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# Functional equations on discrete sets

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**Abstract**

Let $Y(+)$ be a group, $D \subseteq \mathbb{Z}^2$ where $\mathbb{Z}(+, \leqslant)$ denotes the ordered group of all integers, and $\mathbb{Z}^2 := \mathbb{Z} \times \mathbb{Z}$. We shall use the notations $D_x := \{u \in \mathbb{Z} \mid \exists v \in X : (u,v) \in D\}$, $D_y := \{v \in \mathbb{Z} \mid \exists u \in \mathbb{Z} : (u,v) \in D\}$, $D_{x+y} := \{z \in \mathbb{Z} \mid \exists (u,v) \in D : z = u+v\}$. The main purpose of the article is to find sets $D \subseteq \mathbb{Z}^2$ that the general solution of the functional equation $f(x+y) = g(x)+h(y)$ for all $(x,y) \in D$ with unknown functions $f : D_{x+y} \to Y$, $g : D_x \to Y$, $h : D_y \to Y$ is in the form of $f(u) = a(u) + C_1 + C_2$ for all $u \in D_{x+y}$, $g(v) = a(v) + C_1$ for all $v \in D_x$, $h(z) = a(z)+C_2$ for all $z \in D_y$ where $a : \mathbb{Z} \to Y$ is an additive function, $C_1$, $C_2 \in Y$ are constants.

## 1 Introduction

In the sequel we shall use the notations that if $D \subseteq X^2 := X \times X$ where $X = X(+)$ is a groupoid then

$$D_x := \{u \in X \mid \exists v \in \mathbb{G} : (u,v) \in D\},$$
$$D_y := \{v \in Y \mid \exists u \in \mathbb{G} : (u,v) \in D\},$$
$$D_{x+y} := \{z \in X \mid \exists (u,v) \in D : z = u+v\}.$$

We shall also use the following concepts.

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- If $X(+)$ and $Y(+)$ are structures, $a : X \to Y$ is a function such that $a(x+y) = a(x) + a(y)$ for all $x, y \in X$ then the function $a$ is said to be additive function [1], [3], [13].
- If $X(+) = Y(+) = \mathbb{R}(+)$, and the function $a : \mathbb{R} \to \mathbb{R}$ is an additive function then it is said to be Cauchy additive function [5], see also [15] and [9]. It is worth mentioning that A. L. Cauchy was the first who proved that the continuous additive function $a : \mathbb{R} \to \mathbb{R}$ is in the form $a(x) = cx$ for all $x \in \mathbb{R}$ where $c \in \mathbb{R}$ is a constant. It is well-known that if a Cauchy additive function $a$ is not continuous then the graph of $a$ is dense in $\mathbb{R}^2$ which can be easily shown using Hamel bases.
- If $X(+)$, and $Y(+)$ are structures, $D \subseteq X^2$, and the unknown functions $f : D_{x+y} \to Y$, $g : D_x \to Y$, $h : D_y \to Y$ satisfy the functional equation

$$f(x+y) = g(x) + h(y) \qquad (x, y \in D) \tag{PexAdd}$$

  then the equation (PexAdd) is said to be restricted Pexider additive functional equation [13], [17], [10]. In his paper [17] J. Rimán have shown that if the functions $f : D_{x+y} \to E$, $g : D_x \to E$, $h : D_y \to E$ satisfy the functional equation (PexAdd) where $D$ is a nonempty connected open subset of the set $\mathbb{R}^2$, and $E = E(+)$ is an Abelian group then the general solution of equation (PexAdd) is in the following form

$$\begin{aligned} f(u) &= a(u) + C_1 + C_2 \qquad && u \in D_{x+y}, \\ g(v) &= a(v) + C_1 && v \in D_x, \\ h(z) &= a(z) + C_2 && z \in D_y, \end{aligned} \tag{PexAddSol}$$

  where $a : \mathbb{R} \to E$ is an additive function, $C_1$, $C_2 \in E$ are constants. In this case the additive function $a$ is said to be (additive) quasy-extension of functions $f$, $g$, $h$.

The main purpose of this article is to find nonempty sets $D \subseteq \mathbb{Z}^2$ where $\mathbb{Z}(+, \leqslant)$ is the ordered group of all integers, and the general solution of functional equation (PexAdd) with unknown functions $f : D_{x+y} \to Y$, $g : D_x \to Y$, $h : D_y \to Y$, where $Y(+)$ is an Abelian group is in the form of (PexAddSol) where $a : \mathbb{Z} \to Y$ is an additive function, $C_1$, $C_2 \in Y$ are constants.

