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Existence results for a nonlocal q-integro multipoint boundary value problem involving a fractional q-difference equation with dual hybrid terms

Boshra Alharbi, Ahmed Alsaedi, Ravi P. Agarwal and Bashir Ahmad

Abstract

This paper is devoted to the study of a fractional q-difference equation involving dual hybrid terms and equipped with nonlocal multipoint and Riemann-Liouville fractional q-integral boundary conditions. Applying a fixed point approach, we investigate the existence criteria for solutions to the given problem. Examples are constructed for illustrating the obtained results. We emphasize that our results are new in the given configuration, and some new results follow as special cases of the present ones.

1 Introduction

The topic of fractional calculus has evolved an interesting and attractive area of investigation during the last few decades. It has been mainly due to the extensive use of its tools in the mathematical modeling of natural and scientific phenomena such as chaotic synchronization [1], immune systems [2], neural networks [3], fractional diffusion [4], ecology [5], etc. For theoretical

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concepts of fractional calculus, we refer the reader to the text [6]. Keeping in mind the occurrence of initial and boundary value problems in the physical problems, many investigators turned to the investigation of fractional counterpart of these problems. For some recent works on fractional order differential equations, for instance, see the text [7] and articles [8]-[12]. Afterward, the concept of fractional differential equations was extended to fractional q-difference equations. One can find some interesting results on nonlinear boundary value problems involving fractional q-difference and q-integral operators in the research papers [13]-[22] and the references cited therein. Recently, in [23], a coupled system of nonlinear fractional q-integro-difference equations equipped with coupled q-integral boundary conditions was studied, while a Langevin-type q-variant system of nonlinear fractional integro-difference equations with nonlocal boundary conditions was investigated in [24]. More recently, in [26], the authors discussed the existence of solutions for a hybrid Riemann-Liouville fractional q-integro-difference equation with nonlocal q-integral boundary conditions.

Motivated by the foregoing works on fractional boundary value problems of q-difference equations, we consider a fractional q-difference equation involving dual hybrid terms and complemented with nonlocal multipoint Riemann-Liouville fractional q-integral boundary conditions. In precise terms, we investigate the existence and uniqueness of solutions to the problem:

$$\varepsilon D_q^{\vartheta_1}[u(x)-f_1(x,u(x))]+(1-\varepsilon)D_q^{\vartheta_2}[u(x)-f_2(x,u(x))] = g(x,u(x)),\ 0<x<1, \tag{1}$$

$$\begin{aligned} u(0)=0,\ u(1) &= \delta\int_0^{\lambda}\frac{(\lambda-qs)^{(\zeta_1-1)}}{\Gamma_q(\zeta_1)}u(s)d_qs \\ &\quad +(1-\delta)\int_0^{\mu}\frac{(\mu-qs)^{(\zeta_2-1)}}{\Gamma_q(\zeta_2)}u(s)d_qs+\sum_{i=1}^{n}a_iu(\xi_i), \end{aligned} \tag{2}$$

where $D_q^{\vartheta_1}$ and $D_q^{\vartheta_2}$ denote the Riemann-Liouville fractional q-derivative operators of order ϑ_1 and ϑ_2 respectively, $1<\vartheta_1,\vartheta_2\le 2$ with $\vartheta_1-\vartheta_2>0$, $0<q<1$, $0<\varepsilon\le 1$, $0\le\delta\le 1$, $\zeta_1,\zeta_2>0$, $0<\lambda,\mu,\xi_i<1$, $a_i\in\mathbb{R}$ and $f_1,f_2,g:[0,1]\times\mathbb{R}\to\mathbb{R}$ are given continuous functions.

The objective of the present work is investigate the criteria ensuring the existence and uniqueness of solutions to the problem (1)-(2). The existence results for the problem at hand are obtained by means of Krasnoselskii's fixed point theorem [27] and Leray-Schauder's nonlinear alternative [28], while the uniqueness of solutions is established by Banach's fixed point theorem [29].

We arrange the remainder of the paper as follows. In Section 2, we recall some basic concepts related to our present study. Section 3 contains the main results, while examples illustrating the obtained results are presented in Section 4.

