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New Criteria for Functions in a Class of Meromorphically Strongly Starlike Functions

Nak Eun Cho, Inhwa Kim and H. M. Srivastava*

Abstract

The authors propose to investigate some new criteria for a certain class of meromorphically strongly starlike functions in the punctured open unit disk. Some intriguing applications that arise as special cases of the main results, which are presented in this study, are also considered.

1 Introduction

Given two functions f and F, which are analytic in the open unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

the function f is said to be subordinate to the function F in $\mathbb{U}$, which is written as $f \prec F$ or $f(z) \prec F(z)$, if there exists a *Schwarz function* ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \qquad (z \in \mathbb{U})$$

such that

$$f(z) = F\big(\omega(z)\big) \qquad (z \in \mathbb{U}).$$

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*Corresponding Author
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In particular, if the function F is univalent in $\mathbb{U}$, then we have the following equivalence (see [18]):

$$f(z) \prec F(z) \quad (z \in \mathbb{U}) \iff f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

Let Σ be the class of meromorphic and univalent functions f, defined in the punctured open unit disk

$$\mathbb{D} = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\},$$

which are of the form given by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (z \in \mathbb{D}).$$

Suppose that a function $f \in \Sigma$ satisfies the following inequality:

$$\left| \arg \left\{ -\frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2}\alpha \qquad (0 \leqq \alpha < 1;\ z \in \mathbb{U}).$$

Then we say that f is a meromorphically strongly starlike function of order α in $\mathbb{D}$. We find it to be convenient to denote by $\Sigma_s[\alpha]$ the subclass of Σ consisting of all functions f in Σ which are strongly meromorphic starlike of order α in $\mathbb{D}$. In particular, $\Sigma_s[1] \equiv \Sigma^*$, which is the well-known class of meromorphic starlike functions in $\mathbb{D}$ (see [13] and [18]).

In view of the principle of subordination between analytic functions, the above definition is equivalent to

$$\Sigma_s[\alpha] = \left\{ f : f \in \Sigma \quad \text{and} \quad -\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \qquad (z \in \mathbb{U};\ 0 < \alpha \leqq 1) \right\}.$$

In recent years, several authors have obtained numerous intriguing applications in Geometric Function Theory of Complex Analysis related to starlikeness, convexity, close-to-convexity, spiral-likeness, and so on (see, for example, [1], [2], [6], [14], [15], [17], [18], [20] and [22]). The methods and techniques, which are used in these earlier investigations, are based upon the principle of differential subordination between analytic functions. Moreover, by appealing to the generalized Jack's lemma, the Nunokawa lemma and other related results, certain sufficient conditions have been derived in the earlier works [6] and [21]. These developments take into account the concept of the argument, the real part and the imaginary part to determine functions that are multivalently starlike and convex in the open unit disk $\mathbb{U}$. In this paper, we explore certain argument properties for analytic functions, which are established in the open unit disk $\mathbb{U}$ by using some results given by Nunokawa *et al.* [19]. As a consequence of our investigation, we derive sufficient conditions for functions in a class of meromorphically strongly starlike functions, which we have introduced herein. We also demonstrate the applications of these findings in the context of our main results. Our main results are related to various other interesting developments which were explored by many authors (see, for example, [7] to [12], [16], [23] and [24]). Many of these developments have found practical use and applications in the space of analytic and meromorphic functions in $\mathbb{U}$.

2 Main Results

The following lemma due to Nunokawa *et al.* [19] will be needed in proving our results.

Lemma 1. (see [19]) *Let the function p be nonzero analytic for $z \in \mathbb{U}$ with $p(0) = 1$ and $p'(0) = 0$. Suppose also that there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg\{p(z)\}| < \frac{\pi}{2}\alpha \quad \textit{for} \quad |z| < |z_0| \tag{2.1}$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi}{2}\alpha \qquad (\alpha > 0). \tag{2.2}$$

Then

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\alpha k, \tag{2.3}$$

where

$$k \geqq \left(a + \frac{1}{a}\right) \geqq 2 \quad \textit{when} \quad \arg\{p(z_0)\} = \frac{\pi}{2}\alpha \tag{2.4}$$

and

$$k \leqq -\left(a + \frac{1}{a}\right) \leqq -2 \quad \textit{when} \quad \arg\{p(z_0)\} = -\frac{\pi}{2}\alpha, \tag{2.5}$$

where

$$[p(z_0)]^{1/\alpha} = \pm ia \qquad (a > 0). \tag{2.6}$$

By applying Lemma 1, we first state and prove Theorem 1 below.