**Definition 1.1.** Preserving the notations above a nonempty set $D \subseteq \mathbb{Z}^2$ is said to be suitable if the general solution of functional equation (PexAdd) is in the form of (PexAddSol).

The investigation of additive extensions of functions or functional equations was initiated by P. Erdős [12]. Previous results dealt with additive extensions: [4], [7], [18], [14], [2].

The results concerning to the restricted additive (but not Pexider additive) functional equations where $D \subseteq \mathbb{R}^{2N} := \mathbb{R}^N \times \mathbb{R}^N$ is a nonempty connected open set, and $Y(+) = \mathbb{R}^N(+)$ can be found in the book [13]. In the paper [16] can be also found a similar Pexider additive functional equation with similar settings.

The article [6] deals with functions interpreted on a discrete set.

Finally we give a short list of concepts concerning ordering, and ordered structures:

- An $X(\leqslant)$ structure is said to be ordered set if the relation $\leqslant$ is reflexive, anti-symmetric, transitive, and linear (that is, $x \leqslant y$, or $y \leqslant x$ is fulfilled for all $x$, $y \in X$).
- If $X(\leqslant)$ is an ordered set, $a, b \in X$ such that $a < b$ then the set $]a, b[:= \{x \in X \mid a < x < b\}$ is said to be open interval (in X). Based on the interpretation we use the nonempty connected open subsets are not necessarily open intervals, for example the $\{x \in \mathbb{Q} | \sqrt{2} < x < 3\}$ is not an open interval in the ordered group $\mathbb{Q}(+, \leqslant)$.
- An ordered set $X(\leqslant)$ is said to be dense (in itself) if $]a, b[\neq \emptyset$ for all $a, b \in X$ with $a < b$.
- An ordered group $\mathbb{G}(+, \leqslant)$ is said to be Archimedean ordered if for all $x$, $y \in \mathbb{G}_+ := \{x \in \mathbb{G} \mid x > 0\}$ then there exists an $n \in \mathbb{Z}_+$ such that $y < nx := x + \cdots + x$.

## 2 Some algebraic and topological property of the ordered group of all integers

**Theorem 2.1.** *If $X(+, \leqslant)$ is an Archimedean ordered Abelian group which is not dense then the group $X$ is isomorphic to the ordered group $\mathbb{Z}(+)$.*

*Proof.* It is enough to prove that the group $X$ is an infinite order cyclic group $X = \langle x \rangle$ with infinite order, that is, the group $X$ is generated by a single element $x$ of $X$ and the order of the element $x$ is infinite.

Since the ordered group $X$ is not dense thus there exists elements $a, b \in X$ $a < b$ such that $]a, b[= \emptyset$.

**1** First we show that if $a$, $b$, $c$, $d \in X$ such that $a < b$, $c < d$, moreover, $]a, b[= \emptyset$, and $]c, d[= \emptyset$ then $b - a = d - c$. For this it is enough to prove that $b - a \leqslant d - c$. Indirectly assume that $d - c < b - a$. Thus $a < a + (d - c) < a + (b - a) = b$ which contradicts that $]a, b[= \emptyset$.

**2** Let $x = b - a$. It is worth mentioning that there is no positive element of $X$ less than $x$ because if there were an element $r \in X$ that $0 < r < x$ then $a < a+r < a+(b-a) = b$ thus $a+r \in ]a, b[$ which is contradicts that $]a, b[= \emptyset$.
**3** Finally, we shows that $X = \langle x \rangle$, that is, the generator system of the group $X$ is the singleton $\{x\}$. For this let $y \in X$ be an arbitrary element. According to the Euclidean Division Theorem (see [8], and [11]) there exist elements $n \in \mathbb{Z}$, and $r \in X$ such that $y = nx + r$ where $0 \leqslant r < x$ which is only possible if $r = 0$ which completes the proof. □

Since the topology on $\mathbb{Z}$ generated by the open intervals results discrete topology thus we have to break with the usual terminology according to which we interpret the Pexider additive functional equation on well-chained open sets [11]. For this purpose we introduce a new notation

$$[a, b] := \{a, a+1, \dots, b\}$$

for all $a$, $b \in \mathbb{Z}$ with $a \leqslant b$ allowing that $[a, b]$ is a singleton whenever $a = b$.