2 Auxiliary material

Let us begin this section with some basic concepts q-fractional calculus.

For an arbitrary real number $q \in (0,1)$, the $q-$number $[\alpha]_q$ is defined by $[\alpha]_q = \dfrac{1-q^\alpha}{1-q}$, for every $\alpha \in \mathbb{R}$. We define the q-shifted factorial of real number α as $(\alpha;q)_0 = 1$ and $(\alpha;q)_n = \prod_{j=0}^{n-1}(1-\alpha q^j)$ for $n \in \mathbb{N}\cup\{\infty\}$. The q-analogue of the power function $(\alpha-\beta)^n$ with $n \in \mathbb{N}_0 := \{0,1,2,\ldots\}$ is defined by

$$(\alpha-\beta)^{(0)} = 1, \qquad (\alpha-\beta)^{(n)} = \prod_{j=0}^{n-1}(\alpha-\beta q^j),\ \alpha,\beta \in \mathbb{R}.$$

In case ς is real number, we have $(\alpha-\beta)^{(\varsigma)} = \alpha^\varsigma \prod_{j=0}^{\infty}\dfrac{\alpha-\beta q^j}{\alpha-\beta q^{\varsigma+j}}$ and $\alpha^{(\varsigma)} = \alpha^\varsigma$ when $\beta = 0$. If $\varsigma > 0$ and $0 \le \alpha \le \beta \le t$, then $(t-\beta)^{(\varsigma)} \le (t-\alpha)^{(\varsigma)}$.

The q-Gamma function $\Gamma_q(\varrho)$ is defined as

$$\Gamma_q(\varsigma) = \frac{(1-q)^{(\varsigma-1)}}{(1-q)^{\varsigma-1}}, \qquad \varsigma \in \mathbb{R}\setminus\{0,-1,-2,\ldots\},$$

which satisfies the relation $\Gamma_q(\varsigma+1) = [\varsigma]_q\Gamma_q(\varsigma)$ [30].

Definition 2.1 ([30]) The Riemann-Liouville fractional q-integral for a continuous function $u:(0,\infty)\to\mathbb{R}$ of order $\varsigma \ge 0$ is defined by $(I_q^0 u)(t) = u(t)$ and

$$\begin{aligned}(I_q^\varsigma u)(t) &= \frac{1}{\Gamma_q(\varsigma)}\int_0^t (t-qs)^{(\varsigma-1)}u(s)d_qs\\ &= t^\varsigma(1-q)^\varsigma\sum_{k=0}^{\infty} q^k\frac{(q^\varsigma;q)_k}{(q;q)_k}u(tq^k), \quad \varsigma>0,\ t\in(0,\infty).\end{aligned}$$

Also, the Riemann-Liouville fractional q-integral for a continuous function $u:(0,\infty)\to\mathbb{R}$ of order $\varsigma \ge 0$ can be defined as

$$(I_q^\varsigma u)(t) = \frac{1}{\Gamma_q(\varsigma)}\int_0^t (t-qs)^{(\varsigma-1)}u(s)d_qs, \quad \varsigma>0,$$

for $t \in (0,\infty)$, provided that the right-hand side is point-wise defined on $(0,\infty)$ [14, 30].

Recall that $I_q^{\varsigma_1} I_q^{\varsigma_2} u(t) = I_q^{\varsigma_1+\varsigma_2} u(t)$ for $\varsigma_1, \varsigma_2 \in \mathbb{R}^+$ [14, 30] and

$$I_q^{\varsigma_1} t^{\varsigma_2} = \frac{\Gamma_q(\varsigma_2+1)}{\Gamma_q(\varsigma_1+\varsigma_2+1)} t^{\varsigma_1+\varsigma_2}, \quad \varsigma_1 > 0, \quad \varsigma_2 \in (-1,\infty), \ t > 0.$$

If $f \equiv 1$, then $I_q^{\varsigma} 1(t) = \dfrac{1}{\Gamma_q(\varsigma+1)} t^{\varsigma}$ for all $t > 0$.