Theorem 1. *Let the function p be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p'(0) = 0$. If*

$$\left|\arg\left\{\frac{zp'(z)}{p(z)} - p(z)\right\}\right| > \frac{\pi}{2}\delta \qquad (z \in \mathbb{U}),$$

where δ $(1 < \delta < 2)$ is given by

$$\delta = \begin{cases} \alpha + 2 - \frac{2}{\pi}\tan^{-1}\left(\dfrac{2\alpha n(\alpha)\sin\left(\frac{\pi}{2}(1-\alpha)\right)}{m(\alpha) - 2\alpha n(\alpha)\cos\left(\frac{\pi}{2}(1-\alpha)\right)}\right) & (0 < \alpha < \alpha_0) \\ \alpha + \frac{2}{\pi}\tan^{-1}\left(\dfrac{2\alpha n(\alpha)\sin\left(\frac{\pi}{2}(1-\alpha)\right)}{-m(\alpha) + 2\alpha n(\alpha)\cos\left(\frac{\pi}{2}(1-\alpha)\right)}\right) & (\alpha_0 < \alpha < 1) \end{cases} \tag{2.7}$$

and α_0 *is the positive root of the following equation*:

$$m(\alpha) - 2\alpha n(\alpha)\cos\left(\frac{\pi}{2}(1-\alpha)\right) = 0$$

when

$$m(\alpha) = (1+\alpha)^{\frac{\alpha+1}{2}} \quad \textit{and} \quad n(\alpha) = (1-\alpha)^{\frac{\alpha-1}{2}},$$

then

$$|\arg\{p(z)\}| < \frac{\pi}{2}\alpha \qquad (z \in \mathbb{U}).$$

Proof. If there exists a point $z_0 \in \mathbb{U}$ satisfying the conditions (2.1) and (2.2), then Lemma 1 leads us to the result that (2.3) holds true subject to the restrictions (2.4), (2.5) and (2.6).

For the case in which $\arg\{p(z_0)\} = \frac{\pi}{2}\alpha$, we have

$$\begin{aligned} \arg\left\{\frac{z_0 p'(z_0)}{p(z_0)} - p(z_0)\right\} &= \arg\{p(z_0)\} + \arg\left\{\frac{z_0 p'(z_0)}{[p(z_0)]^2} - 1\right\} \\ &= \frac{\pi}{2}\alpha + \arg\left\{\frac{i\alpha k}{(ia)^\alpha} - 1\right\} \\ &\leqq \frac{\pi}{2}\alpha + \arg\left\{\alpha\left(a^{1-\alpha} + a^{-1-\alpha}\right)e^{i\frac{\pi}{2}(1-\alpha)} - 1\right\}. \end{aligned}$$

Now, upon setting

$$g(a) = a^{1-\alpha} + a^{-1-\alpha} \qquad (a > 0;\ 0 < \alpha < 1),$$

we find that g takes on the minimum value at

$$a = \sqrt{\frac{1+\alpha}{1-\alpha}} \quad (0 < \alpha < 1).$$

Therefore, we have

$$
\begin{aligned}
&\arg\left\{\frac{z_0 p'(z_0)}{p(z_0)} - p(z_0)\right\} \\
&\quad \leqq \frac{\pi}{2}\alpha + \arg\left\{\alpha\left[\left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{1-\alpha}{2}} + \left(\frac{1+\alpha}{1-\alpha}\right)^{-\frac{1+\alpha}{2}}\right] e^{i\frac{\pi}{2}(1-\alpha)} - 1\right\} \\
&\quad = \frac{\pi}{2}\alpha + \arg\left\{\frac{2\alpha}{1-\alpha}\left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{1+\alpha}{2}} e^{i\frac{\pi}{2}(1-\alpha)} - 1\right\} \\
&\quad = \frac{\pi}{2}\alpha + \pi - \tan^{-1}\left(\frac{\frac{2\alpha}{1-\alpha}\left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{1+\alpha}{2}} \sin\left(\frac{\pi}{2}(1-\alpha)\right)}{1 - \frac{2\alpha}{1-\alpha}\left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{1+\alpha}{2}} \cos\left(\frac{\pi}{2}(1-\alpha)\right)}\right) \\
&\quad = \frac{\pi}{2}\alpha + \pi - \tan^{-1}\left(\frac{2\alpha n(\alpha)\ \sin\left(\frac{\pi}{2}(1-\alpha)\right)}{m(\alpha) - 2\alpha n(\alpha)\ \cos\left(\frac{\pi}{2}(1-\alpha)\right)}\right) \\
&\quad = \frac{\pi}{2}\delta,
\end{aligned}
$$