**Proposition 2.2.** *If $a$, $b$, $c$, $d \in \mathbb{Z}$ with $a \leqslant b$, and $c \leqslant d$ then*

$$[a, b] + [c, d] = [a + c, b + d].$$

*Proof.* It is clear. □

## 3 Additive functions on $\mathbb{Z}$, and on $\mathbb{Q}$

If $Y(+)$ is an arbitrary group, $c \in Y$ $x \in \mathbb{Z}$ then the element $cx \in Y$ is defined by

$$cx := \begin{cases} c + \cdots + c, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ (-c) + \cdots + (-c) & \text{if } x < 0 \end{cases} . \tag{1}$$

If $Y(+)$ is a group then the element $c \in Y$ is said to be $n$-divisible for same $n \in \mathbb{Z}_+$ if there exists an $y \in Y$ that $c = y + \cdots + y$. If this element $y \in Y$ uniquely exists then the element $c$ is said to be uniquely $n$-divisible. If the all elements of the group $Y(+)$ is $n$-divisible for all $n \in \mathbb{Z}_+$ then the group $Y$ is said to be $n$-divisible for all $n \in \mathbb{Z}_+$, and can be used similarly the case when the group $Y(+)$ is uniquely $n$-divisible for all $n \in \mathbb{Z}_+$. If $p \in \mathbb{Z}$, $q \in \mathbb{Z}_+$, the element $c \in Y$ is uniquely $q$-divisible then $\frac{p}{q}c := p\frac{1}{q}c$ where $y := \frac{1}{q}c$ is the uniquely existing element that $y + \dots y = c$, and $py$ is defined by (1).

**Theorem 3.1.** *Let $Y(+)$ be a group. The function $a : \mathbb{Z} \to Y$ is additive if and only if then there exists an element $c \in Y$ such that $a(x) = cx$ for all $x \in \mathbb{Z}$.*

*Proof.* By mathematical induction can be easily seen that $a(nx) = na(x)$ for all $n$, $x \in \mathbb{Z}$ whence we have that $a(x) = cx$ for all $x \in \mathbb{Z}$ where $c := a(1)$.

Conversely, let $c \in Y$ be an arbitrary element, and define the function $a : \mathbb{Z} \to Y$ by $a(x) := cx$ for all $(x \in \mathbb{Z})$. If $x, y \in \mathbb{Z}$ then we easily obtain that $a(x+y) = c(x+y) = cx + cy = a(x) + a(y)$ whence we have that the function $a$ is additive. □

*Remark* 3.2. If $Y(+)$ is an arbitrary group, $a : \mathbb{Z} \to Y$ is an additive function then the (additive) group $\mathcal{R}_a := \{a(x) | x \in \mathbb{Z}\}$ is Abelian.

Now we consider the additive functions $a : \mathbb{Q} \to Y$ where $\mathbb{Q}(+, \leqslant)$ is the ordered group of all rational numbers, which shows a close analogy with Theorem 3.1.

**Proposition 3.3.** *If $Y(+)$ is a group, $a : \mathbb{Q} \to Y$ is an additive function then the group $\mathcal{R}_a$ is $n$-divisible for all $n \in \mathbb{Z}_+$, and*

$$a\left(\frac{p}{q}x\right) = \frac{p}{q}a(x) \qquad (x \in \mathbb{Q}, p \in \mathbb{Z}, q \in \mathbb{Z}_+). \tag{2}$$

*Proof.* It is clear that $a(nx) = na(x)$ for all $x \in \mathbb{Q}$, and $n \in \mathbb{Z}$.

If $q \in \mathbb{Z}_+$, and $x \in \mathbb{Q}$ then $a(\frac{x}{q}) \in Y$ is an element such that $qa(\frac{x}{q}) = f(x) \in Y$ for all $x \in \mathbb{Q}$ whence we obtain that the group $\mathcal{R}_a$ is $n$-divisible for all $n \in \mathbb{Z}_+$, and by this way we also obtain Equation (2). □

**Proposition 3.4.** *If $Y(+)$ is a group, $a : \mathbb{Q} \to Y$ is an additive function then the group $\mathcal{R}_a$ is torsion free, that is, it is Abelian, and zero is the only element with finite order.*

