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A generalization of n-ary prime subhypermodule

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Abstract

Let (M, f, g) be an (m, n)-hypermodule over an (m, n)-hyperring (R, h, k). A proper subhypermodule N of M is called n-ary 2-absorbing subhypermodule if whenever $g(r_1^{n-1}, m) \subseteq N$ for some $r_1^{n-1} \in R$ and $m \in M$, then either $g(r_1^{n-1}, M) \subseteq N$ or $g(r_i, m, 1_R^{(n-2)}) \subseteq N$ for some $i \in \{1, \ldots, n-1\}$. Various properties of n-ary 2-absorbing subhypermodules are investigated. In particular, it is shown that if N is a subhypermodule of an (m, n)-hypermodule (M, f, g) over an (m, n)-hyperring (R, h, k), then N is n-ary 2-absorbing if and only if whenever $g(I_1, I_2, 1_R^{(n-3)}, L) \subseteq N$ for some hyperideals I_1, I_2 of R and subhypermodule L of M, then either $g(I_1, I_2, 1_R^{(n-3)}, M) \subseteq N$ or $g(I_1, 1_R^{(n-2)}, L) \subseteq N$ or $g(I_2, 1_R^{(n-2)}, L) \subseteq N$. Also, n-ary 2-absorbing subhypermodules in multiplication (m, n)-hypermodules are studied.

The body of the article.

1 Introduction

One of the generalizations of groups is the hypergroups was introduced by Marty, and then the n-ary groups was introduced to be a generalization of hypergroups [14].

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In 1904, at the annual meeting of the Advancement of Science [18], E. Krasner introduced the n-ary algebras. After that, Dörente studied the n-ary groups [16]. Then, Timm and Crombez studied the (m,n)-rings [12, 13]. Afterward, Davvaz et al. introduced the n-ary hypergroup as a generalization of the hypergroup and an extending of an n-ary group ([14]). Thereupon, (m,n)-hyperring was introduced by Mirvakili et al., and then, he introduced (m,n)-rings of the (m,n)-hyperrings (see [23]). After that, he introduced the subclass of (m,n)-hyperrings containing the class of Krasner hyperrings which is the Krasner (m,n)-hyperrings. Then, the notion of (m,n)-hypermodules was introduced by Anvariyeh et al. [8]. Moreover, free and canonical (m,n)-hypermodules were defined by Belali et al. [7, 11].

Badawi in 2007 introduced the concept of 2-absorbing ideals in commutative rings R [10]. Let I be a nonzero proper ideal of R. Then I is called to be 2-absorbing, if for each $d,e,f \in R$ and $Idef \in I$, either $de \in I$ or $df \in I$ or $ef \in I$. After that, in 2011, Anderson et al. defined the n-absorbing ideals for an integer n [6], which is an ideal I of R is an n-absorbing ideal if for $a_1, a_2, \ldots, a_{n+1} \in R$ and $a_1 a_2 \cdots a_{n+1} \in I$, then n of the a_i's whose product is in I (see also [9]).

On the other hand, Ulucak [26] studied 2-absorbing δ-primary and δ-primary hyperideals the generalizations of 2-absorbing and prime hyperideals, respectively. Ameri et al. defined hyperideals, Jacobson radical, n-ary prime hyperideals and primary hyperideals of Krasner (m,n)-hyperrings and nilradical [3]. Moreover, in [4], they introduced the n-ary prime subhypermodules of (m,n)-hypermodules. The notion of (k,n)-absorbing hyperideals was introduced in [17] by Hila et al. Then, 2-absorbing ideals of commutative rings were extended to n-ary 2-absorbing hyperideals in Krasner (m,n)-hyperrings in [5]. In this paper, we introduce the concept of n-ary 2-absorbing subhypermodules of (m,n)-hypermodules over Krasner (m,n)-hyperrings as generalization of n-ary prime subhypermodules ([4]).

Throughout this paper, all hyperrings are commutative Krasner (m,n)-hyperrings with scalar identity and all hypermodules are canonical unitary (m,n)-hypermodules. In Section 2, the notion of n-ary 2-absorbing subhypermodules of (m,n)-hypermodules over Krasner (m,n)-hyperrings are introduced (see Definition 2.1) and some of their basic properties are given. For instance, in Examples 2.9 and 2.13 some examples concerning n-ary 2-absorbing subhypermodules are presented. In Theorem 2.15, it is shown that if N is a subhypermodule, then N is n-ary 2-absorbing subhypermodule if and only if whenever $g(I_1, I_2, 1_R^{(n-3)}, L) \subseteq N$ for hyperideals I_1, I_2 of R and subhypermodule L of M, then either $g(I_1, I_2, 1_R^{(n-3)}, M) \subseteq N$ or $g(I_1, 1_R^{(n-2)}, L) \subseteq N$ or $g(I_2, 1_R^{(n-2)}, L) \subseteq N$. In Section 3, we study n-ary 2-absorbing subhypermodules in multiplication (m,n)-hypermodules. First we give the definition of g-product of two subhypermodules of a multiplication (m,n)-hypermodule (Definition 3.1). Among other results, it is proved (Theorem 3.2) that if N is an n-ary 2-absorbing subhypermodule of a cyclic multiplication faithful (m,n)-hypermodule (M,f,g), then either $\text{rad}_{(m,n)}(N)=P$ where P is an n-ary prime subhypermodule of M such that $g(P^{(2)},1_R^{(n-2)})\subseteq N$ or $\text{rad}_{(m,n)}(N)=P_1\cap P_2$ where P_1,P_2 are distinct n-ary prime subhypermodules of M such that $g(P_1,P_2,1_R^{(n-2)})\subseteq N$ and $g((\text{rad}_{(m,n)}(N))^{(2)},1_R^{(n-2)})\subseteq N$. It is shown (Theorem 3.6) that if N is a subhypermodule of a cyclic multiplication faithful (m,n)-hypermodule (M,f,g), then N is n-ary 2-absorbing if and only if whenever $g(N_1,N_2,N_3,1_R^{(n-3)})\subseteq N$ for some subhypermodules N_1,N_2,N_3 of M, then either $g(N_1,N_2,\,1_R^{(n-2)})\subseteq N$ or $g(N_1,N_3,1_R^{(n-2)})\subseteq N$ or $g(N_2,N_3,1_R^{(n-2)})\subseteq N$.

In this paper, definitions and examples on (m,n)-hyperrings can be found in [3, 17, 19, 23, 25], and for any undefined notations on n-ary structures and hyperstructures, and (m,n)-ary hyperring and hypermodule theory, we refer the reader to [2, 8, 15, 19, 20, 21, 22] and [27].

2 On n-Ary 2-Absorbing Subhypermodules

In this paper, we suppose that (R,h,k) is a commutative Krasner (m,n)-hyperring with scalar identity 1_R and (M,f,g) is an (m,n)-hypermodule over (R,h,k) such that (M,f) is a canonical m-ary hypergroup. In this section, we introduce the notion of n-ary 2-absorbing subhypermodule in the (m,n)-hypermodule M, and some of basic properties of n-ary 2-absorbing subhypermodule are studied.

Definition 2.1. Let N be a proper subhypermodule of the (m,n)-hypermodule (M,f,g) over the (m,n)-hyperring (R,h,k). N is saidn to be n-ary 2-absorbing subhypermodule if whenever $g(r_1^{n-1},\,m)\subseteq N$ for $r_1^{n-1}\in R$ and $m\in M$, then either $g(r_1^{n-1},M)\subseteq N$ or $g(r_i,m,1_R^{(n-2)})\subseteq N$ for some $i\in\{1,\ldots,n-1\}$.

In the following, we need the following lemma to prove Theorem 2.3.

Lemma 2.2. *Let N,N_1 and N_2 be subhypermodules of the (m,n)-hypermodule (M,f,g) over (R,h,k). If $N\subseteq N_1\cup N_2$, then $N\subseteq N_1$ or $N\subseteq N_2$.*

Proof. Suppose that neither $N\subseteq N_1$ nor $N\subseteq N_2$ and look for a contradiction. Then there exists $x\in N\setminus N_1$ and $y\in N\setminus N_2$, and hence $x\in N_2$ and $y\in N_1$, since $N\subseteq N_1\cup N_2$. But N is a subhypermodule of M. Then $f(x,y,0^{(m-2)})\subseteq N\subseteq N_1\cup N_2$. Therefore, for every $a\in f(x,y,0^{(m-2)})$, we have either $a\in N_1$ or $a\in N_2$. If $a\in N_1$, then $x\in f(a,-y,0^{(m-2)})\subseteq N_1$, since (M, f) is a canonical m-ary hypergroup, which is a contradiction. The second possibility leads to a contradiction in a similar way. Thus we must have either $N \subseteq N_1$ or $N \subseteq N_2$. $\square$

In the following result, an equivalent definition for n-ary 2-absorbing subhypermodules is provided.