where δ is given by (2.7). This is a contradiction to the assumption of Theorem 1.

For the case in which $\arg\{p(z_0)\} = -\frac{\pi}{2}\alpha$, by applying the same method as the above, we also have

$$\arg\left\{\frac{z_0 p'(z_0)}{p(z_0)} - p(z_0)\right\} \geqq -\frac{\pi}{2}\delta,$$

where δ is given by (2.7). This also contradicts the hypothesis of Theorem 1. We have thus completed the proof of Theorem 1 by contradiction. □

Corollary 1 below follows easily from Theorem 1.

Corollary 1. *Let $f \in \Sigma$ with $f(z) \neq 0$ in $\mathbb{D}$. If*

$$\left|\arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| > \frac{\pi}{2}\delta \qquad (z \in \mathbb{U}),$$

where δ is given by (2.7), *then*

$$\left|\arg\left\{-\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\alpha \qquad (z \in \mathbb{U}).$$

Proof. Let us define the function p by

$$p(z) = -\frac{zf'(z)}{f(z)} \qquad (z \in \mathbb{U}).$$

Then we observe that the function p is an analytic in $\mathbb{U}$, with $p(0) = 1$ and $p'(0) = 0$. It follows also that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z)} - p(z).$$

Consequently, by applying Theorem 1, we are led to Corollary 1. □

Next we state and prove Theorem 2 below.

Theorem 2. *Let the function p be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p'(0) = 0$. If*

$$\delta_1 < \arg\left\{\frac{zp'(z) - p(z)}{p(z)}\right\} < \delta_2 \qquad (z \in \mathbb{U}),$$

where δ_1 and δ_2 are the solutions of the equations given by

$$\delta_1 = \pi - \tan^{-1}(2\alpha) \qquad (0 < \alpha < 1) \tag{2.8}$$

and

$$\delta_2 = \pi + \tan^{-1}(2\alpha) \qquad (0 < \alpha < 1), \tag{2.9}$$

then

$$|\arg\{p(z)\}| < \frac{\pi}{2}\alpha \qquad (z \in \mathbb{U}).$$

Proof. We assume that there exists a point $z_0 \in \mathbb{U}$ such that the conditions given by (2.1) and (2.2) are satisfied. Then, by using Lemma 1, we obtain (2.3) under the restrictions given by (2.4), (2.5) and (2.6).

For the case in which $\arg\{p(z_0)\} = \frac{\pi}{2}\alpha$, we have

$$\begin{aligned}\arg\left\{\frac{z_0p'(z_0) - p(z_0)}{p(z_0)}\right\} &= \arg\{i\alpha k - 1\}\\ &= \pi - \tan^{-1}(\alpha k)\\ &\leqq \pi - \tan^{-1}(2\alpha).\end{aligned}$$

This contradicts the hypothesis of Theorem 2.

Moreover, for the case in which $\arg\{p(z_0)\} = -\frac{\pi}{2}\alpha$, we find that

$$\arg\left\{\frac{z_0 p'(z_0) - p(z_0)}{p(z_0)}\right\} \geqq \pi + \tan^{-1}(2\alpha).$$

This is also a contradiction to the hypothesis of Theorem 2. This evidently completes the proof of Theorem 2. □

By using the same techniques as in the proof of Corollary 1, we obtain the following result.