*Proof.* Let us assume that $x \in \mathbb{Q}$, and $n \in \mathbb{Z}_+$ such that $na(x) = 0$. Then by Proposition 3.3 we obtain that

$$a(x) = \frac{1}{n}a(nx) = \frac{1}{n}na(x) = 0.$$

□

**Proposition 3.5.** *If $Y(+)$ is a group, $a : \mathbb{Q} \to Y$ is an additive function then $\mathcal{N}_a := \{x \in \mathbb{Q} | a(x) = 0\} \in \{\{0\}, \mathbb{Q}\}$, and the group $\mathcal{R}_a$ is uniquely $n$-divisible for all $n \in \mathbb{Z}_+$.*

*Proof.* 1 Assume that there exist $p$, $q \in \mathbb{Z}_+$ such that $a(\frac{p}{q}) = 0$. Thus by Proposition 3.3 we have that $pa(1) = 0$ thus by Proposition 3.4 we have that $a(1) = 0$ thus by Proposition 3.3 we obtain that $\mathcal{N}_a = \mathcal{Q}$.

2 By Proposition 3.3, and Proposition 3.4 we obtain that the group $\mathcal{R}_a$ is uniquely $n$-divisible for all $n \in \mathbb{Z}_+$. □

*Remark* 3.6. It is easy to see that if $Y(+)$ is a group, and $a : \mathbb{Q} \to Y$ is a non-identically zero additive function then the group $\mathcal{R}_a(+)$ is isomorphic to the group $\mathbb{Q}(+)$.

By Propositions 3.3, 3.4, and 3.5 can be easily obtained the following Theorem.

**Theorem 3.7.** *Let $Y(+)$ be a group.*

1. *If $a : \mathbb{Q} \to Y$ is an additive function then there exists a constant $c \in Y$ such that $a(x) = cx$ for all $x \in \mathbb{Q}$.*
2. *If $c \in Y$ is uniquely $n$-divisible for all $n \in \mathbb{Z}_+$, and the function $a : \mathbb{Q} \to Y$ is defined by $a(x) := cx$ then the function $a$ is additive.*

Now we give some Extension and Uniqueness Theorems for additive functions on integers.

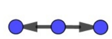
**Theorem 3.8.** *If $Y(+)$ is an Abelian group, $n \in \mathbb{Z}_+$ is a fixed constant, $I := [-n, n]$, the function $f : [-2n, 2n] \to Y$ satisfies the functional equation $f(x + y) = f(x) + f(y)$ for all $x, y \in [-n, n]$ then there exists an additive function $a : \mathbb{Z} \to Y$ such that $a(x) = f(x)$ for all $(x \in I)$.*

*Proof.* If $n = 0$, that is, $I = \{0\}$, then we have that $f(0) = f(0 + 0) = f(0) + f(0)$ whence we obtain that $f(0) = 0$. Choose the constant $c \in Y$ by arbitrarily, and define the function $a : \mathbb{Z} \to Y$ by $a(x) := cx$ for all $x \in \mathbb{Z}$. Thus the function $a$ is an additive extension of the function $f$ from the set $I^2$ to the set $\mathbb{Z}^2$.

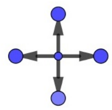
If $n = 1$, that is, $I = \{-1, 0, 1\}$ then define $c := f(1)$. Since $0 = f(0) = f(1+(-1)) = c+f(-1)$, whence we obtain, that $f(-1) = -c$ thus the function $a : \mathbb{Z} \to Y$ defined by $a(x) := cx$ for all $x \in \mathbb{Z}$ is an additive extension of the function $a$.

If $n \geqslant 2$ then the statement can be easily obtained by mathematical induction. □

In the sequel we shall use the well-known concept of neighbourhood.



The numbers $x$, $y \in \mathbb{Z}$ is said to be neighbour if $|x - y| = 1$. Every integers have two neighbour integers.



Similarly, two points $(a_1, a_2)$, and $(b_1, b_2)$ is said to be neighbour if $|b_1 - a_1| + |b_2 - a_2| = 1$.
Every points of $\mathbb{Z}^2$ have four neighbour points.