Definition 2.2 ([30]) The fractional q-derivative of the Riemann-Liouville type of order $\varsigma \geq 0$ is defined by $(D_q^0 u)(t) = u(t)$ and

$$(D_q^{\varsigma} u)(t) = (D_q^n I_q^{n-\varsigma} u)(t), \ \varsigma > 0,$$

where n is the smallest integer greater than or equal to ς, $I_q^{(.)}$ is the Riemann-Liouville fractional q-integral of order $(.)$ and D_q^n is the q-derivative of integer order n.

We can also define the Riemann-Liouville fractional q-derivative of order $\varsigma > 0$ for a function $u : (0,\infty) \to \mathbb{R}$ as

$$D_q^{\varsigma} u(t) = \frac{1}{\Gamma_q(n-\varsigma)} \int_0^t \frac{u(s)}{(t-qs)^{\varsigma-n+1}} d_q s, \qquad n-1 < \varsigma < n,$$

provided that the right-hand side is point-wise defined on $(0,\infty)$ [14, 30].

Now we prove an auxiliary lemma for the linear variant of the problem (1)-(2).

Lemma 2.3 Let $\rho_1, \rho_2, \sigma \in C([0,1],\mathbb{R})$ and

$$\Delta := 1 - \frac{\delta \Gamma_q(\vartheta_1)\ \lambda^{\vartheta_1+\zeta_1-1}}{\Gamma_q(\vartheta_1+\zeta_1)} - \frac{(1-\delta)\Gamma_q(\vartheta_1)\ \mu^{\vartheta_1+\zeta_2-1}}{\Gamma_q(\vartheta_1+\zeta_2)} - \sum_{i=1}^{n} a_i\ \xi_i^{\vartheta_1-1} \neq 0. \quad (3)$$

The function u is a solution for the fractional q-difference boundary value problem

$$\begin{cases} \varepsilon D_q^{\vartheta_1}[u(x)-\rho_1(x)] + (1-\varepsilon) D_q^{\vartheta_2}[u(x)-\rho_2(x)] = \sigma(x), \ 0 < x < 1, \\ u(0) = 0, \ u(1) = \delta \displaystyle\int_0^{\lambda} \frac{(\lambda-qs)^{(\zeta_1-1)}}{\Gamma_q(\zeta_1)} u(s) d_q s \\ \qquad\qquad + (1-\delta) \displaystyle\int_0^{\mu} \frac{(\mu-qs)^{(\zeta_2-1)}}{\Gamma_q(\zeta_2)} u(s) d_q s + \sum_{i=1}^{n} a_i u(\xi_i), \end{cases} \quad (4)$$

if and only if u is a solution for the fractional q-integral equation

$$\begin{aligned} u(x) &= \rho_1(x) + \frac{(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^x (x-qs)^{(\vartheta_1-\vartheta_2-1)}[u(s)-\rho_2(s)]\ d_qs \\ &+ \frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\int_0^x (x-qs)^{(\vartheta_1-1)}\sigma(s)\ d_qs \\ &+ \frac{x^{\vartheta_1-1}}{\Delta}\Big[\frac{\delta}{\Gamma_q(\zeta_1)}\int_0^\lambda (\lambda-qs)^{(\zeta_1-1)}\rho_1(s))\ d_qs \\ &+ \frac{\delta(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1)}\int_0^\lambda (\lambda-qs)^{(\vartheta_1-\vartheta_2+\zeta_1-1)}[u(s)-\rho_2(s)]\ d_qs \\ &+ \frac{\delta}{\varepsilon\Gamma_q(\vartheta_1+\zeta_1)}\int_0^\lambda (\lambda-qs)^{(\vartheta_1+\zeta_1-1)}\sigma(s)\ d_qs \\ &+ \frac{(1-\delta)}{\Gamma_q(\zeta_2)}\int_0^\mu (\mu-qs)^{(\zeta_2-1)}\rho_1(s)\ d_qs \\ &+ \frac{(1-\delta)(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2)}\int_0^\mu (\mu-qs)^{(\vartheta_1-\vartheta_2+\zeta_2-1)}[u(s)-\rho_2(s))]\ d_qs \\ &+ \frac{(1-\delta)}{\varepsilon\Gamma_q(\vartheta_1+\zeta_2)}\int_0^\mu (\mu-qs)^{(\vartheta_1+\zeta_2-1)}\sigma(s)\ d_qs \\ &- \frac{(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^1 (1-qs)^{(\vartheta_1-\vartheta_2-1)}[u(s)-\rho_2(s)]\ d_qs \\ &- \frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\int_0^1 (1-qs)^{(\vartheta_1-1)}\sigma(s)\ d_qs - \rho_1(1) \\ &+ \frac{(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\sum_{i=1}^n a_i\int_0^{\xi_i} (\xi_i-qs)^{(\vartheta_1-\vartheta_2-1)}[u(s)-\rho_2(s)]\ d_qs \\ &+ \frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\sum_{i=1}^n a_i\int_0^{\xi_i} (\xi_i-qs)^{(\vartheta_1-1)}\sigma(s)\ d_qs + \sum_{i=1}^n a_i\rho_1(\xi_i)\Big]. \end{aligned} \tag{5}$$