Corollary 2. *Let $f \in \Sigma$ with $f(z) \neq 0$ in $\mathbb{D}$. Suppose also that the following inequality is satisfied*:

$$\delta_1 < \arg\left\{\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} < \delta_2 \qquad (z \in \mathbb{U}),$$

where δ_1 and δ_2 are the solutions of the equations given by (2.8) *and* (2.9), *respectively. Then*

$$\left|\arg\left\{-\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\alpha \qquad (z \in \mathbb{U}).$$

Theorem 3. *Let the function p be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p'(0) = 0$. Suppose also that*

$$\beta \tan\left(\frac{\pi}{2}\alpha\right) \geqq 2\alpha \qquad (0 < \alpha \leqq 1;\ \beta > 0).$$

If

$$\left|\arg\left\{\frac{1}{zp'(z) - \beta p(z)}\right\}\right| < \pi - \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{2\alpha}{\beta}\right) \qquad (0 < \alpha \leqq 1;\ \beta > 0;\ z \in \mathbb{U}),$$

then

$$|\arg\{p(z)\}| < \frac{\pi}{2}\alpha \qquad (0 < \alpha \leqq 1;\ z \in \mathbb{U}).$$

Proof. Suppose that exists a point $z_0 \in \mathbb{U}$ such that the conditions in (2.1) and (2.2) are satisfied. Then, by using Lemma 1, we obtain (2.3) under the restrictions given by (2.4), (2.5) and (2.6).

For the case in which $\arg\{p(z_0)\} = \frac{\pi}{2}\alpha$, we have

$$\begin{aligned}
\arg\left\{\frac{1}{z_0p'(z_0)-\beta p(z_0)}\right\} &= -\arg\{z_0p'(z_0)-\beta p(z_0)\} \\
&= -\arg\{p(z_0)\} - \arg\left\{\frac{z_0p'(z_0)}{p(z_0)}-\beta\right\} \\
&= -\frac{\pi}{2}\alpha - \arg\{i\alpha k-\beta\} \\
&= \pi - \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{\alpha k}{\beta}\right) \\
&\geqq \pi - \frac{\pi}{2}\alpha + \tan^{-1}\left(\frac{2\alpha}{\beta}\right).
\end{aligned}$$

This is a contradiction to the assumption of Theorem 3.

For the case in which $\arg\{p(z_0)\} = -\frac{\pi}{2}\alpha$, by applying the same method as described above, we obtain

$$\arg\left\{\frac{1}{z_0p'(z_0)-\beta p(z_0)}\right\} \leqq -\pi + \frac{\pi}{2}\alpha - \tan^{-1}\left(\frac{2\alpha}{\beta}\right).$$

This also presents a contradiction to the assumption of Theorem 3. We have thus completed the proof of Theorem 3 by contradiction. □

Taking $\beta = 1$ in Theorem 3, we have the following result.

Corollary 3. *Let $f \in \Sigma$ and suppose that*

$$\tan\left(\frac{\pi}{2}\alpha\right) \geqq 2\alpha \qquad (0 < \alpha \leqq 1).$$

If

$$\left|\arg\left\{\frac{1}{z^2f'(z)}\right\}\right| < \pi - \frac{\pi}{2}\alpha + \tan^{-1}(2\alpha) \qquad (0 < \alpha \leqq 1;\ z \in \mathbb{U}),$$

then

$$|\arg\{zf(z)\}| < \frac{\pi}{2}\alpha \qquad (0 < \alpha \leqq 1;\ z \in \mathbb{U}).$$

Based on the result asserted by Theorem 3, we can now establish a criterion for meromorphically strongly starlikeness as follows.

Corollary 4. *Let* $f \in \Sigma$ *and suppose that*

$$\tan\left(\frac{\pi}{2}\alpha\right) \geqq 2\alpha \quad \textit{and} \quad \gamma \geqq \frac{2}{\pi}\tan^{-1}\alpha \qquad (0 < \alpha, \gamma \leqq 1).$$

If

$$\left|\arg\left\{\frac{1}{z^2 f'(z)}\right\}\right| < \pi - \frac{\pi}{2}\alpha + \tan^{-1}(2\alpha) \qquad (0 < \alpha \leqq 1;\ z \in \mathbb{U}),$$

then

$$\left|\arg\left\{-\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\gamma \qquad (0 < \gamma \leqq 1;\ z \in \mathbb{U}).$$

