t)), t)dw(t), \; t \geq 0 \end{aligned} \tag{1}$$

and with the initial condition

$$x_0 = \varphi = \{\varphi(t) : t \in [-\tau, 0]\} \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^d), \tag{2}$$

where the functions

$$f : R^d \times R^d \times R_+ \to R^d, \; g : R^d \times R^d \times R_+ \to R^{d\times m}, \; u : R^d \times R_+ \to R^d$$

are all Borel-measurable and $x(t)$ is a d-dimensional state process.

The next hypotheses are essential for obtaining the main result of this paper:

$\mathcal{A}_1$ (Linear growth condition): There exists a positive constant K such that, for all $x, y \in R^d$ and all $t \geq 0$,

$$|f(x, y, t)|^2 \leq K(|x|^2 + |y|^2). \tag{3}$$

$\mathcal{A}_2$ (Contractivity condition): There exists a constant $\beta \in (0,1)$ such that, for all $x, y \in R^d$ and all $t \geq 0$,

$$|u(x,t) - u(y,t)| \leq \beta |x - y|. \tag{4}$$

Moreover, we will assume that $u(0,t) = 0$, $t \geq 0$, which, together with (4), implies that

$$|u(x,t)| \leq \beta |x|, \quad x \in R^d. \tag{5}$$

$\mathcal{A}_3$: The delay function $\delta : R_+ \to [0, \tau]$ is differentiable and $\delta'(t) \leq \bar{\delta} < 1$.
$\mathcal{A}_4$: There exists a constant $\eta > 0$ such that

$$|\delta(t) - \delta(s)| \leq \eta |t - s|,\ t, s \geq 0. \tag{6}$$

$\mathcal{A}_5$ (Khasminskii-type condition): There exist constants α_1 and α_2 for which $\alpha_1 > \frac{\alpha_2}{1-\bar{\delta}} > 0$, such that, for all $x, y \in R^d$ and all $t \geq 0$,

$$2(x - u(y,t))^T f(x,y,t) + |g(x,y,t)|^2 \leq -\alpha_1 |x|^2 + \alpha_2 |y|^2. \tag{7}$$

For the purpose of guaranteeing that the θ-Euler-Maruyama method, which will be considered in the sequel, is well defined, we introduce an additional assumption.

$\mathcal{C}_1$ (The one-sided Lipschitz conditions): Let $f \in C(R^d \times R^d; R^d)$ and suppose that there exist constants $\mu_1, \mu_2 > 0$ such that, for all $x, y, z \in R^d$ and all $t \geq 0$,

$$\langle x - y, f(x,z,t) - f(y,z,t) \rangle \leq \mu_1 |x - y|^2, \tag{8}$$
$$\langle x - y, f(z,x,t) - f(z,y,t) \rangle \leq \mu_2 |x - y|^2. \tag{9}$$

The main contribution of the present paper is the extension of the stability result from [16] partly by extending a class of neutral terms u. Precisely, in [16], it is assumed that the Lipschitz constant β of the neutral term u satisfies the condition

$$\beta^2 \in \Big(0, \frac{4}{99([(1-\eta)^{-1}] + 1)}\Big),$$

while in the present paper, the stability result will be obtained for

$$\beta^2 \in \Big(0, \frac{1}{4\big(9(1-\theta)^2 + \theta^2 + 3\big)([(1-\eta)^{-1}] + 1)}\Big), \quad \theta \in \Big(\frac{1}{2}, 1\Big).$$

It is easy to observe that for any $\theta \in \left(\frac{1}{2}, 1\right)$, the scope of β in the present paper is greater than that from [16]. Moreover, it could be observed that assumption

$\mathcal{A}_5$ is slightly weaker then $\mathcal{A}_4$ from [16]. Consequently, the technique which will be used in the sequel differs from the one from the previously cited paper.

The following lemmas are essential for proving the stability result in Section 2. The first one is an elementary inequality which will be applied several times in the proof of that result.

Lemma 1. *For all $a, b > 0$, $p \geq 1$, $c > 0$, we have that*

$$(a+b)^p \leq (1+c)^{p-1}(a^p + c^{1-p}b^p).$$

Lemma 2. *[[12], Lemma 3] Assume that (6) holds. For an arbitrary but fixed $i \in \{0, 1, 2, \ldots\}$, let $i - [\delta(i\Delta)/\Delta] = a$, where $a \in \{-n_*, -n_*+1, ..., 0, 1, ..., i\}$. Then,*

$$\#\{j \in \{0, 1, 2, \ldots\} : j - [\delta(i\Delta)/\Delta] = a\} \leq [(1-\eta)^{-1}] + 1,$$

where $\#S$ denotes the number of elements of the set S.

In a view of [12], one find that the assumptions $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_5$, as well as the local Lipschitz condition on f and g and hypotheses $f(0,0,t) = g(0,0,t) = 0$, $t \geq 0$ imply the existence and uniqueness of the global solution to Eq. (1), which is a.s. exponentially stable. Baring in mind these results from [12] we will determine the conditions under which the θ-Euler-Maruyama solution is a.s. exponentially stable. As will be shown in the sequel, for that purpose we will need, among other conditions, the linear growth condition $\mathcal{A}_2$ on the drift coefficient f. However, as mentioned earlier, we will weaken the condition for the neutral term u comparing to the corresponding condition from [16].

Let us consider the autonomous version of the initial equation (1), that is

$$\begin{aligned} x(t) = \varphi(0) + u(x(t-\delta(t))) - u(x(-\delta(0))) + \int_0^t f(x(s), x(s-\delta(s)))ds \quad (10)\\ + \int_0^t g(x(s), x(s-\delta(s)))dw(s), \quad t \geq 0, \end{aligned}$$

with the initial condition $x(t) = \varphi(t)$, $t \in [-\tau, 0]$. In that sense, instead of the assumptions $\mathcal{A}_1$-$\mathcal{A}_5$, let their autonomous versions hold.

The main result in this paper will be obtained for the θ-Euler-Maruyama solution defined in the paper [16]. Let $\Delta \in (0,1)$ be a step size, such that $\Delta = \tau/n_*$, for some integer $n_* > \tau$. Recall the discrete θ-Euler-Maruyama approximate solution q from [16], corresponding to Eq. (10) defined on the equidistant partition $k\Delta, k = -(n_*+1), -n_*, ..., -1, 0, 1, ...$ Also, set

$$\delta(-\Delta) = \delta(0), \quad q_{-(n_*+1)} = \varphi(-n_*\Delta). \tag{11}$$

Let $[\cdot]$ be the integer part function. Then, the mentioned θ-Euler-Maruyama approximate solution is defined as

$$q_k = \varphi(k\Delta),\ k = -n_*, -n_* + 1, ..., 0, \tag{12}$$

while, for $k \in \{0, 1, 2, ...\}$,

$$\begin{aligned} q_{k+1} = q_k &+ \theta u(q_{k+1-[\delta((k+1)\Delta)/\Delta]}) + (1-\theta)u(q_{k-[\delta(k\Delta)/\Delta]}) \\ &-\theta u(q_{k-[\delta(k\Delta)/\Delta]}) - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) \\ &+\theta f(q_{k+1}, q_{k+1-[\delta((k+1)\Delta)/\Delta]})\Delta + (1-\theta)f(q_k, q_{k-[\delta(k\Delta)/\Delta]})\Delta \\ &+g(q_k, q_{k-[\delta(k\Delta)/\Delta]})\Delta w_k, \end{aligned} \tag{13}$$

where $\Delta w_k = w((k+1)\Delta) - w(k\Delta)$. For convenience, we will use the notation

$$\begin{aligned} z_k &= q_k - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) - \theta u(q_{k-[\delta(k\Delta)/\Delta]}) \\ &\quad -\theta f(q_k, q_{k-[\delta(k\Delta)/\Delta]})\Delta, \\ f_k &= f(q_k, q_{k-[\delta(k\Delta)/\Delta]}), \quad g_k = g(q_k, q_{k-[\delta(k\Delta)/\Delta]}). \end{aligned}$$

In order to conclude that there exists unique θ-Euler-Maruyama approximate solution of Eq. (13), it should be noted that this equation is of the form

$$x = d + \theta\big(\Delta f(x,a)I_{A^c} + \Delta f(x,x)I_A + u(x)I_A\big), \quad x \in R^d, \tag{14}$$

for given $a, d \in R^d$, where $I_A = 1$ if $[\delta((k+1)\Delta)/\Delta] = 0$ and $I_A = 0$, otherwise. The desired conclusion then follows from the next lemma, which will be imposed without proof. The proof can be found in [15].