**Theorem 3.9.** *If $Y(+)$ is an Abelian group, $a$, $b \in \mathbb{Z}$ such that $a < b$, $I := [a, b] \subseteq \mathbb{Z}$, $c_i$, $C_i \in Y$ $(i = 1, 2)$, and $f : I \to Y$ is a function such that*

$$f(x) = c_1 x + C_1 \qquad (x \in I),$$
$$f(x) = c_2 x + C_2 \qquad (x \in I)$$

*then $c_1 = c_2$, and $C_1 = C_2$.*

*Proof.* Since the interval $I$ contains two neighbour elements $x$, $x + 1 \in I$ thus we obtain that $f(x+1) - f(x) = c_1$, and we also obtain that $f(x+1) - f(x) = c_2$, that is, $c_1 = c_2$ whence we obtain that $C_1 = C_2$. □

The above Theorem 3.8, and the Theorem 3.9 shows a close analogy to the adequate Theorems in [11] concerning Archimedean ordered dense Abelian groups.

## 4 Some examples for suitable subsets of $\mathbb{Z}^2$

We introduce some new notations. Let $x_0$, $y_0 \in \mathbb{Z}$, and $n$, $m \in \mathbb{Z}_+ \cup \{0\}$.

- $B(x_0, n) := [x_0 - n, x_n + n]$,
- $B((x_0, y_0), n, m) := B(x_0, n) \times B(y_0, m)$,
- $B((x_0, y_0), n) := B((x_0, y_0), n, n)$.
- A subset $R \subseteq \mathbb{Z}^2$ is said to be $m \times n$ type rectangular if there exists $(x_0, y_0) \in \mathbb{Z}^2$ such that $R := [x_0, x_0 + n] \times [y_0, x_0 + m]$. (Imagine the points of the rectangular arranged in rows and columns, similarly to matrices.)

If $D := B((x_0, y_0), m, n) \subseteq \mathbb{Z}^2$, then $D_x = B(x_0, m)$, $D_y = B(y_0, n)$, and by Proposition 2.2 $D_{x+y} = D_x + D_y = B(x_0 + y_0, m + n)$.

**Theorem 4.1.** *Let $Y(+)$ be an Abelian group, $n \geqslant 1$, and $D := B((x_0, y_0), n)$. If the functions $f : D_{x+y} \to Y$, $g : D_x \to Y$, $h : D_y \to Y$ satisfy the functional* *equation (PexAdd) then it is in the form of (PexAddSol), that is, there exist constants* $c$, $C_1$, $C_2 \in Y$ *such that*

$$\begin{aligned} f(u) &= cu + C_1 + C_2 && (u \in D_{x+y}), \\ g(v) &= cv + C_1 && (v \in D_x), \\ h(z) &= cz + C_2 && (z \in D_y), \end{aligned} \tag{3}$$

*in other words, the* $3 \times 3$ *type rectangles are suitable sets for all* $n \geqslant 3$.

*Proof.* Define the functions $F : B(0, 2n) \to Y$; $G$, $H : B(0, n) \to Y$ by

$$\begin{aligned} F(u) &:= f(u + x_0 + y_0), && (u \in B(0, 2n)), \\ G(v) &:= g(v + x_0), && (v \in B(0, n)), \\ H(z) &:= h(z + y_0) && (z \in B(0, n)). \end{aligned} \tag{4}$$

Thus the functions $F$, $G$, $H$ satisfy the equation

$$F(x + y) = G(x) + H(y) \qquad (x, y \in B(0, n)) \tag{5}$$

From equation (5) we have that

$$\begin{aligned} F(x) &= G(x) + H(0) && (x \in B(0, n)), \\ F(y) &= G(0) + H(y) && (y \in B(0, n)), \\ F(0) &= G(0) + H(0). \end{aligned} \tag{6}$$

From equations (5), and (6) we obtain that

$$F(x + y) - F(0) = (F(x) - F(0)) + (F(y) - F(0)) \qquad (x, y \in B(0, n)) \tag{7}$$

whence by Theorem 3.1, and Theorem 3.8 we obtain that there exists an additive function $\varphi : \mathbb{Z} \to Y$ such that

$$F(u) = \varphi(u) + F(0) \qquad (u \in B(0, n)) \tag{8}$$

thus by equations (4), (6), and (8) we obtain that

$$\begin{aligned} f(u) &= \varphi(u) - \varphi(x_0) - \varphi(y_0) + F(0) && (u \in B(x_0 + y_0, 2n), \\ g(v) &= \varphi(v) - \varphi(x_0) + H(0) && (v \in B(x_0, n)), \\ h(z) &= \varphi(z) - \varphi(y_0) && (z \in B(y_0, n)). \end{aligned} \tag{9}$$