Proof. We define the functions p and P by

$$p(z) = -\frac{zf'(z)}{f(z)} \qquad (z \in \mathbb{U})$$

and

$$P(z) = zf(z) \qquad (z \in \mathbb{U}),$$

respectively. Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg\{p(z)\}| < \frac{\pi}{2}\gamma \qquad \big(|z| < |z_0|\big)$$

and

$$|\arg\{p(z_0)\}| = \frac{\pi}{2}\gamma \qquad \big(0 < \gamma \leqq 1\big).$$

Then we see that

$$p(z)P(z) = -z^2 f'(z) \qquad (z \in \mathbb{U}).$$

For the case in which $\arg\{p(z_0)\} = \frac{\pi}{2}\gamma$, by applying Corollary 3 and utilizing the assumption of Corollary 4, we have

$$\begin{aligned}\arg\left\{\frac{1}{z_0^2 f'(z_0)}\right\} &= -\arg\{z_0^2 f'(z_0)\} \\ &= -\arg\{p(z_0)P(z_0)\} + \arg\{-1\} \\ &= \arg\{-1\} - \arg\{p(z_0)\} - \arg\{P(z_0)\} \\ &= \arg\{-1\} - \frac{\pi}{2}\gamma - \arg\{P(z_0)\} \\ &< -\pi - \frac{\pi}{2}\gamma + \frac{\pi}{2}\alpha \\ &\leqq -\pi + \frac{\pi}{2}\alpha - \tan^{-1}(2\alpha).\end{aligned}$$

This is a contradiction to the assumption of Corollary 4.

For the case in which $\arg\{p(z_0)\} = -\frac{\pi}{2}\gamma$, by applying the same method as described above, we find that

$$\begin{aligned}\arg\left\{\frac{1}{z_0^2 f'(z_0)}\right\} &= \arg\{-1\} + \frac{\pi}{2}\gamma - \arg\{P(z_0)\} \\ &> \pi + \frac{\pi}{2}\gamma - \frac{\pi}{2}\alpha \\ &\geqq \pi - \frac{\pi}{2}\alpha + \tan^{-1}(2\alpha).\end{aligned}$$

This also contradicts the assumption of Corollary 4. We have thus completed the proof of Corollary 4. □

3 Further Applications of Lemma 1

In this section, we begin by applying Lemma 1 in order to establish Theorem 4 below.

Theorem 4. *Let p be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p'(0) = 0$. Suppose that*

$$\Re\left(p(z) - \frac{zp'(z)}{p(z)}\right) > -\frac{2\gamma + \gamma^2}{1+\gamma} \qquad (\gamma > 0;\ z \in \mathbb{U})$$

and that, for an arbitrary real number r $(0 < r < 1)$, $p(z)$ satisfies the following condition:

$$\min_{|z| \leqq r} \left\{\Re\big(p(z)\big)\right\} = \min_{|z_0| = r} \left\{\big(p(z_0)\big)\right\} \neq p(z_0).$$

Then

$$\Re\big(p(z)\big) > -\gamma \qquad (z \in \mathbb{U}).$$

Proof. Let us put

$$q(z) = \frac{p(z) + \gamma}{1 + \gamma} \qquad (z \in \mathbb{U}).$$

Then we see that $q(0) = 1$ and $q'(0) = 0$. Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\Re\big(q(z)\big) > 0 \quad \text{for} \quad |z| < |z_0|$$

and

$$\Re\big(q(z_0)\big) = 0.$$

Then, from the assumption of Theorem 4, we see that $q(z_0) \neq 0$. Therefore, by applying Lemma 1 with $\alpha = 1$, we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik,$$

where

$$k \geqq \left(a + \frac{1}{a}\right) \geqq 2 \quad \text{when} \quad \arg\{q(z_0)\} = \frac{\pi}{2}$$

and

$$k \leqq -\left(a + \frac{1}{a}\right) \leqq -2 \quad \text{when} \quad \arg\{q(z_0)\} = -\frac{\pi}{2},$$

where

$$q(z_0) = \pm ia \quad \text{and} \quad a > 0.$$

For the case in which $\arg\{q(z_0)\} = \frac{\pi}{2}$, we have

$$\begin{aligned}
\Re\left(p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right) &= -\gamma - \Re\left(\frac{(1+\gamma)}{(1+\gamma)ia - \gamma}\ \frac{z_0 q'(z_0)}{q(z_0)}\ q(z_0)\right) \\
&= -\gamma - \Re\left(\left[\frac{-\gamma - (1+\gamma)ia}{\gamma^2 + (1+\gamma)^2 a^2}\right](1+\gamma)ikia\right) \\
&= -\gamma - \frac{\gamma(1+\gamma)}{\gamma^2 + (1+\gamma)^2 a^2} ak \\
&\leqq -\gamma - \left(\frac{1 + a^2}{\alpha^2 + (1+\gamma)^2 a^2}\right)\ \alpha(1+\gamma).
\end{aligned}$$