Lemma 3. *Assume that the condition (4) and the hypothesis $\mathcal{C}_1$ hold. If $\theta((\mu_1 + \mu_2)\Delta + \beta) < 1$, then, there exists unique solution to Eq. (14).*

2 Almost sure exponential stability of the θ-Euler -Maruyama method

In the existing literature one can find results on different aspects of the θ-Euler-Maruyama approximate method for different types of stochastic differential equations (see, for example, [24, 19, 22]). In these papers the authors studied primarily convergence and stability of the approximate solutions under consideration.

The main result of this section is a.s. asymptotic exponential stability of the theta method, that is, of the discrete θ-Euler-Maruyama approximate solution (13) for $\theta \in (\frac{1}{2}, 1)$. In the paper [13], the author established the a.s. exponential stability result for the backward Euler method (that is, for $\theta = 1$) without requiring the linear growth condition on the drift coefficient f. In that sense it should be emphasized that the technique which is used differs than that which will be used in this paper. Also it is important to observe that the technique mentioned above is successfully employed in [2], for a class of highly nonlinear neutral stochastic differential equations with constant delay and Markovian switching. In the cited paper the corresponding θ-Euler-Maruyama approximate equation is implicit only with respect to one argument (present state) of the drift coefficient. So, the main difficulty in the present paper is to treat the implicitness of the method with respect to both drift coefficient f and neutral term u, which is also parameterized by θ. Thus, in this section, we will extend to the certain extent the stability result from [16], applying slightly weaker Khasminskii-type and contractivity conditions comparing to those from the cited paper. On the other hand, the approach which will be used in the present paper requires the application of the linear growth condition on the drift coefficient f. So, in further analysis the following definition of the a.s. exponential stability of the numerical method plays an important role.

Definition 1. *The solution q_k of Eq. (13) is a.s. asymptotically exponentially stable if there exists a constant $\varepsilon > 0$ such that*

$$\limsup_{k\to\infty} \frac{\log|q_k|}{k\Delta} \le -\varepsilon, \ a.s.$$

for any bounded initial condition φ.

In what follows, we will prove the a.s. asymptotic exponential stability of the discrete θ-Euler-Maruyama solution given by (11)-(13).

Theorem 1. *Let the assumptions of Lemma 3 hold together with the assumptions $\mathcal{A}_1$–$\mathcal{A}_5$. Additionally, suppose that*

$$\beta^2 \in \Big(0, \frac{1}{4\big(9(1-\theta)^2+\theta^2+3\big)([(1-\eta)^{-1}]+1)}\Big), \tag{15}$$

$$\begin{aligned}\alpha_1 > & \frac{12([(1-\eta)^{-1}]+1)}{1-4\beta^2(9(1-\theta)^2+\theta^2+3)([(1-\eta)^{-1}]+1)} \\ & \times\Big(\alpha_2 + K\left(\frac{1}{[(1-\eta)^{-1}]+1}+1\right)+5(1-\theta)^2\beta^2\Big),\end{aligned} \tag{16}$$

$$\alpha_1\theta - \theta^2 K - (\alpha_2\theta + \theta^2 K + 4\beta^2(1-\theta)^2)([(1-\eta)^{-1}]+1) > 0. \tag{17}$$

If $\bar{\varepsilon}$ is the unique positive root of the equation

$$\alpha_1\theta - \theta^2 K - (\alpha_2\theta + \theta^2 K + 2\beta^2(1-\theta)^2(e^{\bar{\varepsilon}} + 1))([(1-\eta)^{-1}] + 1)e^{\bar{\varepsilon}\tau} = 0, \quad (18)$$

then, there exists $\Delta^* \in (0, 1]$ such that, for any $\Delta \in (0, \Delta^*)$ and any $\theta \in (\frac{1}{2}, 1)$, the θ-Euler-Maruyama approximate solution (13) is a.s. asymptotically exponentially stable.

Proof. Taking into account (13), we find that

$$\begin{aligned}|z_{k+1}|^2 = |z_k|^2 &+ [2(q_k - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) - \theta u(q_{k-[\delta(k\Delta)/\Delta]}))^T f_k \\ &+ |g_k|^2 + (1-2\theta)|f_k|^2\Delta]\Delta + m_k, \\ &= |z_k|^2 + [2(q_k - u(q_{k-[\delta(k\Delta)/\Delta]}))^T f_k + |g_k|^2 + (1-2\theta)|f_k|^2\Delta]\Delta \\ &\quad + 2(1-\theta)(u(q_{k-[\delta(k\Delta)/\Delta]}) - u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}))^T f_k\Delta + m_k, \quad (19)\end{aligned}$$

where

$$m_k = |g_k\Delta w_k|^2 - |g_k|^2\Delta + 2(z_k + f_k\Delta)^T g_k\Delta w_k. \quad (20)$$

For $c_0 > 0$, such that $0 < C < \frac{\alpha_1}{2(1+c_0)}$, we have

$$\begin{aligned}&(2\theta-1)|f_k|^2\Delta - C|z_k|^2 \\ &\quad = [(2\theta-1)\Delta - C\theta^2\Delta^2]|f_k|^2 \\ &\quad\quad + 2C\theta\Delta(q_k - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) - \theta u(q_{k-[\delta(k\Delta)/\Delta]}))^T f_k \\ &\quad\quad - C|q_k - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) - \theta u(q_{k-[\delta(k\Delta)/\Delta]})|^2 \\ &\quad = a|f_k + b(q_k - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) - \theta u(q_{k-[\delta(k\Delta)/\Delta]})|^2 \\ &\quad\quad - (ab^2 + C)|q_k - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) - \theta u(q_{k-[\delta(k\Delta)/\Delta]})|^2, (21)\end{aligned}$$

where $a = (2\theta-1)\Delta - C\theta^2\Delta^2$ and $b = \frac{C\theta\Delta}{a}$.

For $\theta \in (\frac{1}{2}, 1)$ we can determine small enough Δ^* such that, for each $\Delta \in (0, \Delta^*)$ we have that $a > 0$ and $-(ab^2 + C) \geq -\frac{\alpha_1}{2(1+c_0)}$. Precisely, we get

$$\Delta^* = \frac{2\theta - 1}{C\theta^2}\Big(1 - \frac{2C(1+c_0)}{\alpha_1}\Big). \quad (22)$$

By Lemma 1 and conditions (4) and (5) from $\mathcal{A}_2$ we have that, for any $\Delta \in (0, \Delta^*)$,

$$\begin{aligned}&(2\theta-1)|f_k|^2\Delta - C|z_k|^2 \\ &\quad \geq -\frac{\alpha_1}{2(1+c_0)}|q_k - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) - \theta u(q_{k-[\delta(k\Delta)/\Delta]})|^2 \\ &\quad = -\frac{\alpha_1}{2(1+c_0)}|q_k - u(q_{k-[\delta(k\Delta)/\Delta]}) - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) \\ &\quad\quad + (1-\theta)u(q_{k-[\delta(k\Delta)/\Delta]})|^2\end{aligned}$$

$$
\begin{aligned}
&\geq -\frac{\alpha_1}{1+c_0}|q_k - u(q_{k-[\delta(k\Delta)/\Delta]})|^2 \\
&\quad -\frac{\alpha_1}{1+c_0}|(1-\theta)(u(q_{k-[\delta(k\Delta)/\Delta]}) - u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}))|^2 \\
&\geq -\alpha_1|q_k|^2 + \alpha_2|q_{k-[\delta(k\Delta)/\Delta]}|^2 - \left(\frac{\alpha_1}{c_0}\beta^2 + \alpha_2\right)|q_{k-[\delta(k\Delta)/\Delta]}|^2 \\
&\quad -\frac{\alpha_1}{1+c_0}(1-\theta)^2\beta^2|q_{k-[\delta(k\Delta)/\Delta]} - q_{k-1-[\delta((k-1)\Delta)/\Delta]})|^2. \qquad (23)
\end{aligned}
$$