Proof. Letting u to be a solution of the q-fractional boundary value problem (4), we rewrite the fractional q-difference equation as

$$D_q^{\vartheta_1}[u(x)-\rho_1(x)] = \Big(\frac{\varepsilon-1}{\varepsilon}\Big)D_q^{\vartheta_2}[u(x)-\rho_2(x)] + \frac{1}{\varepsilon}\sigma(x). \tag{6}$$

Applying the Riemann-Liouville fractional q-integral operator of order ϑ_1 to both sides of (6), we get

$$u(x)-\rho_1(x) = \Big(\frac{\varepsilon-1}{\varepsilon}\Big)I_q^{\vartheta_1}D_q^{\vartheta_2}[u(x)-\rho_2(x)] + \frac{1}{\varepsilon}I_q^{\vartheta_1}\sigma(x) + b_1x^{\vartheta_1-1} + b_2x^{\vartheta_1-2},$$

where $b_1, b_2 \in \mathbb{R}$ are arbitrary constants. In view of the fact that $1 < \vartheta_1 < 2$, by the first boundary condition, we have $b_2 = 0$. Thus,

$$u(x) = \rho_1(x) + \Big(\frac{\varepsilon - 1}{\varepsilon}\Big) I_q^{\vartheta_1 - \vartheta_2}[u(x) - \rho_2(x)] + \frac{1}{\varepsilon} I_q^{\vartheta_1}\sigma(x) + b_1 x^{\vartheta_1 - 1}. \tag{7}$$

Let $\Theta \in \{\zeta_1, \zeta_2\}$, then we have

$$\begin{aligned} I_q^{\Theta}u(x) \quad = \quad & \frac{1}{\Gamma_q(\Theta)}\int_0^x (x - qs)^{(\Theta-1)}\rho_1(s)\ d_qs \\ & + \frac{\varepsilon - 1}{\varepsilon\Gamma_q(\vartheta_1 - \vartheta_2 + \Theta)}\int_0^x (x - qs)^{(\vartheta_1 - \vartheta_2 + \Theta - 1)}[u(s) - \rho_2(s)]\ d_qs \\ & + \frac{1}{\varepsilon\Gamma_q(\vartheta_1 + \Theta)}\int_0^x (x - qs)^{(\vartheta_1 + \Theta - 1)}\sigma(s)\ d_qs \\ & + b_1\frac{\Gamma_q(\vartheta_1)}{\Gamma_q(\vartheta_1 + \Theta)} x^{\vartheta_1 + \Theta - 1}. \end{aligned} \tag{8}$$