Since the function g given by

$$g(a) = \frac{1 + a^2}{\gamma^2 + (1+\gamma)^2 a^2} \qquad (\gamma > 0)$$

is a decreasing function for $a > 0$ and $g(a) > 0$ for $a > 0$, we have

$$\Re\left(p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right) \leqq -\gamma - \frac{\gamma(1+\gamma)}{(1+\gamma)^2} = -\frac{2\gamma+\gamma^2}{1+\gamma}.$$

This is a contradiction to the assumption of Theorem 4.

For the case in which $\arg\{q(z_0)\} = -\frac{\pi}{2}$, by applying the same method as used above, we have

$$\Re\left(p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right) \leqq -\frac{2\alpha+\gamma^2}{1+\gamma}.$$

This leads also to a contradiction to the assumption of Theorem 4. We have thus completed the proof of Theorem 4. □

Corollary 5. *Let $f \in \Sigma$ such that $zf(z) \neq 0$ in $\mathbb{U}$. Suppose that*

$$-\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{2\gamma+\gamma^2}{1+\gamma} \qquad (\gamma > 0;\ z \in \mathbb{U}),$$

and that ,for an arbitrary real number r $(0 < r < 1)$, $zf(z)$ satisfies the following condition:

$$\min_{|z| \leqq r} \left\{\Re\big(zf(z)\big)\right\} = \min_{|z_0| = r} \left\{\Re\big(z_0 f(z_0)\big)\right\} \neq z_0 f(z_0).$$

Then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < \gamma \qquad (z \in \mathbb{U}).$$

Finally, we state and prove the following result.

Theorem 5. *Let the function p be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p'(0) = 0$. Suppose that, for an arbitrary real number r $(0 < r < 1)$, the function p satisfies the following condition:*

$$\min_{|z| \leqq r} \left\{\Re\big(p(z)\big)\right\} = \min_{|z_0| = r} \left\{\Re\big(p(z_0)\big)\right\} \neq p(z_0).$$

Then the following condition:

$$\Re\left(p(z) - \frac{zp'(z)}{p(z)}\right) < \begin{cases} \dfrac{2\gamma-\gamma^2}{1-\gamma} & \left(\dfrac{3-\sqrt{5}}{2} < \gamma < \dfrac{1}{2};\ z \in \mathbb{U}\right) \\[2ex] \dfrac{\gamma^2-\gamma+1}{\gamma} & \left(\dfrac{1}{2} < \gamma < 1;\ z \in \mathbb{U}\right) \\[2ex] \dfrac{3}{2} & \left(\gamma = \dfrac{1}{2};\ z \in \mathbb{U}\right) \end{cases}$$

implies that $\Re\big(p(z)\big) > \gamma$ *for all* $z \in \mathbb{U}$.

Proof. Let us set

$$q(z) = \frac{p(z) - \gamma}{1 - \gamma} \qquad (\gamma < 1;\ z \in \mathbb{U}).$$

Then the function q is analytic in $\mathbb{U}$, $q(0) = 1$ and $q'(0) = 0$. If there exists a point $z_0 \in \mathbb{U}$ such that

$$\Re\big(q(z)\big) > 0 \quad \text{for} \quad |z| < |z_0|$$

and

$$\Re\big(q(z_0)\big) = 0,$$

then we find from the assumption of Theorem 5 that $q(z_0) \neq 0$. Hence, by applying Lemma 1 with $\alpha = 1$, we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik,$$

where k is a real number with $2 \leqq |k|$. If $\arg\{q(z_0)\} = \frac{\pi}{2}$, $q(z_0) = ia$ and $a > 0$, then we have $ak > 0$, if k is a positive real number.

If, on the other hand, $\arg\{q(z_0)\} = -\frac{\pi}{2}$, $q(z_0) = -ia$ and $a > 0$, then we have $(-a)k > 0$, if k is a negative real number.