Theorem 2.3. *Let N be a subhypermodule of the (m, n)-hypermodule (M, f, g) over (R, h, k). Then N is an n-ary 2-absorbing subhypermodule of M if and only if for every elements r_1^{n-1} of R and every subhypermodule K of M, $g(r_1^{n-1}, K) \subseteq N$ implies that either $g(r_1^{n-1}, M) \subseteq N$ or $g(r_i, K, 1_R^{(n-2)}) \subseteq N$ for some $i \in \{1, \ldots, n-1\}$.*

Proof. Let $a_1^{n-1} \in R$ and K be a subhypermodule of M such that $g(a_1^{n-1}, K) \subseteq N$. Suppose further that $g(a_1^{n-1}, M) \nsubseteq N$. For each $i \in \{1, \ldots, n-1\}$ set

$$A_i = \{m \in K : g(a_i, 1_R^{(n-2)}, m) \subseteq N\} \text{ and}$$
$$B_i = \{m \in K : g(a_i, 1_R^{(n-2)}, m) \nsubseteq N\}.$$

By [4, Lemma 3.3], it can be easily seen that the sets A_i's,B_i's are subhypermodules of M and $K = A_i \cup B_i$ for every $i \in \{1, \ldots, n-1\}$. Hence either $K \subseteq A_i$ or $K \subseteq B_i$ for every $i \in \{1, \ldots, n-1\}$, by Lemma 2.2, and so either $K = A_i$ or $K = B_i$ for every $i \in \{1, \ldots, n-1\}$. If $K = A_i$ for some $i \in \{1, \ldots, n\}$, then we are done. Hence assume that $K = B_i$ for every $i \in \{1, \ldots, n-1\}$. But N is an n-ary 2-absorbing subhypermodule of M. Then $g(a_1^{n-1}, m) \subseteq N$ for every $m \in K$ and $g(a_1^{n-1}, M) \nsubseteq N$, we must have $g(a_i, 1_R^{(n-2)}, m) \subseteq N$ for some $i \in \{1, \ldots, n-1\}$, a contradiction, as $m \in K = B_i$. Hence $K = A_i$ for some $i \in \{1, \ldots, n-1\}$, and thus $g(a_i, K, 1_R^{(n-2)}) \subseteq N$ for some $i \in \{1, \ldots, n-1\}$. $\square$

Corollary 2.4. *Let N be an n-ary 2-absorbing subhypermodule of the (m, n)-hypermodule (M, f, g) over (R, h, k). Then for every elements r_1^n of R and m of M, if $g(r_1^{n-1}, g(r_n, 1_R^{(n-2)}, m)) \subseteq N$, then either $g(r_1^{n-1}, M) \subseteq N$ or $g(r_i, g(r_n, 1_R^{(n-2)}, m), 1_R^{(n-2)}) \subseteq N$ for some $i \in \{1, \ldots, n-1\}$.*

Proof. Suppose that $g(r_1^{n-1}, g(r_n, 1_R^{(n-2)}, m)) \subseteq N$ for some elements r_1^n of R and m of M such that $g(r_1^{n-1}, M) \nsubseteq N$ and $g(r_i, g(r_n, 1_R^{(n-2)}, m), 1_R^{(n-2)}) \nsubseteq N$ for every $i \in \{1, \ldots, n-1\}$. Hence $g(r_i, m, 1_R^{(n-2)}) \nsubseteq N$ for every $i \in \{1, \ldots, n\}$. But $g(k(r_1, r_n, 1_R^{(n-2)}), r_2^{n-1}, m) \subseteq N$ and N is n-ary 2-absorbing. Therefore $g(k(r_1, r_n, 1_R^{(n-2)}), r_2^{n-1}, M) = g(r_1^{n-1}, g(r_n, 1_R^{(n-2)}, M)) \subseteq N$.

It follows easily from [2, Lemma 3.3] that $g(r_n, 1_R^{(n-2)}, M)$ is a subhypermodule of M and since N is an n-ary 2-absorbing, we conclude by Theorem 2.3 that either

$$g(r_1^{n-1}, M) \subseteq N \text{or} g(r_i, g(r_n, 1_R^{(n-2)}, M), 1_R^{(n-2)}) \subseteq N$$

for some $i \in \{1, \ldots, n-1\}$, a contradiction. □

In [4, Theorem 5.13] the authors showed that if (M, f, g) is a canonical (m, n)-hypermodule over (R, h, k) and N is a primary subhypermodule of M, then $S_N = \{r \in R \mid g(r, 1_R^{(n-2)}, M) \subseteq N\}$ is a prime hyperideal of R. We give an example which shows that this theorem is not true.

Example 2.5. Let $(\mathbb{Z}, f, g)$ be the (m, n)-hypermodule over $(\mathbb{Z}, h, k)$ as in [2, Example 3.5]. Suppose that $p \in \mathbb{Z}$ is a prime number. It follows from [2, Lemma 3.3] that $\langle p^2 \rangle = g(\mathbb{Z}, 1_{\mathbb{Z}}^{(n-3)}, p^{(2)})$ is a subhypermodule of the (m, n)-hypermodule $(\mathbb{Z}, f, g)$. We show that $N = \langle p^2 \rangle$ is a primary subhypermodule of $(\mathbb{Z}, f, g)$ but S_N need not be a prime hyperideal of $(\mathbb{Z}, h, k)$. Let $g(r_1^{n-1}, m) \subseteq \langle p^2 \rangle$ for some $r_1^{n-1}, m \in \mathbb{Z}$ such that $m \notin \langle p^2 \rangle$. Then by the definition of the n-ary hyperoperation g, we have

$$\{r_1 \cdots r_{n-1} \cdot m\} \subseteq g(\mathbb{Z}, 1_{\mathbb{Z}}^{(n-3)}, p^{(2)}) = \{t \cdot p \cdot p : t \in \mathbb{Z}\}.$$

Hence, there exists $t \in \mathbb{Z}$ such that $r_1 \cdots r_{n-1} \cdot m = t \cdot p \cdot p$. But p is a prime number and $m \notin \langle p^2 \rangle$, so that $r_1 \cdots r_{n-1} = s \cdot p = k(p, 1_{\mathbb{Z}}^{(n-2)}, s) \subseteq k(p, 1_{\mathbb{Z}}^{(n-2)}, \mathbb{Z})$ for some $s \in \mathbb{Z}$. This means that $g(r_1^{n-1}, \mathbb{Z}) \subseteq g(k(p, 1_{\mathbb{Z}}^{(n-2)}, \mathbb{Z}), 1_{\mathbb{Z}}^{(n-2)}, \mathbb{Z})$, and so $g(k(r_1^{(2)}, 1_{\mathbb{Z}}^{(n-2)}), \ldots, k(r_{n-1}^{(2)}, 1_{\mathbb{Z}}^{(n-2)}), \mathbb{Z}) \subseteq g(k(p^{(2)}, 1_{\mathbb{Z}}^{(n-3)}, \mathbb{Z}), 1_{\mathbb{Z}}^{(n-2)}, \mathbb{Z}) = g(p^{(2)}, 1_{\mathbb{Z}}^{(n-3)}, g(\mathbb{Z}, 1_{\mathbb{Z}}^{(n-2)}, \mathbb{Z})) = g(p^{(2)}, 1_{\mathbb{Z}}^{(n-3)}, \mathbb{Z})) = \langle p^2 \rangle$, by the definition of g. Thus $N = \langle p^2 \rangle$ is a primary subhypermodule of $(\mathbb{Z}, f, g)$. Now, we show that S_N need not be a prime hyperideal of $(\mathbb{Z}, h, k)$. Since $p \cdot p \in \langle p^2 \rangle = g(\mathbb{Z}, 1_{\mathbb{Z}}^{(n-3)}, p^{(2)})$, we conclude that $p \cdot p \in S_N$. But $p \cdot p \not| p$. Hence $g(\mathbb{Z}, 1_{\mathbb{Z}}^{(n-2)}, p) \nsubseteq N$, and so $p \notin S_N$. Thus S_N is not a prime hyperideal of $(\mathbb{Z}, h, k)$.

We give a modification of this theorem as follows.

Theorem 2.6. *Let N be a subhypermodule of the (m, n)-hypermodule (M, f, g) over (R, h, k). If N is a primary subhypermodule of M, then S_N is a primary hyperideal of R.*

Proof. Let N be a subhypermodule of the (m, n)-hypermodule (M, f, g) over (R, h, k) and let $k(r_1^n) \in S_N$ for $r_1^n \in R$ such that $k(r_2^n, 1_R) \notin \sqrt{S_N}^{(m,n)}$. Then

$g(k(r_2^{(\alpha)}, 1_R^{(n-\alpha)}), \ldots, k(r_n^{(\alpha)}, 1_R^{(n-\alpha)}), M) \nsubseteq N$ for every $\alpha \in \mathbb{N}$. But $k(r_1^n) \in S_N$. Then $g(k(r_1^n), 1_R^{(n-2)}, M) \subseteq N$. Hence $g(r_1, 1_R^{(n-2)}, g(r_2^n, M)) \subseteq N$, and so $g(r_2^n, g(r_1, M, 1_R^{(n-2)})) \subseteq N$. Since $g(k(r_2^{(\alpha)}, 1_R^{(n-\alpha)}), \ldots, k(r_n^{(\alpha)}, 1_R^{(n-\alpha)}), M) \nsubseteq N$ for every $\alpha \in \mathbb{N}$ and N is a primary subhypermodule of M, we conclude that $g(r_1, M, 1_R^{(n-2)}) \subseteq N$. Hence $r_1 \in S_N$, and thus S_N is a primary hyperideal of R. □

We can also see that Theorem 2.6 holds if N is an n-ary 2-absorbing subhypermodule of M.