Now, using the second boundary condition together with (3) and (8), we get

$$\begin{aligned} b_1 \quad = \quad & \frac{1}{\Delta}\Big[\frac{\delta}{\Gamma_q(\zeta_1)}\int_0^\lambda (\lambda - qs)^{(\zeta_1 - 1)}\rho_1(s)\ d_qs \\ & + \frac{\delta(\varepsilon - 1)}{\varepsilon\Gamma_q(\vartheta_1 - \vartheta_2 + \zeta_1)}\int_0^\lambda (\lambda - qs)^{(\vartheta_1 - \vartheta_2 + \zeta_1 - 1)}[u(s) - \rho_2(s)]\ d_qs \\ & + \frac{\delta}{\varepsilon\Gamma_q(\vartheta_1 + \zeta_1)}\int_0^\lambda (\lambda - qs)^{(\vartheta_1 + \zeta_1 - 1)}\sigma(s)\ d_qs \\ & + \frac{(1 - \delta)}{\Gamma_q(\zeta_2)}\int_0^\mu (\mu - qs)^{(\zeta_2 - 1)}\rho_1(s)\ d_qs \\ & + \frac{(1 - \delta)(\varepsilon - 1)}{\varepsilon\Gamma_q(\vartheta_1 - \vartheta_2 + \zeta_2)}\int_0^\mu (\mu - qs)^{(\vartheta_1 - \vartheta_2 + \zeta_2 - 1)}[u(s) - \rho_2(s)]\ d_qs \\ & + \frac{(1 - \delta)}{\varepsilon\Gamma_q(\vartheta_1 + \zeta_2)}\int_0^\mu (\mu - qs)^{(\vartheta_1 + \zeta_2 - 1)}\sigma(s)\ d_qs \\ & - \frac{(\varepsilon - 1)}{\varepsilon\Gamma_q(\vartheta_1 - \vartheta_2)}\int_0^1 (1 - qs)^{(\vartheta_1 - \vartheta_2 - 1)}[u(s) - \rho_2(s)]\ d_qs \\ & - \frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\int_0^1 (1 - qs)^{(\vartheta_1 - 1)}\sigma(s)\ d_qs - \rho_1(1) \\ & + \frac{(\varepsilon - 1)}{\varepsilon\Gamma_q(\vartheta_1 - \vartheta_2)}\sum_{i=1}^n a_i\int_0^{\xi_i} (\xi_i - qs)^{(\vartheta_1 - \vartheta_2 - 1)}[u(s) - \rho_2(s)]\ d_qs \\ & + \frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\sum_{i=1}^n a_i\int_0^{\xi_i} (\xi_i - qs)^{(\vartheta_1 - 1)}\sigma(s)\ d_qs + \sum_{i=1}^n a_i\rho_1(\xi_i)\Big]. \end{aligned}$$

Inserting the value of b_1 in (7) leads to the solution (5). One can obtain the converse of the lemma by direct computation. □

By Lemma 2, we can transform the problem (1)-(2) into an equivalent fixed point problem: $\mathcal{W}u = u$, where $\mathcal{W} : \mathcal{E} \to \mathcal{E}$ is an operator defined by

$$\begin{aligned}
&(\mathcal{W}u)(x)\\
= \; & f_1(x,u(x)) + \frac{(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^x (x-qs)^{(\vartheta_1-\vartheta_2-1)}[u(s)-f_2(s,u(s))]\; d_qs\\
&+\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\int_0^x (x-qs)^{(\vartheta_1-1)}g(s,u(s))\; d_qs\\
&+\frac{x^{\vartheta_1-1}}{\Delta}\Big[\frac{\delta}{\Gamma_q(\zeta_1)}\int_0^\lambda (\lambda-qs)^{(\zeta_1-1)}f_1(s,u(s))\; d_qs\\
&+\frac{\delta(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1)}\int_0^\lambda (\lambda-qs)^{(\vartheta_1-\vartheta_2+\zeta_1-1)}[u(s)-f_2(s,u(s))]\; d_qs\\
&+\frac{\delta}{\varepsilon\Gamma_q(\vartheta_1+\zeta_1)}\int_0^\lambda (\lambda-qs)^{(\vartheta_1+\zeta_1-1)}g(s,u(s))\; d_qs\\
&+\frac{(1-\delta)}{\Gamma_q(\zeta_2)}\int_0^\mu (\mu-qs)^{(\zeta_2-1)}f_1(s,u(s))\; d_qs\\
&+\frac{(1-\delta)(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2)}\int_0^\mu (\mu-qs)^{(\vartheta_1-\vartheta_2+\zeta_2-1)}[u(s)-f_2(s,u(s))]\; d_qs\\
&+\frac{(1-\delta)}{\varepsilon\Gamma_q(\vartheta_1+\zeta_2)}\int_0^\mu (\mu-qs)^{(\vartheta_1+\zeta_2-1)}g(s,u(s))\; d_qs\\
&-\frac{(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^1 (1-qs)^{(\vartheta_1-\vartheta_2-1)}[u(s)-f_2(s,u(s))]\; d_qs\\
&-\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\int_0^1 (1-qs)^{(\vartheta_1-1)}g(s,u(s))\; d_qs - f_1(1,u(1))\\
&+\frac{(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\sum_{i=1}^n a_i\int_0^{\xi_i} (\xi_i-qs)^{(\vartheta_1-\vartheta_2-1)}[u(s)-f_2(s,u(s))]\; d_qs\\
&+\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\sum_{i=1}^n a_i\int_0^{\xi_i} (\xi_i-qs)^{(\vartheta_1-1)}g(s,u(s))\; d_qs\\
&+\sum_{i=1}^n a_i f_1(\xi_i,u(\xi_i))\Big],\; u\in\mathcal{E},\; x\in[0,1]. \qquad (9)
\end{aligned}$$