By Theorem 3.1 we obtain that there exists a constant $c \in Y$ such that $\varphi(x) = cx$ for all $x \in \mathbb{Z}$. Define the constants $C_1$, $C_2 \in Y$ by

$$C_1 := -\varphi(x_0) + F(0), \qquad C_2 := -\varphi(y_0) + H(0). \tag{10}$$

From equations (6), (9), and (10) we obtain that the functions $f$, $g$, $h$ are in the form of (3). ☐

**Theorem 4.2.** **1.** *If the set $D \subseteq \mathbb{Z}^2$ is suitable, then the sets*

$$D + (x, y) := \{(u + x, v + y) \mid (u, v) \in D\} \tag{11}$$

*are also suitable for all $x$, $y \in \mathbb{Z}$.*

**2.** *If the sets $D^1$, $D^2 \subseteq \mathbb{Z}^2$ are suitable, moreover, the set $D^1_{x+y} \cap D^2_{x+y}$ contain two neighbour elements, $D^1_x \cap D^2_x \neq \emptyset$, and $D^1_y \cap D^2_y \neq \emptyset$, then the set $D^1 \cup D^2$ is also suitable.*

*Proof.* **1.** is clear.

**2.** Sine the sets $D^1$, and $D^2$ are suitable thus there exist constants $c_1$, $c_2$, $C^i_1$, $C^i_2 \in Y$ $(i = 1, 2)$ such that

$$f(u) = \begin{cases} c_1 u + C^1_1 + C^1_2 & \left(u \in D^1_{x+y}\right), \\ c_2 u + C^2_1 + C^2_2 & \left(u \in D^2_{x+y}\right), \end{cases}$$
$$g(v) = \begin{cases} c_1 v + C^1_1 & \left(v \in D^1_x\right), \\ c_2 v + C^2_1 & \left(v \in D^2_x\right), \end{cases}$$
$$h(z) = \begin{cases} c_1 z + C^1_2 & \left(z \in D^1_y\right), \\ c_2 z + C^2_2 & \left(z \in D^2_y\right). \end{cases}$$

Since there exist neighbour elements $u$, $u + 1 \in D^1_{x+y} \cap D^2_{x+y}$ thus $f(u + 1) - f(u) = c_1$, and $f(u + 1) - f(u) = c_2$ thus $c_1 = c_2$.

Since $D^1_x \cap D^2_x \neq \emptyset$ thus $C^1_1 = C^2_1$. Similarly, since $D^1_y \cap D^2_y \neq \emptyset$ thus $C^1_2 = C^2_2$ which was to be proved. □

**Theorem 4.3.** *Let $Y = Y(+)$ be an Abelian group, and the set $D := [a, b] \times [c, d] \subseteq \mathbb{Z}^2$ be an $m \times n$ type rectangular such that $m \geqslant 1$, and $n \geqslant 2$. Consider the functional equation (PexAdd) with unknown functions $f : D_{x+y} \to Y$, $g : D_x \to Y$, $h : D_y \to Y$.*

**1.** *If $m = 1$ then the general solution of equation (PexAdd) is in the form*

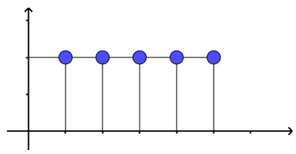
$$\begin{aligned} f(x_k + y_1) &= \alpha_k + \beta_k \\ g(x_k) &= \alpha_k \qquad\qquad (k = 1, 2, \ldots, m), \\ h(y_1) &= \beta_k; \end{aligned} \tag{12}$$

*where $(\alpha_k)$, and $(\beta_k)$ are arbitrary sequences, thus the set $D$ is not suitable.*

**2.** *If $m \geqslant 2$ then the general solution of equation (PexAdd) is in the form of (PexAddSol) where $c$, $C_1$, $C_2 \in Y$ are constants, that is, $D$ is also suitable.*

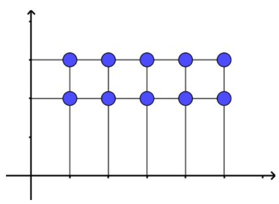
*The case $n \leqslant m$ can be discussed in an analogous way.*

*Proof.* **1.** Let $m = 1$.