Theorem 2.7. *Let N be a subhypermodule of the (m, n)-hypermodule (M, f, g) over (R, h, k). If N is an n-ary 2-absorbing subhypermodule of M, then S_N is an n-ary 2-absorbing hyperideal of R.*

Proof. Let $a_1^n \in R$ such that $k(a_1^n) \in S_N$. It is shown that S_N is an n-ary 2-absorbing hyperideal of R. For each $i \in \{1, \ldots, n-1\}$ set

$$A_{in} = \{m \in M : g(a_i, 1_R^{(n-2)}, g(a_n, m, 1_R^{(n-2)})) \subseteq N\} \text{ and}$$
$$B_{in} = \{m \in M : g(a_i, 1_R^{(n-2)}, g(a_n, m, 1_R^{(n-2)})) \nsubseteq N\}.$$

By [4, Lemma 3.3], it is easy to see that the sets A_{in}'s,B_{in}'s are subhypermodules of M and $M = A_{in} \cup B_{in}$ for every $i \in \{1, \ldots, n-1\}$. Hence, by Lemma 2.2, either $M \subseteq A_{in}$ or $M \subseteq B_{in}$ for every $i \in \{1, \ldots, n-1\}$, and so either $M = A_{in}$ or $M = B_{in}$ for every $i \in \{1, \ldots, n-1\}$. If $M = A_{in}$ for some $i \in \{1, \ldots, n-1\}$, the proof is complete. Hence assume that $M = B_{in}$ for every $i \in \{1, \ldots, n-1\}$. Since N is an n-ary 2-absorbing subhypermodule of M and $g(a_1^{n-1}, g(a_n, m, 1_R^{(n-2)})) \subseteq N$ for every $m \in M$, we must have either $g(a_1^{n-1}, M) \subseteq N$ or $g(a_i, 1_R^{(n-2)}, g(a_n, m, 1_R^{(n-2)})) \subseteq N$ for some $i \in \{1, \ldots, n-1\}$. Since $m \in B_{in}$, one may assume that $g(a_1^{n-1}, M) \subseteq N$. Now use this argument $n-3$ more times to see that $g(a_1^2, 1_R^{(n-3)}, M) \subseteq N$. Therefore $M = A_{12}$, which is a contradiction. Hence $M = A_{ij}$ for some $i, j \in \{1, \ldots, n\}$ such that $i \neq j$, and thus S_N is an n-ary 2-absorbing hyperideal of R. □

Corollary 2.8. *Let N be a subhypermodule of the (m, n)-hypermodule (M, f, g) over (R, h, k). If N is an n-ary 2-absorbing subhypermodule of M, then $N_m = \{r \in R : g(r, m, 1_R^{(n-2)}) \subseteq N\}$ is an n-ary 2-absorbing hyperideal of R for every $m \in M \setminus N$.*

Proof. Using [23, Lemma 3.3], it is easily seen that the set N_m is a hyperideal of R. Let $a_1^n \in R$ and $k(a_1^n) \in N_m$. It is shown that N_m is an n-ary 2-absorbing hyperideal of R. Since N is an n-ary 2-absorbing subhypermodule of M and $g(a_1^{n-1}, g(a_n, m, 1_R^{(n-2)})) \subseteq N$, either $g(a_1^{n-1}, M) \subseteq$

N or $g(a_i, 1_R^{(n-2)}, g(a_n, m, 1_R^{(n-2)})) \subseteq N$ for some $i \in \{1, \ldots, n-1\}$. If $g(a_i, 1_R^{(n-2)}, g(a_n, m, 1_R^{(n-2)})) \subseteq N$ for some $i \in \{1, \ldots, n-1\}$, then we are done. If $g(a_1^{n-1}, M) \subseteq N$, then by Theorem 2.7, N_m is an n-ary 2-absorbing hyperideal of R, as S_N is an n-ary 2-absorbing hyperideal of R. □

For nontrivial n-ary 2-absorbing subhypermodules see the following examples.

Example 2.9. Let (R, h, k) be the Krasner $(2,4)$-hyperring such that $R = \{0,1,2,3\}$, with the 2-ary hyperoperation h and the 4-ary operation k defined as follows:

h	0	1	2	3
0	0	1	2	3
1	1	{0, 1}	3	{2, 3}
2	2	3	0	1
3	3	{2, 3}	1	{0, 1}

$$k(x_1, x_2, x_3, x_4) = \begin{cases} 2 & \text{if } x_1^4 \in \{2,3\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $M = \{0,1,2,3,4\}$ be a set. We define the 2-ary hyperoperation f and the 4-ary external hyperoperation g on M as follows:

f	0	1	2	3	4
0	0	1	2	3	4
1	1	{0, 1}	3	{2, 3}	{3, 4}
2	2	3	0	1	2
3	3	{2, 3}	1	{0, 1}	1
4	4	{3, 4}	2	1	0

$$g(x_1, x_2, x_3, x_4) = \begin{cases} 2 & \text{if } x_1^3 \in \{2,3\} \text{ and } x \in \{2,3,4\}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\{0\}$ and $\{0,2\}$ are 4-ary 2-absorbing subhypermodules of the $(2,4)$-hypermodule (M, f, g) over the Krasner $(2,4)$-hyperring (R, h, k).

Example 2.10. Suppose that $(R, +, \cdot)$ is a Krasner hyperring such that R is an integral hyperdomain with the ordinary multiplication operation $\cdot$. Suppose also that R endowed with the following m-ary hyperoperation h and n-ary operation k is a Krasner (m,n)-hyperring:

$$h(x_1, x_2, \ldots, x^m) = \sum_{i=1}^{m} x_i \text{ and } k(x_1, x_2, \ldots, x^n) = x_1 \cdots x_n.$$

If we regard (R, h, k) as an (m,n)-hypermodule over itself, the subhypermodule $\{0\}$ is an n-ary 2-absorbing subhypermodule of R.

In the following theorem, one may see that that either the hyperideal N_m defined in Corollary 2.8 is a prime hyperideal of R or there is an element $a \in R$ such that N_{am} is a prime hyperideal of R whenever N is an n-ary 2-absorbing subhypermodule of M and $m \in M \setminus N$.

Theorem 2.11. *Let N be a subhypermodule of the (m,n)-hypermodule (M,f,g) over (R,h,k). If N is an n-ary 2-absorbing subhypermodule of M, then for every $m \in M \setminus N$ either $N_m = \{r \in R : g(r,m,1_R^{(n-2)}) \subseteq N\}$ is a prime hyperideal of R or $N_{am} = \{r \in R : g(r,a,m,1_R^{(n-3)}) \subseteq N\}$ is a prime hyperideal of R for some $a \in R$.*

Proof. Let N be an n-ary 2-absorbing subhypermodule of M. It follows from Theorem 2.7 that S_N is an n-ary 2-absorbing hyperideal of R, and so either $\sqrt{S_N}^{(m,n)} = P$ is an n-ary prime hyperideal of R or $\sqrt{S_N}^{(m,n)} = P_1 \cap P_2$, where P_1, P_2 are the only distinct n-ary prime hyperideals of R that are minimal over S_N, by [5, Theorem 3.7]. First assume that $\sqrt{S_N}^{(m,n)} = P$ is an n-ary prime hyperideal of R. If $P \subseteq N_m$ and $k(a_1^n) \in N_m$ for some $a_1^n \in R$ and $m \in M \setminus N$, then $g(k(a_1^n), m, 1_R^{(n-2)}) = g(a_1^{n-1}, g(a_n, m, 1_R^{(n-2)})) \subseteq N$. Since N is an n-ary 2-absorbing subhypermodule of M, either $g(a_1^{n-1}, M) \subseteq N$ or $g(a_i, 1_R^{(n-2)}, g(a_n, m, 1_R^{(n-2)})) = g(a_i, a_n, m, 1_R^{(n-3)}) \subseteq N$ for some $i \in \{1, \ldots, n-1\}$. If $g(a_1^{n-1}, M) \subseteq N$, then $k(a_1^{n-1}, 1_R) \in S_N \subseteq P$, and so $a_i \in N_m$ for some $i \in \{1, \ldots, n-1\}$, as $P \subseteq N_m$ and P is prime. Let $g(a_i, 1_R^{(n-2)}, g(a_n, m, 1_R^{(n-2)})) = g(a_i, a_n, m, 1_R^{(n-3)}) \subseteq N$, for some $i \in \{1, \ldots, n-1\}$. Since N is an n-ary 2-absorbing subhypermodule of M, either $g(a_1, a_n, M, 1_R^{(n-3)}) \subseteq N$ or $a_i \in N_m$ or $a_n \in N_m$. Therefore N_m is a prime hyperideal of R, as $S_N \subseteq P \subseteq N_m$. If $P \nsubseteq N_m$, then there exists $a \in P \setminus N_m$, and so $g(a, m, 1_R^{(n-2)}) \nsubseteq N$. It follows from [5, Theorem 3.7 (i)] that $k(P^{(2)}, 1_R^{(n-2)}) \subseteq S_N \subseteq N_m$. Hence $k(P, a, 1_R^{(n-2)}) \subseteq N_m$, and so $P \subseteq N_{am} = \{r \in R : g(r, a, m, 1_R^{(n-3)}) \subseteq N\}$. By a similar argument as above, N_{am} is a prime hyperideal of R for some $a \in R$.