Applying $\mathcal{A}_4$, the estimate (23) becomes

$$
\begin{aligned}
&(2\theta-1)|f_k|^2\Delta - C|z_k|^2 \\
&\quad \geq 2(q_k - u(q_{k-[\delta(k\Delta)/\Delta]}))^T f_k + |g_k|^2 - \left(\frac{\alpha_1}{c_0}\beta^2 + \alpha_2\right)|q_{k-[\delta(k\Delta)/\Delta]}|^2 \\
&\quad -\frac{\alpha_1}{1+c_0}(1-\theta)^2\beta^2|q_{k-[\delta(k\Delta)/\Delta]} - q_{k-1-[\delta((k-1)\Delta)/\Delta]})|^2. \qquad (24)
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
&2(q_k - u(q_{k-[\delta(k\Delta)/\Delta]}))^T f_k + |g_k|^2 + (1-2\theta)|f_k|^2\Delta \\
&\quad \leq -C|z_k|^2 + \left(\frac{\alpha_1}{c_0}\beta^2 + \alpha_2\right)|q_{k-[\delta(k\Delta)/\Delta]}|^2 \\
&\quad +\frac{\alpha_1}{1+c_0}(1-\theta)^2\beta^2|q_{k-[\delta(k\Delta)/\Delta]} - q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2. \qquad (25)
\end{aligned}
$$

Substituting (25) into (19) and applying the assumption $\mathcal{A}_1$, we have

$$
\begin{aligned}
&|z_{k+1}|^2 \\
&\leq |z_k|^2 - C\Delta|z_k|^2 + \left(\frac{\alpha_1}{c_0}\beta^2 + \alpha_2\right)|q_{k-[\delta(k\Delta)/\Delta]}|^2\Delta \\
&\quad +\frac{\alpha_1}{1+c_0}(1-\theta)^2\beta^2|q_{k-[\delta(k\Delta)/\Delta]} - q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2\Delta \\
&\quad +2(1-\theta)(u(q_{k-[\delta(k\Delta)/\Delta]}) - u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}))^T f_k\Delta + m_k \\
&\leq |z_k|^2 - C\Delta|z_k|^2 + \left(\frac{\alpha_1}{c_0}\beta^2 + \alpha_2\right)|q_{k-[\delta(k\Delta)/\Delta]}|^2\Delta \\
&\quad +\frac{2\alpha_1}{1+c_0}(1-\theta)^2\beta^2|q_{k-[\delta(k\Delta)/\Delta]}|^2\Delta + \frac{2\alpha_1}{1+c_0}(1-\theta)^2\beta^2|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2\Delta \\
&\quad +2(1-\theta)\beta|q_{k-[\delta(k\Delta)/\Delta]} - q_{k-1-[\delta((k-1)\Delta)/\Delta]}||f_k|\Delta + m_k \\
&\leq |z_k|^2 - C\Delta|z_k|^2 + \left(\frac{\alpha_1}{c_0}\beta^2 + \alpha_2 + \frac{2\alpha_1}{1+c_0}(1-\theta)^2\beta^2\right)|q_{k-[\delta(k\Delta)/\Delta]}|^2\Delta \\
&\quad +\frac{2\alpha_1}{1+c_0}(1-\theta)^2\beta^2|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2\Delta
\end{aligned}
$$

$$
\begin{aligned}
&+(1-\theta)^2\beta^2|q_{k-[\delta(k\Delta)/\Delta]} - q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2\Delta + |f_k|^2\Delta + m_k \\
&\quad\le |z_k|^2 - C\Delta|z_k|^2 + \left(\frac{\alpha_1}{c_0}\beta^2 + \alpha_2 + \frac{2\alpha_1}{1+c_0}(1-\theta)^2\beta^2\right)|q_{k-[\delta(k\Delta)/\Delta]}|^2\Delta \\
&+\frac{2\alpha_1}{1+c_0}(1-\theta)^2\beta^2|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2\Delta + 2(1-\theta)^2\beta^2|q_{k-[\delta(k\Delta)/\Delta]}|^2\Delta \\
&+2(1-\theta)^2\beta^2|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2\Delta + K|q_k|^2\Delta + K|q_{k-[\delta(k\Delta)/\Delta]}|^2\Delta + m_k \\
&= |z_k|^2 - C\Delta|z_k|^2 + \left[\frac{\alpha_1}{c_0}\beta^2 + \alpha_2 + K + \left(\frac{2\alpha_1}{1+c_0} + 2\right)(1-\theta)^2\beta^2\right]|q_{k-[\delta(k\Delta)/\Delta]}|^2\Delta \\
&+\left(\frac{2\alpha_1}{1+c_0} + 2\right)(1-\theta)^2\beta^2|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2\Delta + K|q_k|^2\Delta + m_k. \qquad (26)
\end{aligned}
$$

Then for any arbitrary constant $A > 1$,

$$
\begin{aligned}
&A^{(k+1)\Delta}|z_{k+1}|^2 - A^{k\Delta}|z_k|^2 \\
&\le A^{(k+1)\Delta}\Big[|z_k|^2(1-C\Delta) \\
&\qquad +\left[\frac{\alpha_1}{c_0}\beta^2 + \alpha_2 + K + \left(\frac{2\alpha_1}{1+c_0} + 2\right)(1-\theta)^2\beta^2\right]|q_{k-[\delta(k\Delta)/\Delta]}|^2\Delta \\
&\qquad +\left(\frac{2\alpha_1}{1+c_0} + 2\right)(1-\theta)^2\beta^2|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2\Delta + K|q_k|^2\Delta + m_k\Big] \\
&\quad -A^{k\Delta}|z_k|^2 \\
&= A^{(k+1)\Delta}|z_k|^2(1 - C\Delta - A^{-\Delta}) \\
&\quad +A^{(k+1)\Delta}\left[\frac{\alpha_1}{c_0}\beta^2 + \alpha_2 + K + \left(\frac{2\alpha_1}{1+c_0} + 2\right)(1-\theta)^2\beta^2\right]|q_{k-[\delta(k\Delta)/\Delta]}|^2\Delta \\
&\quad +A^{(k+1)\Delta}\left(\frac{2\alpha_1}{1+c_0} + 2\right)(1-\theta)^2\beta^2|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2\Delta \\
&\quad +A^{(k+1)\Delta}K|q_k|^2\Delta + A^{(k+1)\Delta}m_k. \qquad (27)
\end{aligned}
$$

For simplicity, denote

$$
\begin{aligned}
&R_1(\Delta) = 1 - C\Delta - A^{-\Delta}, \\
&R_2 = \frac{\alpha_1}{c_0}\beta^2 + \alpha_2 + K + \left(\frac{2\alpha_1}{1+c_0} + 2\right)(1-\theta)^2\beta^2, \\
&R_3 = \left(\frac{2\alpha_1}{1+c_0} + 2\right)(1-\theta)^2\beta^2.
\end{aligned}
$$

Consequently, we have that

$$\begin{aligned}
&A^{k\Delta}|z_k|^2\\
&\le |z_0|^2 + R_1(\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|z_i|^2 + R_2\Delta\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-[\delta(i\Delta)/\Delta]}|^2\\
&+R_3\Delta\sum_{i=0}^{k-1}A^{(i+1)\Delta}|q_{i-1-[\delta((i-1)\Delta)/\Delta]}|^2+K\Delta\sum_{i=0}^{k-1}A^{(i+1)\Delta}|q_i|^2+M_k, \qquad (28)
\end{aligned}$$

where

$$M_k = \sum_{i=0}^{k-1} A^{(i+1)\Delta} m_i$$

is a martingale with $M_0 = 0$.