We now consider the following three cases:

Case 1. For the case when

$$\frac{3 - \sqrt{5}}{2} < \gamma < \frac{1}{2},$$

we have

$$\begin{aligned}
\Re\left(p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right) &= \gamma - \Re\left(\frac{(1-\gamma)}{\gamma + (1-\gamma)ia}\ \frac{z_0 q'(z_0)}{q(z_0)}\ q(z_0)\right) \\
&= \gamma - \Re\left(\frac{(1-\gamma)}{\gamma + (1-\gamma)ia} ikia\right) \\
&= \gamma + \Re\left(\frac{\gamma - (1-\gamma)ia}{\gamma^2 + (1-\gamma)^2 a^2}\ (1-\gamma)ak\right) \\
&= \gamma + \frac{\gamma(1-\gamma)}{\gamma^2 + (1-\gamma)^2 a^2}\ ak \\
&\geqq \gamma + \gamma(1-\gamma)\ \frac{a^2 + 1}{(1-\gamma)^2 a^2 + \gamma^2}.
\end{aligned}$$

Thus, upon setting

$$g(a) = \frac{a^2+1}{(1-\gamma)^2 a^2 + \gamma^2} \qquad (a > 0),$$

we obtain

$$g'(a) = \frac{2(2\gamma-1)a}{[(1-\gamma)^2 a^2 + \gamma^2]^2}.$$

Hence the function $g(a)$ is a decreasing function for $a > 0$. Therefore, we have

$$\Re\left(p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right) \geqq \gamma + \frac{\gamma}{1-\gamma} = \frac{2\gamma-\gamma^2}{1-\gamma}.$$

This is a contradiction to the assumption of Case 1, so it completes the demonstration of Case 1.

Case 2. For the case when $\frac{1}{2} < \gamma < 1$, by applying the same method and reason as those used above, we have

$$\Re\left(p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right) \geqq \gamma + \gamma(1-\gamma)\ \frac{a^2+1}{(1-\gamma)^2 a^2 + \gamma^2}$$

and the function g given by

$$g(a) = \frac{a^2+1}{(1-\gamma)^2 a^2 + \gamma^2}$$

is an increasing function for $a > 0$. Therefore, we see that

$$\Re\left(p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right) \geqq \gamma + \gamma(1-\gamma)\ \frac{1}{\gamma^2} = \frac{\gamma^2-\gamma+1}{\gamma}.$$

This is a contradiction to the assumption of Case 2, and so we complete the proof of Case 2.

Case 3. For the case when $\gamma = \frac{1}{2}$, the function g given by

$$g(a) = \frac{a^2+1}{(1-\gamma)^2 a^2 + \gamma^2}$$

is a constant function for $a > 0$. Therefore, we have

$$\Re\left(p(z_0) - \frac{z_0 p'(z_0)}{p(z_0)}\right) \geqq \frac{3}{2}.$$

This is also a contradiction to the assumption of Case 3, and so we complete the proof of Case 3. □

4 Conclusion

In this paper, we have systematically studied a class of functions p with $p(0) = 1$ and $p'(0) = 0$, which are analytic in the open unit disk $\mathbb{U}$. We have also derived some argument criteria which are associated with meromorphically starlike and strongly starlike functions as special cases of the main results presented in this paper. Furthermore, we have investigated some implications associated with the real parts of analytic functions described above.

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Nak Eun Cho,
Department of Applied Mathematics,
Pukyong National University,
Busan 48513, Republic of Korea.
Email: necho@pknu.ac.kr

Inhwa Kim,
Anheuser-Bush School of Business,
Harris-Stowe State University,
St. Louis, Missouri 63103, U.S.A.
Email: kimi@hssu.edu

H. M. Srivastava,
Department of Mathematics and Statistics, University of Victoria,
Victoria, British Columbia V8W 3R4, Canada;
Department of Medical Research, China Medical University Hospital,
China Medical University, Taichung 40402, Taiwan;
Center for Converging Humanities, Kyung Hee University,
26 Kyungheedae-ro, Dongdaemun-gu, Seoul 02447, Republic of Korea;
Department of Applied Mathematics, Chung Yuan Christian University,
Chung-Li, Taoyuan City 320314, Taiwan;
Department of Mathematics and Informatics, Azerbaijan University,
71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan;
Section of Mathematics, International Telematic University Uninettuno,
I-00186 Rome, Italy.
Email: harimsri@math.uvic.ca