Now, assume that $\sqrt{S_N}^{(m,n)} = P_1 \cap P_2$, where P_1, P_2 are the only distinct n-ary prime hyperideals of R that are minimal over S_N. If $P_1 \subseteq N_m$, then by a similar argument as above, N_m is a prime hyperideal of R. Suppose that $P_1 \nsubseteq N_m$. Then there exists $a \in P_1 \setminus N_m$, and so $g(a, m, 1_R^{(n-2)}) \nsubseteq N$. It follows from [5, Theorem 3.7 (ii)] that $k(P_1, P_2, 1_R^{(n-2)}) \subseteq S_N \subseteq N_m$. Hence $k(P_2, a, 1_R^{(n-2)}) \subseteq N_m$, and so $P_2 \subseteq N_{am} = \{r \in R : g(r, a, m, 1_R^{(n-3)}) \subseteq N\}$. By a similar argument as above, N_{am} is a prime hyperideal of R for some $a \in R$. □

Theorem 2.12. *Let N be a subhypermodule of the (m,n)-hypermodule (M,f,g)*

over (R,h,k). Then N is an n-ary 2-absorbing subhypermodule of M, if N satisfies one of the following conditions:

(i) *N is a prime subhypermodule of M.*

(ii) *N is the intersection of two prime subhypermodules of M.*

(iii) *N is a primary subhypermodule of M with the properties that*

$$k\big(k(r_1^{(\alpha)},1_R^{(n-\alpha)}),\ldots,k(r_{n-1}^{(\alpha)},1_R^{(n-\alpha)}),1_R\big)\notin S_N$$

for every $r_1^{n-1}\in R$ and $\alpha>1$.

Proof. There is nothing to prove, if N is a prime subhypermodule of M. Suppose that $N=N_1\cap N_2$, where N_1,N_2 are two prime subhypermodules of M, and suppose that $g(r_1^{n-1},m)\subseteq N$ for $r_1^{n-1}\in R$ and $m\in M$. If $m\in N_1\cap N_2$ or $g(r_1^{n-1},M)\subseteq N_1\cap N_2$, then there is nothing to prove. Assume that $m\in N_1$ and $g(r_1^{n-1},M)\subseteq N_2$. This means that $r_i\in S_{N_2}$ for some $i\in\{1,\ldots,n-1\}$, as S_{N_2} is a prime hyperideal of R, by [2, Theorem 4.3]. Therefore $g(r_i,m,1_R^{(n-2)})\subseteq N=N_1\cap N_2$, and hence N is an n-ary 2-absorbing subhypermodule of M. Suppose that N is a primary subhypermodule of M and $g(r_1^{n-1},m)\subseteq N$ for $r_1^{n-1}\in R$ and $m\in M$ such that $g(r_i,m,1_R^{(n-2)})\nsubseteq N$ for every $i\in\{1,\ldots,n-1\}$. If $m\in N$, then we are done. Assume that $m\in M\setminus N$. Then either $g(k(r_1^{(t)},1_R^{(n-t)}),\ldots,k(r_{n-1}^{(t)},1_R^{(n-t)}),M)\subseteq N$ for $t\leq n$ or $g(k_{(l)}(r_1^{(t)}),\ldots,k_{(l)}(r_{n-1}^{(t)}),M)\subseteq N$ for $t>n$ such that $t=l(n-1)+1$. The first possibility implies that $k(k(r_1^{(t)},1_R^{(n-t)}),\ldots,k(r_{n-1}^{(t)},1_R^{(n-t)}),1_R)\in S_N$, but $k(k(r_1^{(\alpha)},1_R^{(n-\alpha)}),\ldots,k(r_{n-1}^{(\alpha)},1_R^{(n-\alpha)}),1_R)\notin S_N$ for every $r_1^{n-1}\in R$ and $\alpha>1$, by hypothesis. Hence $t=1$, and so

$$g\big(k(r_1,1_R^{(n-1)}),\ldots,k(r_{n-1},1_R^{(n-1)}),M\big)=g(r_1^{n-1},M)\subseteq N.$$

The second case is proved similarly. Thus N is an n-ary 2-absorbing subhypermodule of M. $\square$

Example 2.13. Let $(\mathbb{Z},f,g)$ be the (m,n)-hypermodule over (R,h,k) with the following hyperoperations and operation

$$\begin{aligned}f(x_1^m)&=\bigoplus_{i=1}^{m}x_i=\\&=\{x_1^m,x_{i_1}+x_{i_2},\ldots,x_{i_1}+x_{i_2}+\cdots\\&+x_{i_m}|1\leq i_1\neq i_2\neq\cdots\neq i_m\leq m\}\\g(s_1^{n-1},x)&=(\bigotimes_{i=1}^{n-1}s_i)\odot x=\{(\prod_{i=1}^{n-1}s_i)\cdot x\},\end{aligned}$$

where $x \oplus y = \{x, y, x+y\}$, $z \odot x = \{z \cdot x\}$ and $x \oplus y = x \cdot y$, for $x_1^m, s_1^n, x, y, z \in \mathbb{Z}$, as in [2, Example 3.5]. Suppose that $p, q \in \mathbb{Z}$ are prime numbers.

(1) It follows from [2, Example 4.2] and Theorem 2.12 that $\langle p\rangle \cap \langle q\rangle = g(\mathbb{Z}, 1_{\mathbb{Z}}^{(n-2)}, p) \cap g(\mathbb{Z}, 1_{\mathbb{Z}}^{(n-2)}, q)$ is an n-ary 2-absorbing subhypermodule of the (m, n)-hypermodule $(\mathbb{Z}, f, g)$.

(2) $\langle pq\rangle = g(\mathbb{Z}, 1_{\mathbb{Z}}^{(n-3)}, p, q) = \{t \cdot p \cdot q : t \in \mathbb{Z}\}$ is an n-ary 2-absorbing subhypermodule of the (m, n)-hypermodule $(\mathbb{Z}, f, g)$. To see this, let $g(r_1^{n-1}, m) \subseteq \langle pq\rangle$ for some $r_1^{n-1}, m \in \mathbb{Z}$ such that $m \notin \langle pq\rangle$. Then there exists $r \in \mathbb{Z}$ such that $r_1 \cdots r_{n-1} \cdot m = r \cdot p \cdot q$. But p is a prime number, then either $p \mid r_i$ for some $i \in \{1, \ldots, n-1\}$ or $p \mid m$. Assume that $p \mid r_i$ for some $i \in \{1, \ldots, n-1\}$. Then $r_i = p \cdot s = k(p, 1_{\mathbb{Z}}^{(n-2)}, s) \subseteq k(p, 1_{\mathbb{Z}}^{(n-2)}, \mathbb{Z})$ for some $s \in \mathbb{Z}$, and so $r_1 \cdots r_{i-1} \cdot p \cdot s \cdot r_{i+1} \cdots r_{n-1} \cdot m = r \cdot p \cdot q = k(p, q, r, 1_{\mathbb{Z}}^{(n-3)})$. This means that either $q \mid r_j$ for some $j \in \{1, \ldots, n-1\} \setminus \{i\}$ or $q \mid m$ or $q \mid s$. If $q \mid r_j$, then $p \cdot q \mid r_i \cdot r_j$, and so $g(r_1^{n-1}, \mathbb{Z}) \subseteq \langle pq\rangle$. If $q \mid m$, then $g(r_i, m, 1_R^{(n-2)}) \subseteq \langle pq\rangle$ for some $i \in \{1, \ldots, n-1\}$. If $q \mid s$, then $p \cdot q \mid r_i$, and so $g(r_i, m, 1_R^{(n-2)}) \subseteq \langle pq\rangle$ for some $i \in \{1, \ldots, n-1\}$. Thus $\langle pq\rangle$ is an n-ary 2-absorbing subhypermodule of $(\mathbb{Z}, f, g)$.

By Theorem 2.11, if N is an n-ary 2-absorbing subhypermodule of M such that $\sqrt{S_N}^{(m,n)}$ is a prime hyperideal of R, then $N_m = \{r \in R : g(r, m, 1_R^{(n-2)}) \subseteq N\}$ may be a prime hyperideal of R for every $m \in M \setminus N$. The following theorem shows that $\sqrt{N_m}^{(m,n)}$ should be a prime hyperideal of R.

Theorem 2.14. *Let N be an n-ary 2-absorbing subhypermodule of the (m, n)-hypermodule (M, f, g) over (R, h, k) such that $\sqrt{S_N}^{(m,n)} = P$ is a prime hyperideal of R. If $m \in M \setminus N$, then $\sqrt{N_m}^{(m,n)}$ is a prime hyperideal of R containing P. Moreover, if $\sqrt{S_N}^{(m,n)} = P \cap Q$ for some prime hyperideals P, Q such that $P \subseteq \sqrt{N_m}^{(m,n)}$, then $\sqrt{N_m}^{(m,n)}$ is a prime hyperideal of R.*

Proof. Let $k(a_1^n) \in \sqrt{N_m}^{(m,n)}$ for some $a_1^n \in R$ and $m \in M \setminus N$. Then either $k\big(k(a_1^{(t)}, 1_R^{(n-t)}), \ldots, k(a_n^{(t)}, 1_R^{(n-t)})\big) \in N_m$ for $t \leq n$ or $k\big(k_{(l)}(a_1^{(t)}), \ldots, k_{(l)}(a_n^{(t)})\big) \in N_m$ for $t > n$ such that $t = l(n-1)+1$. The first possibility implies that

$$g(k(a_1^{(t)}, 1_R^{(n-t)}), \ldots, k(a_{n-1}^{(t)}, 1_R^{(n-t)}), g(a_n^{(t)}, 1_R^{(n-t-1)}, m)) \subseteq N.$$

But N is n-ary 2-absorbing subhypermodule. Then either

$$g\big(k(a_1^{(t)}, 1_R^{(n-t)}), \ldots, k(a_{n-1}^{(t)}, 1_R^{(n-t)}), M\big) \subseteq N$$

or $g(k(a_i^{(t)}, 1_R^{(n-t)}), g(a_n^{(t)}, 1_R^{(n-t-1)}, m), 1_R^{(n-2)}) \subseteq N$ for some $i \in \{1, \ldots, n-1\}$. Hence either $a_i \in \sqrt{S_N}^{(m,n)} \subseteq \sqrt{N_m}^{(m,n)}$ for some $i \in \{1, \ldots, n-1\}$, $\sqrt{S_N}^{(m,n)}$ is a prime hyperideal or $g(k(a_i^{(t)}, 1_R^{(n-t)}), k(a_n^{(t)}, 1_R^{(n-t)}), M, 1_R^{(n-3)}) \subseteq N$ for some $i \in \{1, \ldots, n-1\}$ or $g(a_i^{(t)}, 1_R^{(n-t-1)}), m) \subseteq N$ for some $i \in \{1, \ldots, n\}$. Thus $\sqrt{N_m}^{(m,n)}$ is a prime hyperideal of R. The second possibility is similar.