By the definition of z_k, condition (7) and Lemma 1, for any $c_0 > 0$, we get

$$\begin{aligned}
|z_k|^2 &\ge |q_k - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) - \theta u(q_{k-[\delta(k\Delta)/\Delta]})|^2\\
&\quad -2\theta\Delta(q_k - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) - \theta u(q_{k-[\delta(k\Delta)/\Delta]}))^T f_k\\
&= |q_k - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) - \theta u(q_{k-[\delta(k\Delta)/\Delta]})|^2\\
&\quad -2\theta\Delta(q_k - u(q_{k-[\delta(k\Delta)/\Delta]}))^T f_k\\
&\quad -2\theta(1-\theta)\Delta(u(q_{k-[\delta(k\Delta)/\Delta]}) - u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}))^T f_k\\
&\ge |q_k - (1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) - \theta u(q_{k-[\delta(k\Delta)/\Delta]})|^2\\
&\quad -\alpha_2\theta\Delta|q_{k-[\delta(k\Delta)/\Delta]}|^2\\
&\quad -(1-\theta)^2\Delta|u(q_{k-[\delta(k\Delta)/\Delta]}) - u(q_{k-1-[\delta((k-1)\Delta)/\Delta]})|^2 - \theta^2\Delta|f_k|^2\\
&\ge \frac{1}{1+c_0}|q_k|^2 - \frac{1}{c_0}|(1-\theta)u(q_{k-1-[\delta((k-1)\Delta)/\Delta]}) + \theta u(q_{k-[\delta(k\Delta)/\Delta]})|^2\\
&\quad +\alpha_1\theta\Delta|q_k|^2 - \alpha_2\theta\Delta|q_{k-[\delta(k\Delta)/\Delta]}|^2\\
&\quad -(1-\theta)^2\beta^2\Delta|q_{k-[\delta(k\Delta)/\Delta]} - q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2\\
&\quad -\theta^2K\Delta(|q_k|^2 + |q_{k-[\delta(k\Delta)/\Delta]}|^2)\\
&\ge \frac{1}{1+c_0}|q_k|^2 - \frac{2(1-\theta)^2\beta^2}{c_0}|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2 - \frac{2\theta^2\beta^2}{c_0}|q_{k-[\delta(k\Delta)/\Delta]}|^2\\
&\quad +\alpha_1\theta\Delta|q_k|^2 - \alpha_2\theta\Delta|q_{k-[\delta(k\Delta)/\Delta]}|^2 - (1-\theta)^2\beta^2\Delta(2|q_{k-[\delta(k\Delta)/\Delta]}|^2\\
&\quad +2|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2) - \theta^2K\Delta(|q_k|^2 + |q_{k-[\delta(k\Delta)/\Delta]}|^2)\\
&= \left(\frac{1}{1+c_0} + \alpha_1\theta\Delta - \theta^2K\Delta\right)|q_k|^2\\
&\quad + \left(-\frac{2\theta^2\beta^2}{c_0} - \alpha_2\theta\Delta - 2(1-\theta)^2\beta^2\Delta - \theta^2K\Delta\right)|q_{k-[\delta(k\Delta)/\Delta]}|^2
\end{aligned}$$

$$+\left(-\frac{2(1-\theta)^2\beta^2}{c_0} - 2(1-\theta)^2\beta^2\Delta\right)|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2. \tag{29}$$

Moreover, one can observe that $R_1(0) = 0$ and $R_1^{'}(\Delta) = -C + A^{-\Delta}\log A$. So, we will assume that $1 < A < e^C$, such that $R_1^{'}(\Delta) < 0$, $\Delta \in (0, \Delta^*)$, which implies that $R_1(\Delta) < 0$, $\Delta \in (0, \Delta^*)$. Then, substituting (29) into (28) we get that, for any $\Delta \in (0, \Delta^*)$,

$$A^{k\Delta}|z_k|^2 \le |z_0|^2 + K_1(\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_i|^2 + K_2(\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-[\delta(i\Delta)/\Delta]}|^2$$
$$+K_3(\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-1-[\delta((i-1)\Delta)/\Delta]}|^2 + M_k, \tag{30}$$

where

$$K_1(\Delta) = R_1(\Delta)\left(\frac{1}{1+c_0} + \alpha_1\theta\Delta - \theta^2 K\Delta\right) + K\Delta,$$
$$K_2(\Delta) = R_1(\Delta)\left(-\frac{2\theta^2\beta^2}{c_0} - \alpha_2\theta\Delta - 2(1-\theta)^2\beta^2\Delta - \theta^2 K\Delta\right) + R_2\Delta,$$
$$K_3(\Delta) = R_1(\Delta)\left(-\frac{2(1-\theta)^2\beta^2}{c_0} - 2(1-\theta)^2\beta^2\Delta\right) + R_3\Delta.$$

Observing that $K_3(\Delta) > 0$, $\Delta \in (0, \Delta^*)$, bearing in mind (11), we find that

$$K_3(\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-1-[\delta((i-1)\Delta)/\Delta]}|^2$$
$$\le K_3(\Delta)A^{\Delta}|q_{-1-[\delta(0)/\Delta]}|^2 + K_3(\Delta)A^{\Delta}\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-[\delta(i\Delta)/\Delta]}|^2. \tag{31}$$

So, the expression (30) becomes

$$A^{k\Delta}|z_k|^2 \le |z_0|^2 + K_1(\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_i|^2$$
$$+(K_2(\Delta) + K_3(\Delta)A^{\Delta})\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-[\delta(i\Delta)/\Delta]}|^2$$
$$+K_3(\Delta)A^{\Delta}|q_{-1-[\delta(0)/\Delta]}|^2 + M_k. \tag{32}$$

On the basis of Lemma 2, the second sum on the right-hand side of (32), can be estimate as

$$\begin{aligned}\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_{i-[\delta(i\Delta)/\Delta]}|^2 &\leq A^{n_*\Delta}\sum_{i=0}^{k-1} A^{(i-[\delta(i\Delta)/\Delta]+1)\Delta}|q_{i-[\delta(i\Delta)/\Delta]}|^2 \\ &\leq ([(1-\eta)^{-1}]+1)A^{n_*\Delta}\sum_{i=-n_*}^{k-1} A^{(i+1)\Delta}|q_i|^2. \qquad (33)\end{aligned}$$

Since $n_*\Delta = \tau$, the expression can be (32) becomes

$$A^{k\Delta}|z_k|^2 \leq X + h(\Delta)\sum_{i=0}^{k-1} A^{(i+1)\Delta}|q_i|^2 + M_k, \qquad (34)$$

where

$$\begin{aligned} X &= |z_0|^2 + K_3(\Delta)A^{\Delta}|q_{-1-[\delta(0)/\Delta]}|^2 \\ &\quad + (K_2(\Delta) + K_3(\Delta)A^{\Delta})([(1-\eta)^{-1}]+1)A^{\tau}\sum_{i=-n_*}^{-1} A^{(i+1)\Delta}|\varphi(i\Delta)|^2 \\ &< \infty, \qquad (35)\end{aligned}$$

and

$$h(\Delta) = K_1(\Delta) + (K_2(\Delta) + K_3(\Delta)A^{\Delta})([(1-\eta)^{-1}]+1)A^{\tau}. \qquad (36)$$