Here $\mathcal{E}$ is the Banach space of all continuous real valued functions defined on $[0,1]$ endowed with the norm $\|u\| = \sup_{x\in[0,1]}|u(x)|,\; u\in\mathcal{E}$.

In the forthcoming analysis, we set the notation:

$$
\begin{aligned}
\upsilon_1 &= 1+\frac{\delta\ \lambda^{\zeta_1}}{|\Delta|\Gamma_q(\zeta_1+1)}+\frac{(1-\delta)\ \mu^{\zeta_2}}{|\Delta|\Gamma_q(\zeta_2+1)}+\frac{1}{|\Delta|}+\frac{1}{|\Delta|}\sum_{i=1}^{n}a_i,\\
\upsilon_2 &= \frac{1}{\varepsilon}\left[\frac{1}{\Gamma_q(\vartheta_1+1)}+\frac{\delta\ \lambda^{\vartheta_1+\zeta_1}}{|\Delta|\Gamma_q(\vartheta_1+\zeta_1+1)}+\frac{(1-\delta)\ \mu^{\vartheta_1+\zeta_2}}{|\Delta|\Gamma_q(\vartheta_1+\zeta_2+1)}\right.\\
&+ \left.\frac{1}{|\Delta|\Gamma_q(\vartheta_1+1)}+\frac{1}{|\Delta|\Gamma_q(\vartheta_1+1)}\sum_{i=1}^{n}a_i\ \xi_i^{\vartheta_1}\right],\\
\upsilon_3 &= \frac{|\varepsilon-1|}{\varepsilon}\left[\frac{1}{\Gamma_q(\vartheta_1-\vartheta_2+1)}+\frac{\delta\ \lambda^{\vartheta_1-\vartheta_2+\zeta_1}}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1+1)}\right.\\
&+ \frac{(1-\delta)\ \mu^{\vartheta_1-\vartheta_2+\zeta_2}}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2+1)}\\
&+ \left.\frac{1}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+1)}+\frac{1}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+1)}\sum_{i=1}^{n}a_i\ \xi_i^{\vartheta_1-\vartheta_2}\right].
\end{aligned} \tag{10}
$$

3 Main Results

We begin this section with an existence result for the problem (1)-(2), which is proved with the aid of Krasnoselskii's fixed point theorem [27].

Theorem 3.1. Assume that

(A_1) there exist $\varphi_i \in C([0,1],\mathbb{R}^+), i=1,2,3$, such that

$$|f_1(x,u)| \le \varphi_1(x),\ |f_2(x,u)| \le \varphi_2(x),\ |g(x,u)| \le \varphi_3(x),\ \forall (x,u) \in [0,1]\times\mathbb{R},$$

and $\|\varphi_i\| = \sup_{x\in[0,1]} |\varphi_i(x)|$.

If $\upsilon_3 < 1$, where υ_3 is given in (10), then the fractional hybrid q-difference equation (1) with q-integral nonlocal boundary conditions (2) has at least one solution on $[0,1]$.