The "Moreover" statement is clear if $P \subseteq \sqrt{N_m}^{(m,n)}$. We note that if P and Q are not contained in $\sqrt{N_m}^{(m,n)}$, then $\sqrt{N_m}^{(m,n)}$ need not be prime, as by Example 2.13, $N = \langle pq \rangle = g(\mathbb{Z}, 1_{\mathbb{Z}}^{(n-3)}, p, q)$ is an n-ary 2-absorbing subhypermodule of the (m,n)-hypermodule $(\mathbb{Z}, f, g)$ such that $p, q \in \mathbb{Z}$ are prime numbers. If we take $m = 1_{\mathbb{Z}}$, then $\sqrt{N_m}^{(m,n)} = \langle p \rangle \cap \langle q \rangle = g(\mathbb{Z}, 1_{\mathbb{Z}}^{(n-2)}, p) \cap g(\mathbb{Z}, 1_{\mathbb{Z}}^{(n-2)}, q)$ is not a prime hyperideal of R. □

The next theorem shows that if N is an n-ary 2-absorbing subhypermodule of the (m,n)-hypermodule (M, f, g) over (R, h, k), and $g(I_1, I_2, 1_R^{(n-3)}, L) \subseteq N$ for some hyperideals I_1, I_2 of R and subhypermodule L of M, then either $g(I_1, I_2, 1_R^{(n-3)}, M) \subseteq N$ or $g(I_1, 1_R^{(n-2)}, L) \subseteq N$ or $g(I_2, 1_R^{(n-2)}, L) \subseteq N$.

Theorem 2.15. *Let N be a subhypermodule of the (m,n)-hypermodule (M, f, g) over (R, h, k). Then N is n-ary 2-absorbing if and only if $g(I_1, I_2, 1_R^{(n-3)}, L) \subseteq N$ for some hyperideals I_1, I_2 of R and subhypermodule L of M, then one of the following conditions holds:*

(i) $g(I_1, I_2, 1_R^{(n-3)}, M) \subseteq N$.
(ii) $g(I_1, 1_R^{(n-2)}, L) \subseteq N$.
(iii) $g(I_2, 1_R^{(n-2)}, L) \subseteq N$.

Proof. Let N be an n-ary 2-absorbing subhypermodule of the (m,n)-hypermodule (M, f, g) over (R, h, k) and let $g(I_1, I_2, 1_R^{(n-3)}, L) \subseteq N$ for some hyperideals I_1, I_2 of R and subhypermodule L of M such that non of

$$g(I_1, I_2, 1_R^{(n-3)}, M) \subseteq N, g(I_1, 1_R^{(n-2)}, L) \subseteq N$$

and $g(I_2, 1_R^{(n-2)}, L) \subseteq N$ is hold. Then there exist $a_1 \in I_1$ and $a_2 \in I_2$ such that $g(a_1, 1_R^{(n-2)}, L)$ and $g(a_2, 1_R^{(n-2)}, L)$ which are not contained in N. This means that $g(a_1, a_2, 1_R^{(n-3)}, M) \subseteq N$, by Theorem 2.3 as N is 2-absorbing.

Thus $g(I_1, I_2, 1_R^{(n-3)}, M) \nsubseteq N$, and so there exist $r_1 \in I_1$ and $r_2 \in I_2$ such that $g(r_1, r_2, 1_R^{(n-3)}, M) \nsubseteq N$. But $g(r_1, r_2, 1_R^{(n-3)}, L) \subseteq N$, and hence either $g(r_1, 1_R^{(n-2)}, L) \subseteq N$ or $g(r_2, 1_R^{(n-2)}, L) \subseteq N$. Consider three following cases.

Case one: Suppose that $g(r_1, 1_R^{(n-2)}, L) \subseteq N$ and $g(r_2, 1_R^{(n-2)}, L) \nsubseteq N$. Since $g(a_1, r_2, 1_R^{(n-3)}, L) \subseteq N$ and $g(r_2, 1_R^{(n-2)}, L), g(a_1, 1_R^{(n-2)}, L)$ are not contained in N, we conclude that $g(a_1, r_2, 1_R^{(n-3)}, M) \subseteq N$.

Also, $g(h(r_1, a_1, 0^{(m-2)}), r_2, 1_R^{(n-3)}, L) \subseteq N$, $g(r_1, 1_R^{(n-2)}, L) \subseteq N$ and $g(a_1, 1_R^{(n-2)}, L) \nsubseteq N$. Therefore $g(h(r_1, a_1, 0^{(m-2)}), 1_R^{(n-2)}, L) \nsubseteq N$.

Hence there exists $u_1 \in h(r_1, a_1, 0^{(m-2)})$ such that $g(u_1, 1_R^{(n-2)}, L) \nsubseteq N$. Again, since $g(u_1, r_2, 1_R^{(n-3)}, L) \subseteq N, g(r_2, 1_R^{(n-2)}, L) \nsubseteq N$ and $g(u_1, 1_R^{(n-2)}, L) \nsubseteq N$, we conclude that $g(u_1, r_2, 1_R^{(n-3)}, M) \subseteq N$. It follows that

$$\begin{aligned} g(r_1, r_2, 1_R^{(n-3)}, M) &\subseteq g(h(u_1, -a_1, 0^{(m-2)}), r_2, 1_R^{(n-3)}, M) = \\ &f(g(-a_1, r_2, 1_R^{(n-3)}, M), g(u_1, r_2, 1_R^{(n-3)}, M), 0^{(m-2)}) \subseteq N, \end{aligned}$$

a contradiction.

Case two: Suppose that $g(r_1, 1_R^{(n-2)}, L) \nsubseteq N$ and $g(r_2, 1_R^{(n-2)}, L) \subseteq N$. By a similar argument as in the previous case, $g(r_1, r_2, 1_R^{(n-3)}, M) \subseteq N$ which is also a contradiction.

Case three: Suppose that $g(r_1, 1_R^{(n-2)}, L) \subseteq N$ and $g(r_2, 1_R^{(n-2)}, L) \subseteq N$. Since $g(r_2, 1_R^{(n-2)}, L) \subseteq N$ and $g(a_2, 1_R^{(n-2)}, L) \nsubseteq N$, we conclude that $g(h(r_2, a_2, 0^{(m-2)}), 1_R^{(n-2)}, L) \nsubseteq N$. Hence there exists $u_2 \in h(r_2, a_2, 0^{(m-2)})$ such that $g(u_2, 1_R^{(n-2)}, L) \nsubseteq N$. But $g(a_1, u_2, 1_R^{(n-3)}, L) \subseteq N$, $g(u_2, 1_R^{(n-2)}, L)$, $g(a_1, 1_R^{(n-2)}, L)$ are not contained in N and N is 2-absorbing.

Thus $g(a_1, u_2, 1_R^{(n-3)}, M) \subseteq N$. It is not hard to see that $g(r_1, 1_R^{(n-2)}, L) \subseteq N$ and $g(a_1, 1_R^{(n-2)}, L) \nsubseteq N$ implies that $g(h(r_1, a_1, 0^{(m-2)}), 1_R^{(n-2)}, L) \nsubseteq N$.

Hence there exists $u_1 \in h(r_1, a_1, 0^{(m-2)})$ such that $g(u_1, 1_R^{(n-2)}, L) \nsubseteq N$, and since $g(u_1, a_2, 1_R^{(n-3)}, L) \subseteq N$ and $g(a_2, 1_R^{(n-2)}, L) \nsubseteq N$, $g(u_1, a_2, 1_R^{(n-3)}, M) \subseteq N$. But $g(u_1, u_2, 1_R^{(n-3)}, L) \subseteq N$ and both of $g(u_1, 1_R^{(n-2)}, L)$ and $g(u_2, 1_R^{(n-2)}, L)$ are not contained in N. Then $g(u_1, u_2, 1_R^{(n-3)}, M) \subseteq N$. Therefore

$$\begin{aligned} g(r_1, r_2, 1_R^{(n-3)}, M) &\subseteq g(h(u_1, -a_1, 0^{(m-2)}), h(u_2, -a_2, 0^{(m-2)}), 1_R^{(n-3)}, M) \\ &= f(g(-a_1, u_2, 1_R^{(n-3)}, M), g(a_1, a_2, 1_R^{(n-3)}, M), \\ &\quad g(u_1, u_2, 1_R^{(n-3)}, M), g(u_1, -a_2, 1_R^{(n-3)}, M), 0^{(m-4)}) \\ &\subseteq N. \end{aligned}$$

Hence $g(r_1, r_2, 1_R^{(n-3)}, M) \subseteq N$, which is a contradiction. Thus $g(I_1, I_2, 1_R^{(n-3)}, M) \subseteq N$.