The proof is evident

**2.** Let $m = 2$, and let us assume that the functions $f$, $g$, $h$ satisfy equation (PexAdd).



Use the notation $[a, b] := \{x_1, x_2, \dots x_n\}$, and $[c, d] := \{y_1, y_2\}$.

It is easy to see that

$$x_{k+1} = x_k + 1, \quad \text{for all} \quad (k = 1, 2, \dots, n-1) \quad \text{and} \quad y_2 = y_1 + 1.$$

By equitation (PexAdd) we have that

$$\begin{aligned} f(x_k + y_1) &= g(x_k) + h(y_1) \\ f(x_k + y_2) &= g(x_k) + h(y_2) \end{aligned} \qquad (k = 1, 2, \dots, n). \tag{13}$$

Since $f(x_{k+1} + y_1) = f(x_k + y_2)$ thus by equation (13) we obtain that

$$g(x_{k+1}) - g(x_k) = h(y_2) - h(y_1) \qquad (k = 1, 2, \dots, n-1),$$

that is, the sequence $g(x_k)$ $(k = 1, 2, \dots, m)$ is an arithmetic sequence with difference $d = h(y_2) - h(y_1)$ whence we obtain that

$$\begin{aligned} g(x_k) &= g(x_1) + (k-1)\,(h(y_2) - h(y_1)) \qquad k \in \{1, 2, \dots, n\}; \\ h(y_l) &= h(y_1) + (l-1)\,(h(y_2) - h(y_1)) \qquad l \in \{1, 2\}. \end{aligned} \tag{14}$$

Let $u \in [a, b]$. Then there exists a number $k \in \{1, \dots, n\}$ such that

$$u = x_k. \tag{15}$$

Since $x_1 = a$ thus $x_k = a + k - 1$ whence we obtain that

$$k - x_k = k - (a + k - 1) = -a + 1. \tag{16}$$

Thus by equations (14), (15), and (16) we obtain that

$$\begin{aligned} g(u) &\overset{(15)}{=} g(x_k) \overset{(14)}{=} g(x_1) + (k-1)\,(h(y_2) - h(y_1)) \\ &= g(x_1) + (k-1-u+u)\,(h(y_2) - h(y_1)) \\ &= (h(y_2) - h(y_1))\,u + (g(x_1) + (k-1-u)(h(y_2) - h(y_1))) \\ &\overset{(15)}{=} (h(y_2) - h(y_1))\,u + (g(x_1) + (k-1-x_k)(h(y_2) - h(y_1))) \\ &\overset{(16)}{=} (h(y_2) - h(y_1))\,u + (g(x_1) - a(h(y_2) - h(y_1))) := cu + C_1, \end{aligned}$$

where $c := (h(y_2) - h(y_1))$, and $C_1 := g(x_1) - a(h(y_2) - h(y_1))$.

Let $v \in [c, d]$. Then there exists a number $l \in \{1, 2\}$ such that

$$v = y_l. \tag{17}$$

Since $y_1 = b$ thus $y_l = b + l - 1$ whence we obtain that

$$l - y_l = l - (b + l - 1) = -b + 1. \tag{18}$$

Thus by Equations (14), (17), and (18) we obtain that

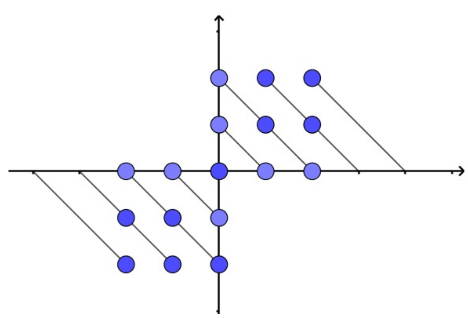
$$\begin{aligned} h(v) &\overset{(17)}{=} h(y_l) \overset{(14)}{=} h(y_1) + (l-1)\,(h(y_2) - h(y_1)) \\ &= h(y_1) + (l-1-v+v)\,(h(y_2) - h(y_1)) \\ &= (h(y_2) - h(y_1))\,v + (h(y_1) + (l-1-v)(h(y_2) - h(y_1))) \\ &\overset{(17)}{=} (h(y_2) - h(y_1))\,v + (h(y_1) + (l-1-y_l)(h(y_2) - h(y_1))) \\ &\overset{(18)}{=} (h(y_2) - h(y_1))\,v + (h(y_1) - b(h(y_2) - h(y_1))) := cv + C_2, \end{aligned}$$

where $C_2 := h(y_1) - b(h(y_2) - h(y_1))$.