Note that

$$\begin{aligned} &h(\Delta) \\ &= R_1(\Delta)\left(\frac{1}{1+c_0} - A^{\tau}\frac{2\beta^2((1-\theta)^2A^{\Delta}+\theta^2)([(1-\eta)^{-1}]+1)}{c_0}\right) \\ &\quad + \Big[K + (R_2 + R_3A^{\Delta})([(1-\eta)^{-1}]+1)A^{\tau} + R_1(\Delta) \\ &\quad\quad \times(\alpha_1\theta - \theta^2K - (\alpha_2\theta + \theta^2K + 2\beta^2(1-\theta)^2(A^{\Delta}+1))([(1-\eta)^{-1}]+1)A^{\tau})\Big]\Delta \\ &= \Delta\Big\{\frac{R_1(\Delta)}{\Delta}\left(\frac{1}{1+c_0} - A^{\tau}\frac{2\beta^2((1-\theta)^2A^{\Delta}+\theta^2)([(1-\eta)^{-1}]+1)}{c_0}\right) \\ &\quad + K + (R_2 + R_3A^{\Delta})([(1-\eta)^{-1}]+1)A^{\tau} + R_1(\Delta) \\ &\quad\quad \times\Big[\alpha_1\theta - \theta^2K - (\alpha_2\theta + \theta^2K + 2\beta^2(1-\theta)^2(A^{\Delta}+1))([(1-\eta)^{-1}]+1)A^{\tau}\Big]\Big\}. \qquad (37)\end{aligned}$$

Then, for any $\Delta \in (0, \Delta^*)$, we have that

$$h(\Delta) \leq \Delta\Big\{\frac{R_1(\Delta)}{\Delta}\left(\frac{1}{1+c_0} - A^{\tau}\frac{2\beta^2((1-\theta)^2A+\theta^2)([(1-\eta)^{-1}]+1)}{c_0}\right)$$

$$+K + (R_2 + R_3 A)([(1-\eta)^{-1}] + 1)A^{\tau} + R_1(\Delta)h_1(A)\Big\},$$

where

$$h_1(A) \\ = \alpha_1\theta - \theta^2 K - (\alpha_2\theta + \theta^2 K + 2\beta^2(1-\theta)^2(A+1))([(1-\eta)^{-1}] + 1)A^{\tau}. \quad (38)$$

We find that

$$\begin{aligned} h_1'(A) &= [-(\alpha_2\theta + \theta^2 K + 2\beta^2(1-\theta)^2(A+1))\tau A^{\tau-1} - 2\beta^2(1-\theta)^2 A^{\tau}] \\ &\quad \times([(1-\eta)^{-1}] + 1) \\ &< 0. \end{aligned}$$

On the other hand,

$$h_1(1) = \alpha_1\theta - \theta^2 K - (\alpha_2\theta + \theta^2 K + 4\beta^2(1-\theta)^2)([(1-\eta)^{-1}] + 1).$$

On the basis of the assumption (17), we have that $h_1(1) > 0$. Bearing in mind that Eq.(18) has the unique positive root $\bar{\varepsilon} = \log \bar{A}$, we conclude that

$$\begin{aligned} h_1(e^{\varepsilon}) &= \alpha_1\theta - \theta^2 K - (\alpha_2\theta + \theta^2 K + 2\beta^2(1-\theta)^2(e^{\varepsilon}+1))([(1-\eta)^{-1}] + 1)e^{\varepsilon\tau} \\ &> 0, \end{aligned}$$

whenever $\varepsilon \in (0, \bar{\varepsilon})$ and, thus, whenever $A \in (1, \bar{A} \wedge e^C)$.

Let us denote $a(\Delta) = \frac{R_1(\Delta)}{\Delta}$, $\Delta \in (0, \Delta^*)$, such that $\lim_{\Delta\to 0} a(\Delta) = \log A - C < 0$ for any $A \in (1, \bar{A} \wedge e^C)$ and $a'(\Delta) = \frac{b(\Delta)}{\Delta^2}$, where

$$b(\Delta) = -1 + A^{-\Delta}(1 + \Delta \log A).$$

Having in mind that $b(0) = 0$ and

$$b'(\Delta) = -\Delta A^{-\Delta} \log^2 A < 0,\ \Delta \in (0, \Delta^*),\ A \in (1, \bar{A} \wedge e^C),$$

we find that $b(\Delta) < 0$, $\Delta \in (0, \Delta^*)$, $A \in (1, \bar{A} \wedge e^C)$. Consequently, we have that $a'(\Delta) < 0$, that is, $a(\Delta)$ is decreasing function on $(0, \Delta^*)$ for any $A \in (1, \bar{A} \wedge e^C)$. Obviously, as $\log A - C > a(\Delta)$, $\Delta \in (0, \Delta^*)$, for any $A \in (1, \bar{A} \wedge e^C)$, if we show that

$$\begin{aligned} h_2 &:= (\log A - C)\left(\frac{1}{1+c_0} - A^{\tau}\frac{2\beta^2((1-\theta)^2 A + \theta^2)([(1-\eta)^{-1}] + 1)}{c_0}\right) \\ &\quad + K + (R_2 + R_3 A)([(1-\eta)^{-1}] + 1)A^{\tau} \le 0, \end{aligned} \quad (39)$$

then we have $h(\Delta) < 0$, for $\Delta \in (0, \Delta^*)$ and $A \in (1, \bar{A} \wedge e^C)$. Thus, if

$$\begin{aligned}
&h_2 - \log A \left(\frac{1}{1+c_0} - A^{\tau} \frac{2\beta^2((1-\theta)^2 A + \theta^2)([(1-\eta)^{-1}] + 1)}{c_0} \right) \\
&\quad - \left(C \frac{2\beta^2((1-\theta)^2 A + \theta^2)([(1-\eta)^{-1}] + 1)}{c_0} + (R_2 + R_3 A)([(1-\eta)^{-1}] + 1) \right) (A^{\tau} - 1) \\
&\quad = K + (R_2 + R_3 A)([(1-\eta)^{-1}] + 1) \\
&\quad -C \left(\frac{1}{1+c_0} - \frac{2\beta^2((1-\theta)^2 A + \theta^2)([(1-\eta)^{-1}] + 1)}{c_0} \right) < 0, \qquad (40)
\end{aligned}$$

for some $c_0 > 0$ and $A \in (1, \bar{A} \wedge e^C)$, then $h_2 \le 0$ for the same c_0 and some A close to 1.

So, first observe that

$$\begin{aligned}
&C \left(\frac{1}{1+c_0} - \frac{2\beta^2((1-\theta)^2 A + \theta^2)([(1-\eta)^{-1}] + 1)}{c_0} \right) \\
&\quad > K + \left(\frac{\alpha_1}{c_0}\beta^2 + \alpha_2 + K + \left(\frac{2\alpha_1}{1+c_0} + 2 \right)(1-\theta)^2 \beta^2 (A+1) \right) \\
&\quad \times ([(1-\eta)^{-1}] + 1) \\
&\quad \Leftrightarrow C \left(\frac{1}{(1+c_0)([(1-\eta)^{-1}] + 1)} - \frac{2\beta^2((1-\theta)^2 A + \theta^2)}{c_0} \right) \\
&\quad > \frac{K}{[(1-\eta)^{-1}] + 1} + \frac{\alpha_1}{c_0}\beta^2 + \alpha_2 + K + \left(\frac{2\alpha_1}{1+c_0} + 2 \right)(1-\theta)^2 \beta^2 (A+1) \\
&\quad \Leftrightarrow C \left(\frac{1}{(1+c_0)([(1-\eta)^{-1}] + 1)} - \frac{2\beta^2((1-\theta)^2 A + \theta^2)}{c_0} \right) \\
&\quad - \frac{\alpha_1}{c_0}\beta^2 - \frac{2\alpha_1}{1+c_0}(1-\theta)^2 \beta^2 (A+1) \\
&\quad > \alpha_2 + K \left(\frac{1}{[(1-\eta)^{-1}] + 1} + 1 \right) + 2(1-\theta)^2 \beta^2 (A+1). \qquad (41)
\end{aligned}$$

We will choose $c_0 = 1$ and $C = \frac{\alpha_1}{3(1+c_0)} = \frac{\alpha_1}{6}$. So, on the basis of (41), we need to show that, for the appropriate choice of A,

$$\begin{aligned}
&\alpha_1 \left(\frac{1}{12([(1-\eta)^{-1}] + 1)} - \frac{2\beta^2((1-\theta)^2 A + \theta^2)}{6} - \beta^2 - (1-\theta)^2 \beta^2 (A+1) \right) \\
&\quad > \alpha_2 + K \left(\frac{1}{[(1-\eta)^{-1}] + 1} + 1 \right) + 2(1-\theta)^2 \beta^2 (A+1). \qquad (42)
\end{aligned}$$

In a view of the assumption (15), that is

$$\beta^2 \in \left(0, \frac{1}{4\big(9(1-\theta)^2 + \theta^2 + 3\big)([(1-\eta)^{-1}] + 1)}\right),$$

we conclude that the expression multiplying α_1 in (42) is positive for any $A \in \left(1, \frac{3}{2} \wedge \bar{A} \wedge e^C\right)$. Moreover, on the basis of the assumption (16) we find that (42) holds for any $A \in \left(1, \frac{3}{2} \wedge \bar{A} \wedge e^C\right)$, which yields $h_2 \leq 0$, as desired.