Proof. In order to verify the hypotheses of Krasnoselskii's fixed point theorem

[27], we introduce two operators $\mathcal{W}_1, \mathcal{W}_2 : B_\rho \to \mathcal{E}$ as

$$\begin{aligned}(\mathcal{W}_1 u)(x) &= \frac{(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^x (x-qs)^{(\vartheta_1-\vartheta_2-1)}u(s)\ d_qs \\ &+\frac{x^{\vartheta_1-1}}{\Delta}\Big[\frac{\delta(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1)}\int_0^\lambda (\lambda-qs)^{(\vartheta_1-\vartheta_2+\zeta_1-1)}u(s)\ d_qs \\ &+\frac{(1-\delta)(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2)}\int_0^\mu (\mu-qs)^{(\vartheta_1-\vartheta_2+\zeta_2-1)}u(s)\ d_qs \\ &-\frac{(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^1 (1-qs)^{(\vartheta_1-\vartheta_2-1)}u(s)\ d_qs \\ &+\frac{(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\sum_{i=1}^n a_i\int_0^{\xi_i} (\xi_i-qs)^{(\vartheta_1-\vartheta_2-1)}u(s)\ d_qs\Big], \qquad (11)\end{aligned}$$

$$\begin{aligned}(\mathcal{W}_2 u)(x) &= f_1(x,u(x)) - \frac{(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^x (x-qs)^{(\vartheta_1-\vartheta_2-1)}f_2(s,u(s))\ d_qs \\ &+\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\int_0^x (x-qs)^{(\vartheta_1-1)}g(s,u(s))\ d_qs \\ &+\frac{x^{\vartheta_1-1}}{\Delta}\Big[\frac{\delta}{\Gamma_q(\zeta_1)}\int_0^\lambda (\lambda-qs)^{(\zeta_1-1)}f_1(s,u(s))\ d_qs \\ &-\frac{\delta(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1)}\int_0^\lambda (\lambda-qs)^{(\vartheta_1-\vartheta_2+\zeta_1-1)}f_2(s,u(s))\ d_qs \\ &+\frac{\delta}{\varepsilon\Gamma_q(\vartheta_1+\zeta_1)}\int_0^\lambda (\lambda-qs)^{(\vartheta_1+\zeta_1-1)}g(s,u(s))\ d_qs \\ &+\frac{(1-\delta)}{\Gamma_q(\zeta_2)}\int_0^\mu (\mu-qs)^{(\zeta_2-1)}f_1(s,u(s))\ d_qs \\ &-\frac{(1-\delta)(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2)}\int_0^\mu (\mu-qs)^{(\vartheta_1-\vartheta_2+\zeta_2-1)}f_2(s,u(s))\ d_qs \\ &+\frac{(1-\delta)}{\varepsilon\Gamma_q(\vartheta_1+\zeta_2)}\int_0^\mu (\mu-qs)^{(\vartheta_1+\zeta_2-1)}g(s,u(s))\ d_qs \\ &+\frac{(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^1 (1-qs)^{(\vartheta_1-\vartheta_2-1)}f_2(s,u(s))\ d_qs \\ &-\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\int_0^1 (1-qs)^{(\vartheta_1-1)}g(s,u(s))\ d_qs - f_1(1,u(1))\end{aligned}$$

$$
\begin{aligned}
&-\frac{(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\sum_{i=1}^{n}a_i\int_0^{\xi_i}(\xi_i-qs)^{(\vartheta_1-\vartheta_2-1)}f_2(s,u(s))\ d_qs\\
&+\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\sum_{i=1}^{n}a_i\int_0^{\xi_i}(\xi_i-qs)^{(\vartheta_1-1)}g(s,u(s))\ d_qs\\
&+\sum_{i=1}^{n}a_if_1(\xi_i,u(\xi_i))\Big],
\end{aligned}
\tag{12}
$$

where $B_\rho := \{u \in \mathcal{E} : \|u\| \leq \rho\}$ is a closed, bounded, convex and nonempty subset of Banach space $\mathcal{E}$ with

$$
\rho \geq \frac{\|\varphi_1\|\ \upsilon_1 + \|\varphi_2\|\ \upsilon_3 + \|\varphi_3\|\ \upsilon_2}{1-\upsilon_3}, \quad \upsilon_3 < 1, \tag{13}
$$