Conversely, let N be a subhypermodule of M and let $g(r_1^{n-1}, m) \subseteq N$ for some $r_1^{n-1} \in R$ and $m \in M$. Then

$$g\Big(k(R, r_1^{n-2}, 1_R), k(R, r_{n-1}, 1_R^{(n-2)}), g(R, 1_R^{(n-2)}, m), 1_R^{(n-3)}\Big) \subseteq N.$$

By given hypothesis, we have either

$$g\Big(k(R, r_1^{n-2}, 1_R), k(R, r_{n-1}, 1_R^{(n-2)}), 1_R^{(n-3)}, M\Big) \subseteq N$$

or

$$g\Big(k(R, r_1^{n-2}, 1_R), g(R, 1_R^{(n-2)}, m), 1_R^{(n-2)}\Big) \subseteq N.$$

The first possibility implies $g(r_1^{n-1}, M) \subseteq N$ and the second possibility implies $g(r_{n-1}, 1_R^{(n-2)}, m) \subseteq N$, and so assume that $g(k(R, r_1^{n-2}, 1_R), g(R, 1_R^{(n-2)}, m), 1_R^{(n-2)}) \subseteq N$, which means that

$$g\Big(k(R, r_1^{n-3}, 1_R), k(R, r_{n-2}, 1_R^{(n-2)}), g(R, 1_R^{(n-2)}, m), 1_R^{(n-3)}\Big) \subseteq N.$$

By a similar argument, $g(r_1^{n-1}, M) \subseteq N$ or

$g(r_i, 1_R^{(n-2)}, m) \subseteq N$ for some $i \in \{1, \ldots, n-2\}$. Continue in this way: after $n-2$ steps, we get either $g(r_1^{n-1}, M) \subseteq N$ or $g(r_1, 1_R^{(n-2)}, m) \subseteq N$. Thus N is an n-ary 2-absorbing subhypermodule of M. □

We end this section with the following corollary.

Corollary 2.16. *Let N be a subhypermodule of the (m, n)-hypermodule (M, f, g) over (R, h, k) and I a hyperideal of R. If N is n-ary 2-absorbing, then $N_I = \{m \in M : g(I, m, 1_R^{(n-2)}) \subseteq N\}$ is an n-ary 2-absorbing subhypermodule of M. Moreover, $\{m \in M : g(I^{(n-1)}, m) \subseteq N\} = \{m \in M : g(I^{(n-2)}, m, 1_R) \subseteq N\}$ for every $n \geq 4$.*

Proof. Let $g(a_1^{n-1}, m) \subseteq N_I$ for some $a_1^{n-1} \in R$ and $m \in M$. Then we have $g(I, k(a_1^{n-1}, 1_R), m, 1_R^{(n-3)}) \subseteq N$. Since N is n-ary 2-absorbing, Theorem 2.15 implies that $g(I, m, 1_R^{(n-2)}) \subseteq N$ or $g(k(a_1^{n-1}, 1_R), m, 1_R^{(n-2)}) \subseteq N$ or $g(I, k(a_1^{n-1}, 1_R), M, 1_R^{(n-3)}) \subseteq N$. If $g(I, m, 1_R^{(n-2)}) \subseteq N$, then $m \in N_I$ and so we are done. If $g(I, k(a_1^{n-1}, 1_R), M, 1_R^{(n-3)}) = g(I, g(a_1^{n-1}, M), 1_R^{(n-2)}) \subseteq N$, then $g(a_1^{n-1}, M) \subseteq N_I$, which means that N_I is n-ary 2-absorbing. If $g(k(a_1^{n-1}, 1_R), m, 1_R^{(n-2)}) \subseteq N$, then $g(a_1^{n-1}, m) \subseteq N$. Since N is n-ary 2-absorbing, either $g(a_i, m, 1_R^{(n-2)}) \subseteq N \subseteq N_I$ for some $i \in \{1, \ldots, n-1\}$ or $g(a_1^{n-1}, M) \subseteq N \subseteq N_I$. Thus N_I is an n-ary 2-absorbing subhypermodule of M.

For the "Moreover" statement, we show that

$$\begin{aligned} N_{I^2} &= \{m \in M : g(I^{(2)}, m, 1_R^{(n-3)}) \subseteq N\} \\ &= \{m \in M : g(I^{(3)}, m, 1_R^{(n-4)}) \subseteq N\} = N_{I^3}. \end{aligned}$$

Let $m \in N_{I^3}$. Then $g(I^{(2)}, g(I, m, 1_R^{(n-2)}),\ 1_R^{(n-3)}) \subseteq N$. But N is n-ary 2-absorbing. Then, by Theorem 2.15, either $g(I^{(2)}, m, 1_R^{(n-3)}) \subseteq N$ or $g(I^{(2)}, M, 1_R^{(n-3)}) \subseteq N$, and so $m \in N_{I^2}$. Therefore $N_{I^3} = N_{I^2}$, and hence $\{m \in M : g(I^{(n-1)}, m) \subseteq N\} = \{m \in M : g(I^{(n-2)}, m, 1_R) \subseteq N\}$ for every $n \geq 2$. □

3 n-Ary 2-Absorbing Subhypermodules in Multiplication (m, n)-Hypermodules

In this section n-ary 2-absorbing subhypermodules in multiplication (m, n)-hypermodules over Krasner (m, n)-hyperrings are studied. Recall from [11, page 111] that if X is an (m, n)-ary subhypermodule of a canonical (m, n)-ary hypermodule M, then $\langle X \rangle$ is the (m, n)-ary subhypermodule generated by elements of X. If M is generated by a single element x, then M is called a cyclic (m, n)-hypermodule and we write $M = \langle x \rangle = g(R, x, 1_R^{(n-2)})$.

First, the following definition is given.

Definition 3.1. Let $N = g(I, M, 1_R^{(n-2)})$ and $K = g(J, M, 1_R^{(n-2)})$ be subhypermodules of the (m, n)-hypermodule (M, f, g) over (R, h, k) for some hyperideals I and J of R. The g-product of N and K denoted by $g(N, K, 1_R^{(n-2)})$, is defined by $g(I, J, 1_R^{(n-2)}, M)$.

It is clear from [23, Lemma 3.4] and from the definition of subhypermodules of multiplication (m, n)-hypermodules introduced in [2] that $g(N, K, 1_R^{(n-2)}) = g(I, J, 1_R^{(n-2)}, M)$ is a subhypermodule of M contained in $N \cap K$.

Let N be a subhypermodule of the (m, n)-hypermodule (M, f, g) over (R, h, k). The radical of subhypermodule N of M was defined in [2, page 170] as the intersection of all n-ary prime subhypermodules of M containing N and denoted by $\mathrm{rad}_{(m,n)}(N)$. It is shown in [2, Theorem 4.6] that if M is a multiplication (m, n)-hypermodule, then $\mathrm{rad}_{(m,n)}(N) = g(\sqrt{S_N}^{(m,n)}, 1_R^{(n-2)}, M)$.

Theorem 3.2. *Let N be an n-ary 2-absorbing subhypermodule of the cyclic multiplication faithful (m, n)-hypermodule (M, f, g) over (R, h, k). Then either $\mathrm{rad}_{(m,n)}(N) = P$ where P is an n-ary prime subhypermodule of M such* *that* $g(P^{(2)}, 1_R^{(n-2)}) \subseteq N$ *or* $rad_{(m,n)}(N) = P_1 \cap P_2$ *where* P_1, P_2 *are distinct* n*-ary prime subhypermodules of* M *such that* $g(P_1, P_2, 1_R^{(n-2)}) \subseteq N$ *and* $g((rad_{(m,n)}(N))^{(2)}, 1_R^{(n-2)}) \subseteq N$.

Proof. Let N be an n-ary 2-absorbing subhypermodule of the cyclic multiplication faithful (m,n)-hypermodule (M,f,g) over (R,h,k). It follows from Theorem 2.7 that S_N is an n-ary 2-absorbing hyperideal of R, and so, either $\sqrt{S_N}^{(m,n)} = p$ is an n-ary prime hyperideal of R such that $k(p^{(2)}, 1_R^{(n-2)}) \subseteq S_N$ or $\sqrt{S_N}^{(m,n)} = p_1 \cap p_2$, $k(p_1, p_2, 1_R^{(n-2)}) \subseteq S_N$ and $k((\sqrt{S_N}^{(m,n)})^{(2)}, 1_R^{(n-2)}) \subseteq S_N$ where p_1, p_2 are the only distinct n-ary prime hyperideals of R that are minimal over S_N, by [5, Theorem 3.7]. First assume that $\sqrt{S_N}^{(m,n)} = p$ is an n-ary prime hyperideal of R. But M is multiplication. Hence, we conclude by [2, Theorem 4.6] that $\text{rad}_{(m,n)}(N) = g(p, 1_R^{(n-2)}, M)$, and so, by [2, Corollary 4.5], $P = \text{rad}_{(m,n)}(N)$ is an n-ary prime subhypermodule of M and

$$\begin{aligned} g(P^{(2)}, 1_R^{(n-2)}) & g(g(p, 1_R^{(n-2)}, M), g(p, 1_R^{(n-2)}, M), 1_R^{(n-2)}) \\ &= g(k(p^{(2)}, 1_R^{(n-2)}), M, 1_R^{(n-2)}) \subseteq g(S_N, M, 1_R^{(n-2)}) = N \end{aligned}$$

by [2, Remark 3.2]. Now assume that $\sqrt{S_N}^{(m,n)} = p_1 \cap p_2$, $k(p_1, p_2, 1_R^{(n-2)}) \subseteq S_N$ and $k((\sqrt{S_N}^{(m,n)})^{(2)},\ 1_R^{(n-2)}) \subseteq S_N$, where p_1, p_2 are the only distinct n-ary prime hyperideals of R that are minimal over S_N. Then, by [2, Corollary 4.5], $g(p_1, 1_R^{(n-2)}, M)$ and $g(p_2, 1_R^{(n-2)}, M)$ are prime subhypermodules of N and $\text{rad}_{(m,n)}(N) = g(\sqrt{S_N}^{(m,n)}, M, 1_R^{(n-2)}) = g(p_1 \cap p_2, M, 1_R^{(n-2)}) \subseteq g(p_1, 1_R^{(n-2)}, M) \cap g(p_2, 1_R^{(n-2)}, M)$.