The case $n \geqslant 3$ can be easily obtained by statement **2** of Theorem 4.2. □

## 5 Additional Examples, Results and Problems

*Example* 5.1. Let $D^1 := \{0, 1, 2\}^2$, $D^2 := \{-2, -1, 0\}^2$, $D := D^1 \cup D^2$.



$$\begin{aligned} D^1_x = D^1_y &= \{0, 1, 2\}, \\ D^1_{x+y} &= \{0, 1, 2, 3, 4\}, \\ D^2_x = D^2_y &= \{-2, -1, 0\}, \\ D^2_{x+y} &= \{-4, -3, -2, -1, 0\}. \end{aligned}$$

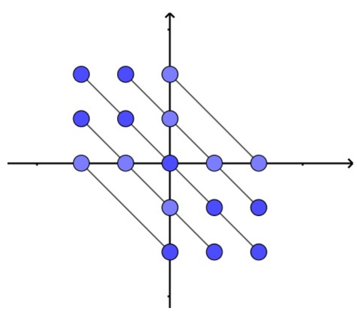
Let us assume that the functions $f$, $g$, $h$ satisfy the equation (PexAdd). By Theorem 4.3 the sets $D^1$, and $D^2$ are suitable.

Since $0 \in D^1_x \cap D^2_x$ thus $C^1_1 = C^2_1$; similarly, since $0 \in D^1_y \cap D^2_y$ thus $C^1_2 = C^2_2$ whence we obtain that the general solution of functional equation (PexAdd) is

$$f(u) = \begin{cases} c_1 u + C_1 + C_2, & \text{if } u \in D^1_{x+y}; \\ c_2 u + C_1 + C_2, & \text{if } u \in D^2_{x+y}; \end{cases}$$
$$g(v) = \begin{cases} c_1 v + C_1, & \text{if } v \in D^1_x; \\ c_2 v + C_1, & \text{if } v \in D^2_y; \end{cases}$$
$$h(z) = \begin{cases} c_1 z + C_2, & \text{if } z \in D^1_x; \\ c_2 z + C_2, & \text{if } z \in D^2_y; \end{cases},$$

where $c$, $C_1$, $C_2 \in Y$ are constants which shows that the set $D$ is not suitable, and the concept of well-chainedness described in [10] is not appropriate for the suitable sets.

*Example* 5.2. Let $D^1 := \{-2, -1, 0\} \times \{0, 1, 2\}$, $D^2 := \{0, 1, 2\} \times \{-2, -1, 0\}$, and $D := D^1 \cup D^2$.



$$D^1_x = D^2_y = \{0, 1, 2\},$$
$$D^1_y = D^2_x = \{-2, -1, 0\},$$
$$D^1_{x+y} = D^2_{x+y} = \{-2, -1, 0, 1, 2\}.$$

Let us assume that the functions $f$, $g$, $h$ satisfy the equation (PexAdd). By Theorem 4.3 the sets $D^1$, and $D^2$ are suitable. Since $D^1_{x+y} \cap D^2_{x+y}$ contains two neighbour elements, $D^1_x \cap D^2_x \neq \emptyset$, and $D^1_y \cap D^2_y \neq \emptyset$ thus by Theorem 4.2 the set $D$ is suitable.

**Definition 5.3.** Define the family $\mathcal{D} \subseteq \mathbb{Z}^2$ by

**1** The $2 \times 2$ type rectangles are elements of the family $\mathcal{D}$;

**2** If $D \in \mathcal{D}$, and $(x, y) \in \mathbb{Z}^2$ then $D + (x, y) \in \mathcal{D}$ where the set $D + (x, y)$ is defined by (11).

**3** If $D^1$, $D^2 \in \mathcal{D}$, $D^1_{x+y} \cap D^2_{x+y}$ contains two neighbour elements, $D^1_x \cap D^2_x \neq \emptyset$, and $D^1_y \cap D^2_y \neq \emptyset$ then the set $D^1 \cup D^2$ is also element of the family $\mathcal{D}$.

By Theorem 4.3, and Theorem 4.2 the all elements of the family $\mathcal{D}$ are suitable, although the authors of this paper think there are some suitable sets that do not belong to the family $\mathcal{D}$.

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