Taking into account (34), one can observe that, for any $\Delta \in (0, \Delta^*)$, we have that

$$A^{k\Delta}|z_k|^2 \leq X + M_k. \tag{43}$$

Applying the discrete semimartingale convergence theorem (see [18]), we find that

$$\limsup_{k\to\infty} A^{k\Delta}|z_k|^2 \leq \limsup_{k\to\infty}(X + M_k) < \infty \;\; a.s.$$

Substituting (43) into (29), we obtain

$$\begin{aligned}
&\left(\frac{1}{1+c_0} + \alpha_1\theta\Delta - \theta^2 K\Delta\right) A^{k\Delta}|q_k|^2 \\
&\leq \left(\frac{2\theta^2\beta^2}{c_0} + \alpha_2\theta\Delta + 2(1-\theta)^2\beta^2\Delta + \theta^2 K\Delta\right) A^{k\Delta}|q_{k-[\delta(k\Delta)/\Delta]}|^2 \\
&\quad + \left(\frac{2(1-\theta)^2\beta^2}{c_0} + 2(1-\theta)^2\beta^2\Delta\right) A^{k\Delta}|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2 \\
&\quad + X + M_k,
\end{aligned} \tag{44}$$

where $c_0 = 1$. Because of that, for any $\gamma \in \left(0, \log\left(\frac{3}{2} \wedge \bar{A}\right) \wedge C\right)$, there exists an integer k_1, such that for any integer $k_2 > k_1$,

$$\begin{aligned}
&\left(\frac{1}{1+c_0} + \alpha_1\theta\Delta - \theta^2 K\Delta\right) \sup_{k_1\leq k\leq k_2} e^{\gamma k\Delta}|q_k|^2 \\
&\leq \left(\frac{2\theta^2\beta^2}{c_0} + \alpha_2\theta\Delta + 2(1-\theta)^2\beta^2\Delta + \theta^2 K\Delta\right) \sup_{k_1\leq k\leq k_2} e^{\gamma k\Delta}|q_{k-[\delta(k\Delta)/\Delta]}|^2 \\
&\leq \left(\frac{2\theta^2\beta^2}{c_0} + \alpha_2\theta\Delta + 2(1-\theta)^2\beta^2\Delta + \theta^2 K\Delta\right) e^{\gamma\tau} \sup_{k_1\leq k\leq k_2} e^{\gamma(k-[\delta(k\Delta)/\Delta])\Delta} \\
&\quad \times |q_{k-[\delta(k\Delta)/\Delta]}|^2 + \left(\frac{2(1-\theta)^2\beta^2}{c_0} + 2(1-\theta)^2\beta^2\Delta\right) e^{\gamma(\tau+1)} \\
&\quad \times \sup_{k_1\leq k\leq k_2} e^{\gamma(k-1-[\delta((k-1)\Delta)/\Delta])\Delta}|q_{k-1-[\delta((k-1)\Delta)/\Delta]}|^2 + X + M_k \\
&\leq \left(\frac{2\theta^2\beta^2}{c_0} + \alpha_2\theta\Delta + 2(1-\theta)^2\beta^2\Delta + \theta^2 K\Delta\right) \\
&\quad \times \left(e^{\gamma\tau} \sup_{k_1-n_*\leq k\leq k_1} e^{\gamma k\Delta}|q_k|^2 + e^{\gamma\tau} \sup_{k_1\leq k\leq k_2} e^{\gamma k\Delta}|q_k|^2\right)
\end{aligned}$$

$$+\left(\frac{2(1-\theta)^2\beta^2}{c_0} + 2(1-\theta)^2\beta^2\Delta\right)$$
$$\times\left(e^{\gamma(\tau+1)} \sup_{k_1-n_*-1\le k\le k_1} e^{\gamma k\Delta}|q_k|^2 + e^{\gamma(\tau+1)} \sup_{k_1\le k\le k_2} e^{\gamma k\Delta}|q_k|^2\right) + X + M_k.$$

Then, substituting $c_0 = 1$, we get

$$H \sup_{k_1\le k\le k_2} e^{\gamma k\Delta}|q_k|^2$$
$$\le \left(2\theta^2\beta^2 + \alpha_2\theta\Delta + 2(1-\theta)^2\beta^2\Delta + \theta^2 K\Delta\right) e^{\gamma\tau} \sup_{k_1-n_*\le k\le k_1} e^{\gamma k\Delta}|q_k|^2$$
$$+\left(2(1-\theta)^2\beta^2 + 2(1-\theta)^2\beta^2\Delta\right) e^{\gamma(\tau+1)} \sup_{k_1-n_*-1\le k\le k_1} e^{\gamma k\Delta}|q_k|^2 + X + M_k, \quad (45)$$

where

$$H = \frac{1}{2} + \alpha_1\theta\Delta - \theta^2 K\Delta - \left(2\theta^2\beta^2 + \alpha_2\theta\Delta + 2(1-\theta)^2\beta^2\Delta + \theta^2 K\Delta\right) e^{\gamma\tau}$$
$$- \left(2(1-\theta)^2\beta^2 + 2(1-\theta)^2\beta^2\Delta\right) e^{\gamma(\tau+1)}$$
$$> \frac{1}{2} - 2\beta^2\left(\theta^2 + (1-\theta)^2\right)e^{\gamma(\tau+1)}$$
$$+\Delta\Big[\alpha_1\theta - \theta^2 K - \left(\alpha_2\theta + 4(1-\theta)^2\beta^2 + \theta^2 K\right) e^{\gamma(\tau+1)}\Big]. \quad (46)$$

Then, for any $\gamma \in \left(0, \frac{1}{\tau+1}\log\left([(1-\eta)^{-1}] + 1\right) \wedge \log(\frac{3}{2} \wedge \bar{A}) \wedge C\right)$ and any $\Delta \in (0, \Delta^*)$, from (46) follows that

$$H > \frac{1}{2} - 2\beta^2\left(\theta^2 + (1-\theta)^2\right)\left([(1-\eta)^{-1}] + 1\right)$$
$$+\Delta\Big[\alpha_1\theta - \theta^2 K - \left(\alpha_2\theta + 4(1-\theta)^2\beta^2 + \theta^2 K\right)\left([(1-\eta)^{-1}] + 1\right)\Big].$$

Bearing in mind the previous inequality, on the basis of conditions (15) and (17), we conclude that $H > 0$, such that (45) yields

$$\sup_{k_1\le k\le k_2} e^{\gamma k\Delta}|q_k|^2$$
$$\le \frac{1}{H}\Bigg[\left(2\theta^2\beta^2 + \alpha_2\theta\Delta + 2(1-\theta)^2\beta^2\Delta + \theta^2 K\Delta\right)e^{\gamma\tau} \sup_{k_1-n_*\le k\le k_1} e^{\gamma k\Delta}|q_k|^2$$
$$+\left(2(1-\theta)^2\beta^2 + 2(1-\theta)^2\beta^2\Delta\right)e^{\gamma(\tau+1)} \sup_{k_1-n_*-1\le k\le k_1} e^{\gamma k\Delta}|q_k|^2 + X + M_k\Bigg]. \quad (47)$$