Now let $x \in g(p_1, 1_R^{(n-2)}, M) \cap g(p_2, 1_R^{(n-2)}, M)$. Then $x = g(x_1, 1_R^{(n-2)}, m) = g(x_2, 1_R^{(n-2)}, m)$ for some $x_1 \in p_1$, $x_2 \in p_2$ and $m \in M$.

Hence $0 \in g(h(x_1, -x_2, 0^{(m-2)}), 1_R^{(n-2)}, m)$, which means that $h(x_1, -x_2, 0^{(m-2)}) \subseteq F_m = \{0\}$. Therefore $0 \in h(x_1, -x_2, 0^{(m-2)})$, and so $x_2 \in h(x_1, 0^{(m-1)}) \subseteq p_1$. Thus $x = g(x_2, 1_R^{(n-2)}, m) \subseteq g(p_1 \cap p_2, M, 1_R^{(n-2)})$.

Hence

$$\begin{aligned} rad_{(m,n)}(N) &= g(\sqrt{S_N}^{(m,n)}, M, 1_R^{(n-2)}) \\ &= g\big(p_1 \cap p_2, M, 1_R^{(n-2)}\big) \\ &= g\big(p_1, 1_R^{(n-2)}, M\big) \cap g\big(p_2, 1_R^{(n-2)}, M\big) \end{aligned}$$

is the intersection of two n-ary prime subhypermodules of M by [2, Corollary

4.5]. Moreover,

$$\begin{aligned} g\Big(g(p_1,1_R^{(n-2)},M), g(p_2,1_R^{(n-2)},M), 1_R^{(n-2)}\Big) \\ = g\Big(k(p_1,p_2,1_R^{(n-2)}), M, 1_R^{(n-2)}\Big) \\ \subseteq g\big(S_N, M, 1_R^{(n-2)}\big) = N, \end{aligned}$$

by [2, Remark 3.2], and thus

$$\begin{aligned} & g\Big((rad_{(m,n)}(N))^{(2)}, 1_R^{(n-2)}\Big) \\ & = g\Big((g(\sqrt{S_N}^{(m,n)}, M, 1_R^{(n-2)}))^{(2)}, 1_R^{(n-2)}\Big) \\ & = g\Big(k\big((\sqrt{S_N}^{(m,n)}\big)^{(2)}, 1_R^{(n-2)}), M, 1_R^{(n-2)}\Big) \\ & \subseteq g\big(S_N, M, 1_R^{(n-2)}\big) = N. \end{aligned}$$

□

Corollary 3.3. *Let N be an n-ary 2-absorbing subhypermodule of the cyclic multiplication faithful (m,n)-hypermodule (M,f,g) over (R,h,k).*

Then $rad_{(m,n)}(N)$ is an n-ary 2-absorbing subhypermodule of M.

Proof. Let N be an n-ary 2-absorbing subhypermodule of M. It follows from Theorem 3.2 that either $\text{rad}_{(m,n)}(N) = P$ where P is an n-ary prime subhypermodule of M or $\text{rad}_{(m,n)}(N) = P_1 \cap P_2$ where P_1, P_2 are distinct n-ary prime subhypermodules of M. Hence $\text{rad}_{(m,n)}(N)$ is an n-ary 2-absorbing subhypermodule of M, by Theorem 2.12. □

Corollary 3.4. *Let N be an n-ary primary subhypermodule of the cyclic multiplication faithful (m,n)-hypermodule (M,f,g) over (R,h,k) such that $\sqrt{S_N}^{(m,n)} = P$ is an n-ary prime hyperideal of R. Then N is n-ary 2-absorbing if and only if $g(P^{(2)}, M, 1_R^{(n-3)}) \subseteq N$.*

Proof. First, assume that N is an n-ary 2-absorbing subhypermodule of M. Since $\sqrt{S_N}^{(m,n)} = P$ is an n-ary prime hyperideal of R, it follows from [2, Theorem 4.6] that $\text{rad}_{(m,n)}(N) = g(P, 1_R^{(n-2)}, M)$, and so $\text{rad}_{(m,n)}(N)$ is an n-ary prime subhypermodule of M, by [2, Corollary 4.5]. But N is an n-ary 2-absorbing subhypermodule of M. By Theorem 3.2, $g(P^{(2)}, M, 1_R^{(n-3)}) \subseteq N$. Now, assume that $g(P^{(2)}, M, 1_R^{(n-3)}) \subseteq N$ and let $g(a_1^{n-1}, m) \subseteq N$ for some $a_1^{n-1} \in R$ and $m \in M$. S uppose further that $g(a_i, m, 1_R^{(n-2)}) \nsubseteq N$ for every $i \in \{1, \ldots, n-1\}$. Then $k(a_1^{i-1}, a_{i-1}^{n-1}, 1_R^{(2)}) \in \sqrt{S_N}^{(m,n)} = P$, for every $i \in \{1, \ldots, n-1\}$, which is prime, and so there exists $j \in \{1, \ldots, n-1\}$

such that $j \neq i$ and $a_j \in P$. But $g(a_j, m, 1_R^{(n-2)}) \nsubseteq N$ and N is primary. Then there exists $l \in \{1, \ldots, n-1\}$ such that $l \neq j$ and $a_l \in P$. The inclusion $g(P^{(2)}, M, 1_R^{(n-3)}) \subseteq N$ implies that $g(a_j, a_l, M, 1_R^{(n-3)}) \subseteq N$. Hence $g(a_1^{n-1}, M) \subseteq g(a_j, a_l, M, 1_R^{(n-3)}) \subseteq N$, and thus N is an n-ary 2-absorbing subhypermodule of M. □

In [24, Example 3.6], the polynomial Krasner (m,n)-hyperring was introduced. Let x be an indeterminate and R a Krasner (m,n)-hyperring. Then $R[x]$ is called the Krasner (m,n)-hyperring of polynomials of x over R.

Suppose that

$$(a_0, a_1, \ldots, a_k, \ldots) = f\Big(g(a_k, x^{(k)}), g(a_{k-1}, x^{(k-1)}), \ldots, g(a_1, x, 1_R^{(n-2)}), a_0\Big)$$

is a sequence with coefficients in R, and a sequence of elements of $R[x]$ such as $(a_{01}, a_{11}, \ldots, a_{t1}, \ldots), \ldots, (a_{0m}, a_{1m}, \ldots, a_{tm}, \ldots)$ is denoted, for all $m \in \mathbb{N}$, by $(a_0, a_1, \ldots, a_t, \ldots)_1^m$. By [24, Example 3.6], one may see that $(R[x], F, G)$ with the m-ary hyperoperation F and the n-ary hyperoperation G defined as follows:

$$F\big((a_0, a_1, \ldots, a_t, \ldots)_1^m\big) = \{(c_0, c_1, \ldots, c_t, \ldots) : c_k \in f(a_{k1}, a_{k2}, \ldots, a_{km})\}$$
$$G\big((a_0, a_1, \ldots, a_t, \ldots)_1^n\big) = \{(d_0, d_1, \ldots, d_t, \ldots) : d_k \in f_{(k)}(g(a_{i_1 1}, \ldots, a_{i_n n})^{(z)})\}$$

is a Krasner (m,n)-hyperring where $i_1 + \cdots + i_n = k$ and $z = k(m-1)+1$.

Example 3.5. Let $(\mathbb{Z}, f, g)$ be the (m,n)-hypermodule over (R, h, k) as in [2, Example 3.5]. Suppose also that $R = \mathbb{Z}[x,y]$ where x, y are indeterminates and $(R[x,y], F, G)$ with m-ary hyperoperation F and the n-ary hyperoperation G defined above is a Krasner (m,n)-hyperring. Assume that

$$P_1 = G\big((2, x, 0 \ldots, 0, \ldots), R, (1, \ldots, 1, \ldots)_3^n\big),$$

$$P_2 = G\big((2, y, 0 \ldots, 0, \ldots), R, (1, \ldots, 1, \ldots)_3^n\big)$$

are n-ary prime hyperideals of R, and let $I = G(P_1, P_2, (1, \ldots, 1, \ldots)_3^n) = G\big(J, R, (1, \ldots, 1, \ldots)_3^n\big)$ such that
$J = \big(G((2, 0 \ldots, 0, \ldots), G((2, x, 0 \ldots, 0, \ldots), G((2, y, 0 \ldots, 0, \ldots), G((x, y, 0, \ldots, 0, \ldots)\big)$. If we regard $(R[x,y], F, G)$ as an (m,n)-hypermodule over itself, the subhypermodule I is an n-ary 2-absorbing subhypermodule of R and
$\mathrm{rad}_{(m,n)}(N) = P_1 \cap P_2 = G\big((2, x, y, 0 \ldots, 0, \ldots), R, (1, \ldots, 1, \ldots)_3^n\big)$.