Letting $k_2 \to +\infty$ in (47), we obtain

$$\begin{aligned}
&\sup_{k_1 \le k \le \infty} e^{\gamma k \Delta} |q_k|^2 \\
&\le \frac{1}{H} \Bigg[\left(2\theta^2\beta^2 + \alpha_2\theta\Delta + 2(1-\theta)^2\beta^2\Delta + \theta^2 K\Delta\right) e^{\gamma\tau} \sup_{k_1 - n_* \le k \le k_1} e^{\gamma k \Delta}|q_k|^2 \\
&\quad + \left(2(1-\theta)^2\beta^2 + 2(1-\theta)^2\beta^2\Delta\right) e^{\gamma(\tau+1)} \sup_{k_1 - n_* - 1 \le k \le k_1} e^{\gamma k \Delta}|q_k|^2 + X + M_k \Bigg] < \infty,
\end{aligned}$$

which yields

$$\limsup_{k\to\infty} e^{\gamma k \Delta}|q_k|^2 < \infty,$$

whenever

$$\gamma \in \left(0, \frac{1}{\tau+1}\log\left([(1-\eta)^{-1}] + 1\right) \wedge \log(\frac{3}{2} \wedge \bar{A}) \wedge C\right), \; \Delta \in (0, \Delta^*).$$

Thus, (44) gives

$$\limsup_{k\to\infty} e^{\gamma k \Delta}|q_k|^2 \le \frac{X + M_k}{H}.$$

Consequently,

$$\limsup_{k\to\infty} \frac{\log(e^{\gamma k \Delta}|q_k|^2)}{k\Delta} = 0,$$

which gives

$$\limsup_{k\to\infty} \frac{\log|q_k|}{k\Delta} \le -\frac{\gamma}{2},$$

for any $\gamma \in \left(0, \frac{1}{\tau+1}\log\left([(1-\eta)^{-1}] + 1\right) \wedge \log(\frac{3}{2} \wedge \bar{A}) \wedge C\right)$ and any $\Delta \in (0, \Delta^*)$. $\square$

Remark 1. *In the proof of the previous theorem, the conditions of Lemma 3 are not used explicitly. Their role was to guarantee the existence and uniqueness of the solution under consideration.*

3 Numerical example and simulations

In this section, in order to illustrate the previous theoretical results, we present an example which will be completed by the appropriate numerical simulations.

Example 1. *We will consider the following scalar neutral stochastic differential equation with time-dependent delay*

$$d\Big[x(t) - \frac{1}{50}\sin x(t-\delta(t))\Big] = -\frac{1}{48}x(t)dt + \frac{1}{20\sqrt{6}}\frac{x(t-\delta(t))}{1+x^2(t-\delta(t))}\cos x(t)dw(t), \quad t \in [0,50], \tag{48}$$

satisfying the initial condition $\varphi(t) = 1$, $t \in [-\tau, 0]$, where $\tau = 0.5$ and $\varphi \in C^b_{\mathcal{F}_0}([-\tau,0]; R)$. Obviously, the drift coefficient $f(x,y) = -\frac{1}{48}x$ satisfies the linear growth condition $\mathcal{A}_1$ for $K = \frac{1}{48^2}$, while the neutral term $u(x) = \frac{1}{50}\sin x$, $x \in R$ satisfies the assumption $\mathcal{A}_2$ for $\beta = \frac{1}{50}$. Assume that the delay function is of the form $\delta(t) = \frac{1}{4} - \frac{1}{4}\sin t$, $t \in [0, 50]$. Then,

$$\delta'(t) = -\frac{1}{4}\cos t \leq \frac{1}{4} = \bar{\delta}, \quad |\delta(t) - \delta(s)| \leq \frac{1}{4}|t-s|, \quad t, s \in [0, 50],$$

and we find that $\mathcal{A}_3$ and $\mathcal{A}_4$ hold with $\eta = \frac{1}{4}$. In order to verify $\mathcal{A}_5$, note that

$$\begin{aligned} &2(x-u(y))f(x,y) + |g(x,y)|^2 \\ &\quad = -\frac{1}{24}x^2 + \frac{1}{1\,200}x\sin y + \frac{1}{2\,400}\frac{y^2}{(1+y^2)^2}\cos^2 x \\ &\quad \leq -\frac{1}{24}x^2 + \frac{1}{2\,400}x^2 + \frac{1}{2\,400}y^2 + \frac{1}{2\,400}y^2 \\ &\quad \leq -\frac{33}{800}x^2 + \frac{1}{1\,200}y^2, \end{aligned}$$

that is, $\alpha_1 = \frac{33}{800}$ and $\alpha_2 = \frac{1}{1\,200}$. Moreover, we have that

$$\frac{\alpha_2}{1-\bar{\delta}} = \frac{1}{900} < \frac{33}{800} = \alpha_1.$$

Since $\mathcal{C}_1$ trivially holds for any positive μ_1 and μ_2, we will choose $\mu_1 = \mu_2 = \frac{1}{5}$, such that $\theta((\mu_1+\mu_2)\Delta + \beta) < 1$ for any $\Delta \in (0,1)$. Thus, on the basis of Lemma 3, we conclude that the corresponding θ-Euler-Maruyama approximate equations have unique solutions. Bearing in mind (13), for $\theta = \frac{3}{4}$, we have *that* $q_k = \varphi(k\Delta)$, $k = -n_*, -n_* + 1, ..., 0$, *while, for* $k = 0, 1, 2, ...$,

$$\begin{aligned} q_{k+1} = q_k &+ \frac{3}{200} \sin q_{k+1-[\delta((k+1)\Delta)/\Delta]} - \frac{1}{100} \sin q_{k-[\delta(k\Delta)/\Delta]} \\ &- \frac{1}{200} \sin q_{k-1-[\delta((k-1)\Delta)/\Delta]} - \frac{1}{64} q_{k+1}\Delta - \frac{1}{192} q_k \Delta \\ &+ \frac{1}{20\sqrt{6}} \frac{q_{k-[\delta(k\Delta)/\Delta]}}{1 + q^2_{k-[\delta(k\Delta)/\Delta]}} \cos q_k \Delta w_k. \end{aligned} \tag{49}$$

Following the proof of Theorem 1 we first observe from (22) that, since $C = \frac{\alpha_1}{6}$, *we have that* $\Delta^* = 1$. *So, we will proceed the verification of the stability result for Eq.(49) for any* $\Delta \in (0, \Delta^*)$.

In Figure 1 several trajectories corresponding to (49) are plotted with step-size $\Delta = 0.01$.

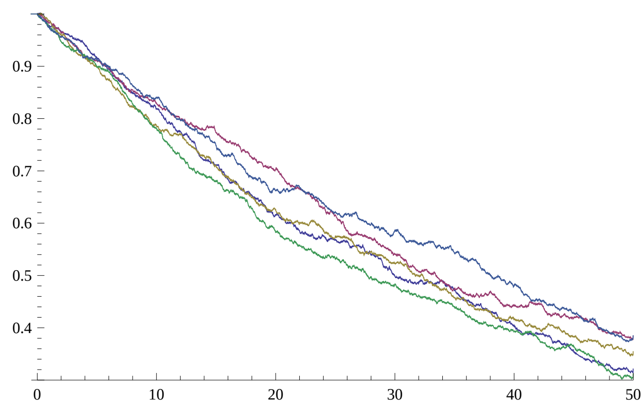

Figure 1: Trajectories of the θ–Euler–Maruyama solution with $\Delta = 0.01$

Noting that

$$\frac{1}{4\big(9(1-\theta)^2 + \theta^2 + 3\big)\big([(1-\eta)^{-1}] + 1\big)} = \frac{1}{33},$$

we find that (15) holds. Moreover,

$$\begin{aligned} \frac{33}{800} &= \alpha_1 \\ &> \frac{12([(1-\eta)^{-1}] + 1)}{1 - 4\beta^2(9(1-\theta)^2 + \theta^2 + 3)([(1-\eta)^{-1}] + 1)} \\ &\quad \times \left(\alpha_2 + K\left(\frac{1}{[(1-\eta)^{-1}] + 1} + 1\right) + 5(1-\theta)^2\beta^2\right) = 0.03914, \end{aligned}$$

that is, the condition (16) is fulfilled. Finally, since

$$\alpha_1\theta - \theta^2 K - (\alpha_2\theta + \theta^2 K + 4\beta^2(1-\theta)^2)([(1-\eta)^{-1}] + 1) = 0.0287,$$

the assumption (17) is satisfied. Thus, on the basis of Theorem 1, for

$$\gamma \in \Big(0, \frac{1}{\tau+1}\log\big([(1-\eta)^{-1}] + 1\big) \wedge \log(\frac{3}{2} \wedge \bar{A}) \wedge C\Big),$$

we have that

$$\limsup_{k\to\infty} \frac{\log|q_k|}{k\Delta} \leq -\frac{\gamma}{2} \; a.s,$$

where $\bar{A} = \log\bar{\varepsilon}$, while $\bar{\varepsilon}$ is the unique positive root of Eq.(18). Direct computation gives

$$\frac{1}{\tau+1}\log\big([(1-\eta)^{-1}] + 1\big) = 0.4621, \quad C = \frac{\alpha_1}{6} = \frac{11}{1\,600}, \quad \bar{A} = 34.1444,$$

such that Theorem 1 holds for $\gamma \in \Big(0, \frac{11}{1\,600}\Big)$. In order to illustrate the almost sure exponential stability of the θ-Euler-Maruyama solution, we plotted a trajectory of the ratio $\frac{\log|q_k|}{k\Delta}$ against the line $z = -\frac{11}{1\,600}$, which can be seen in Figure 2.

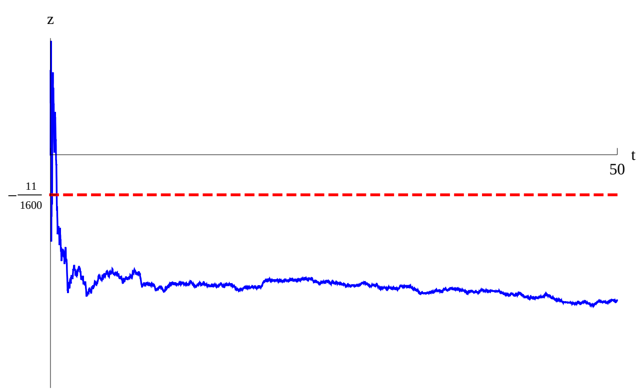

Figure 2: $\frac{\log|q_k|}{k\Delta}$ against the line $z = -\frac{11}{1\,600}$

Acknowledgement

The second author's research is supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia, Contract number 451-03-47/2023-01/200124.

References

[1] M. Jovanović, S. Janković, Neutral stochastic functional differential equations with additive pertubations, Appl. Math. Comput. 213 (2009) 370–379.

[2] G. Lan, C. Yuan, Exponential stability of the exact solutions and θ–EM approximations to neutral SDDEs with Markov switching, J. Comput. Appl. Math. 285 (2015) 230-242.

[3] Q. Luo, X. Mao, Y. Shen, New criteria on exponential stability of neutral stochastic differential delay equations, Systems and Control Letters 55 (2006) 826–834.

[4] X. Mao, M. J. Rassias, Khasminskii-type theorems for stochastic differential delay equations, Stochastic analysis and applications (2005) 1045–1069.

[5] X. Mao, Stochastics differential equations and their applications, Horwood Publishing Limited, 1997.

[6] X. Mao, Asymptotic properties of neutral stochastic differential delay equations, Stochastics and Stochastics Reports 68 (2000) 273–295.

[7] X. Mao, Numerical solutions of stochastic differential delay equations under the generalized Khasminskii-type conditions, Appl. Math. Comput. 217 (2011) 5512–5524.

[8] X. Mao, L. Szpruch, Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally Lipschitz continuous coefficients, J. Comput. Appl. Math. 238 (2013) 14–28.

[9] M. Milošević, M. Jovanović, S. Janković, An approximate method via Taylor series for stochastic differential equations, J. Math. Anal. Appl. 363 (2010) 128–137.

[10] M. Milošević, M. Jovanović, An application of Taylor series in the approximation of solutions to stochastic differential equations with time-dependent delay, J. Comput. Appl. Math. 235 (2011) 4439–4451.

[11] M. Milošević, Highly nonlinear neutral stochastic differential equations with time-dependent delay and the Euler-Maruyama method, Mathematical and Computer Modelling 54 (2011) 2235–2251.

[12] M. Milošević, Almost sure exponential stability of solutions to highly nonlinear neutral stochastics differential equations with time-dependent delay and Euler-Maruyama approximation, Mathematical and Computer Modelling 57 (2013) 887–899.

[13] M. Milošević, Implicit numerical methods for highly nonlinear neutral stochastics differential equations with time-dependent delay, Appl. Math. and Comput. 244 (2014) 741–760.

[14] M. Obradović, M. Milošević, Stability of a class of neutral stochastic differential equations with unbounded delay and Markovian switching and the Euler-Maruyama method, J. Comput. Appl. Math. 309 (2017) 244–266.

[15] M. Obradović, M. Milošević, Almost sure exponential stability of the θ-Euler-Maruyama method for neutral stochastic differential equations with time-dependent delay when $\theta \in [0, \frac{1}{2}]$, Filomat 31:18 (2017) 5629-5645.

[16] M. Obradović, M. Milošević, Almost sure exponential stability of the θ-Euler-Maruyama method, when $\theta \in (\frac{1}{2}, 1)$, for neutral stochastic differential equations with time-dependent delay under nonlinear growth conditions, Calcolo 56:9 (2019).

[17] M. Obradović, Implicit numerical methods for neutral stochastic differential equations with unbounded delay and Markovian switching. Appl. Math. Comput. 347 (2019) 664-687.

[18] A. N. Shiryaev, Probability, Springer, Berlin, 1996.

[19] L. Tan, C. Yuan, Convergence rates of theta-method for NSDDEs under non-globally Lipschitz continuous coefficients, Bulletin of Mathematical Sciences, 09 (03), (2019) 1950006 (32 pages).

[20] F. Wu, X. Mao, L. Szpruch, Almost sure exonential stability of numerical solutions for stochastic delay differential equations, Numer. Math. 115 (2010) 681–697.

[21] M. Xue, S. Zhou, S. Hu, Stability of nonlinear neutral stochastic functional differential equations, J. Appl. Math. Volume 2010, Article ID 425762, 26 pages doi:10.1155/2010/425762

[22] Z. Yan, A. Xiao, X. Tang, Strong convergence of the split-step theta method for neutral stochastic delay differential equations, Applied Numerical Mathematics 120 (2017) 215–232.

[23] Z. Yu, Almost surely asymptotic stability of exact and numerical solutions for neutral stochastic pantograph equations, Abstract and Applied Analysis (2011) doi:10.1155/2011/14079.

[24] J. Zhao, Y. Yi, Y. Xu, Strong convergence and stability of the split-step theta method for highly nonlinear neutral stochastic delay integro-differential equation, Applied Numerical Mathematics 157 (2020) 385–404.

Maja Obradović,
Faculty of Sciences and Mathematics,
University of Niš,
Višegradska 33, 18 000 Niš, Serbia.
Email: maja.obradovic@pmf.edu.rs

Marija Milošević,
Faculty of Sciences and Mathematics,
University of Niš,
Višegradska 33, 18 000 Niš, Serbia.
Email: marija.milosevic@pmf.edu.rs