We end this paper with the following Theorem.

Theorem 3.6. *Let N be a subhypermodule of the cyclic multiplication faithful (m,n)-hypermodule (M,f,g) over (R,h,k). Then N is n-ary 2-absorbing if and only if whenever $g(N_1,N_2,N_3,1_R^{(n-3)}) \subseteq N$ for some subhypermodules N_1,N_2,N_3 of M, then one of the following conditions holds:*
(i) $g(N_1,N_2,\ 1_R^{(n-2)}) \subseteq N$.
(ii) $g(N_1,N_3,1_R^{(n-2)}) \subseteq N$.
(iii) $g(N_2,N_3,1_R^{(n-2)}) \subseteq N$.

Proof. Let N be an n-ary 2-absorbing subhypermodule of the (m,n)-hypermodule M and let $g(N_1,N_2,N_3,\ 1_R^{(n-3)}) \subseteq N$ for some subhypermodules N_1,N_2 $,N_3$ of M. Since M is multiplication (m,n)-hypermodule, there exist hyperideals I_1,I_2 and I_3 of R such that $N_1 = g(I_1,M,1_R^{(n-2)})$, $N_2 = g(I_2,M,1_R^{(n-2)})$ and $N_3 = g(I_3,M,1_R^{(n-2)})$. Hence

$$\begin{aligned} g(N_1,N_2,N_3,1_R^{(n-3)}) &= g\Big(g(I_1,M,1_R^{(n-2)}),\\ &\qquad g\big(g(I_2,M,1_R^{(n-2)}),I_3,M,1_R^{(n-2)}\big),1_R^{(n-3)}\Big)\\ &\subseteq N, \end{aligned}$$

and so $g(I_1,I_2,1_R^{(n-3)},g(I_3,1_R^{(n-2)},M)) \subseteq N$. But N is an n-ary 2-absorbing subhypermodule of M. By Theorem 2.15, either $g(I_1,I_2,1_R^{(n-3)},M) \subseteq N$ or $g(I_1,1_R^{(n-2)},g(I_3,1_R^{(n-2)},M)) \subseteq N$ or $g(I_2,1_R^{(n-2)},g(I_3,1_R^{(n-2)},M)) \subseteq N$. Thus either $g(N_1,N_2,1_R^{(n-2)}) \subseteq N$ or $g(N_1,N_3,1_R^{(n-2)}) \subseteq N$ or $g(N_2,N_3,1_R^{(n-2)}) \subseteq N$. Conversely, suppose that $g(I_1,I_2,1_R^{(n-3)},L) \subseteq N$ for some hyperideals I_1,I_2 of R and subhypermodule L of M. But M is multiplication (m,n)-hypermodule. Then there exists a hyperideal I_3 of R such that $L = g(I_3,M,1_R^{(n-2)})$, and so, by Definition 3.1, $g(g(I_1,M,1_R^{(n-2)}),g(I_2,M,1_R^{(n-2)}),g(I_3,M,1_R^{(n-2)}),1_R^{(n-3)}) \subseteq N$.

Hence, by hypothesis, either $g(I_1,I_2,1_R^{(n-2)},M) \subseteq N$ or $g(I_1,I_3,1_R^{(n-2)},M) = g(I_1,1_R^{(n-2)},L) \subseteq N$ or $g(I_2,I_3,1_R^{(n-2)},M) = g(I_2,1_R^{(n-2)},L) \subseteq N$. Thus N is an n-ary 2-absorbing subhypermodule of M, by Theorem 2.15. □

4 Conclusion

This research contributes to the idea of n-ary 2-absorbing subhypermodule of an (m,n)-hypermodule M, and gives a description of these subhypermodules. Also, we studied n-ary 2-absorbing subhypermodules in multiplication (m,n)-hypermodules over Krasner (m,n)-hyperrings. In the future, this work will be expanded to explore the concept of $(\mathbf{k}, n)$-absorbing subhypermodule of an (m, n)-hypermodule M, with the following definition: a $(\mathbf{k}, n)$-absorbing subhypermodule is a proper subhypermodule N of M having the property that if whenever $g(r_1^{\mathbf{k}(n-1)}, m) \subseteq N$ for $r_1^{\mathbf{k}(n-1)} \in R$ and $m \in M$, then either $g(r_1^{\mathbf{k}(n-1)}, M) \subseteq N$ or there are $(\mathbf{k}-1)(n-1)$ of the r_i's whose g-product with m is in N. We intend to study properties of this notion, as a future work.

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References

[1] Ameri R., Norouzi M. Corrigendum to "On multiplication (m, n)-hypermodules" [European Journal of Combinatorics, 44 (2015), 153–171], European Journal of Combinatorics 44, part B (2015), 172–174. https://doi.org/10.1016/j.ejc.2014.11.005

[2] Ameri R. and Norouzi M. On multiplication (m, n)-hypermodules, European Journal of Combinatorics, 44 (2015), 153–171.

[3] Ameri R., Norouzi M. Prime and primary hyperideals in Krasner (m, n)-hyperrings, European Journal of Combinatorics, (2013), 379–390.

[4] Ameri R., Norouzi M., Leoreanu-Fotea V. On prime and primary subhypermodules of (m, n)-hypermodules, European Journal of Combinatorics, 44 (2015), 175–190.

[5] Anbarloei M. n-ary 2-absorbing and 2-absorbing primary hyperideals in Krasner (m, n)-hyperrings, Matemat. Vesnik, (2018), 1–13.

[6] Anderson D. F. , Badawi A. On n-absorbing ideals of commutative rings, Comm. in Alg., 39(5) (2011), 1646–1672.

[7] Anvariyeh S. M., Mirvakili S. Canonical (m, n)-hypermodules over Krasner (m, n)-hyperrings, Iranian J. Math. Sci. Inf., 7 (2) (2012), 17–34.

[8] Anvariyeh S. M., Mirvakili S., Davvaz B. Fundamental relation on (m, n)-ary hypermodules over (m, n)-ary hyperrings, Ars Combin., 94 (2010), 273–288.

[9] Badawi A. n-Absorbing Ideals of Commutative Rings and Recent Progress on Three Conjectures: A Survey, In Rings, Polynomials, and Modules. Springer, Cham, (2017), 33–52.

[10] Badawi A. On 2-absorbing ideals of commutative rings, Bull. Austral. Math. Soc. 75 (2007), no. 3, 417–429.

[11] Belali Z., Anvariyeh S. M., Mirvakili S. Free and cyclic (m, n)-ary hypermodules, Tamkang J. Math., 42 (1) (2011), 105–118.

[12] Crombez G. On (m, n)-rings, Abh. Math. Semin. Univ. Hamburg, 37 (1972), 180–199.

[13] Crombez G., Timm J. On (m, n)-quotient rings, Abh. Math. Semin. Univ. Hamburg, 37 (1972), 200–203.

[14] Davvaz B., Vougiouklis T. n-ary hypergroups, Iran. J. Sci. Technol., 30 (A2) (2006), 165–174.

[15] Davvaz B. A new view of fundamental relations on hyperrings, Proceedings of Tenth Int. Congress on Algebraic Hyperstructures and Applications, AHA 2008, pp. 43–55.

[16] Dörente W. Untersuchungen über einen verallgemeinerten Gruppenbegriff, Math. Z., 29 (1928), 1–19.

[17] Hila K., Naka K., Davvaz B. On (k, n)-absorbing hyperideals in Krasner (m, n)-hyperrings, Q. J. Math., 69 (2018), 1035–1046.

[18] Krasner E. An extension of the group concept (reported by L.G. Weld), Bull. Amer. Math. Soc., 10 (1904), 290–291.

[19] Leoreanu V. Canonical n-ary hypergroups, Ital. J. Pure Appl. Math., 24 (2008).

[20] Leoreanu-Fotea V., Davvaz B. n-hypergroups and binary relations, European J. Combin., 29 (2008), 1027–1218.

[21] Leoreanu-Fotea V., Davvaz B. Roughness in n-ary hypergroups, Inform. Sci., 178 (2008), 4114–4124.

[22] Mirvakili S., Davvaz B. Constructions of (m, n)-hyperrings, Matematicki Vesnik, 67 (2015), 1–16.

[23] Mirvakili S., Davvaz B. Relations on Krasner (m, n)-hyperrings, European J. Combin., 31 (2010), 790–802.

[24] Norouzi, M., Cristea, I. A note on composition (m, n)-hyperrings, An. oStiinot. Univ. Ovidius Constanota Ser. Mat., 25 (2), (2017), 101–122.

[25] Ostadhadi-Dehkordi S., Davvaz B. A Note on Isomorphism Theorems of Krasner (m, n)-hyperrings, Arab. J. Math., 5 (2016), 103–115.

[26] Ulucak G. On expansions of prime and 2-absorbing hyperideals in multiplicative hyperrings, Turkish Journal of Mathematics, 43 (3), (2019), 1504–1517.

[27] Yassine, A., Nikmehr,M. J., Nikandish, R. n-Ary k-absorbing hyperideals in Krasner (m, n)-hyperrings, Afr. Mat., 33(19), (2022).

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