υ_1, υ_2 and υ_3 are given in (10). Observe that the operator $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$, where $\mathcal{W}$ is given by (9).
For any $u, v \in B_\rho$, by the assumption (A_1), we have

$$
\begin{aligned}
&|\mathcal{W}_1u(x)+\mathcal{W}_2v(x)|\\
\leq\ &\sup_{x\in[0,1]}\Bigg\{|f_1(x,v(x))|+\frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^{x}(x-qs)^{(\vartheta_1-\vartheta_2-1)}|u(s)|\ d_qs\\
&+\frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^{x}(x-qs)^{(\vartheta_1-\vartheta_2-1)}|f_2(s,v(s))|\ d_qs\\
&+\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\int_0^{x}(x-qs)^{(\vartheta_1-1)}|g(s,v(s))|\ d_qs\\
&+\frac{x^{\vartheta_1-1}}{|\Delta|}\Big[\frac{\delta}{\Gamma_q(\zeta_1)}\int_0^{\lambda}(\lambda-qs)^{(\zeta_1-1)}|f_1(s,v(s))|\ d_qs\\
&+\frac{\delta|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1)}\int_0^{\lambda}(\lambda-qs)^{(\vartheta_1-\vartheta_2+\zeta_1-1)}|u(s)|\ d_qs\\
&+\frac{\delta|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1)}\int_0^{\lambda}(\lambda-qs)^{(\vartheta_1-\vartheta_2+\zeta_1-1)}|f_2(s,v(s))|\ d_qs\\
&+\frac{\delta}{\varepsilon\Gamma_q(\vartheta_1+\zeta_1)}\int_0^{\lambda}(\lambda-qs)^{(\vartheta_1+\zeta_1-1)}|g(s,v(s))|\ d_qs\\
&+\frac{(1-\delta)}{\Gamma_q(\zeta_2)}\int_0^{\mu}(\mu-qs)^{(\zeta_2-1)}|f_1(s,v(s))|\ d_qs\\
&+\frac{(1-\delta)|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2)}\int_0^{\mu}(\mu-qs)^{(\vartheta_1-\vartheta_2+\zeta_2-1)}|u(s)|\ d_qs
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{(1-\delta)|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2)}\int_0^{\mu}(\mu-qs)^{(\vartheta_1-\vartheta_2+\zeta_2-1)}|f_2(s,v(s))|\ d_qs \\
& +\frac{(1-\delta)}{\varepsilon\Gamma_q(\vartheta_1+\zeta_2)}\int_0^{\mu}(\mu-qs)^{(\vartheta_1+\zeta_2-1)}|g(s,v(s))|\ d_qs \\
& +\frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^{1}(1-qs)^{(\vartheta_1-\vartheta_2-1)}|u(s)|\ d_qs \\
& +\frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^{1}(1-qs)^{(\vartheta_1-\vartheta_2-1)}|f_2(s,v(s))|\ d_qs \\
& +\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\int_0^{1}(1-qs)^{(\vartheta_1-1)}|g(s,v(s))|\ d_qs+|f_1(1,v(1))| \\
& +\frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\sum_{i=1}^{n}a_i\int_0^{\xi_i}(\xi_i-qs)^{(\vartheta_1-\vartheta_2-1)}|u(s)|\ d_qs \\
& +\frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\sum_{i=1}^{n}a_i\int_0^{\xi_i}(\xi_i-qs)^{(\vartheta_1-\vartheta_2-1)}|f_2(s,v(s))|\ d_qs \\
& +\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\sum_{i=1}^{n}a_i\int_0^{\xi_i}(\xi_i-qs)^{(\vartheta_1-1)}|g(s,v(s))|\ d_qs+\sum_{i=1}^{n}a_i|f_1(\xi_i,v(\xi_i))|\Big]\Bigg\} \\
\leq\ & \|\varphi_1\|\sup_{x\in[0,1]}\Bigg\{1+\frac{x^{(\vartheta_1-1)}}{|\Delta|}\Big[\frac{\delta}{\Gamma_q(\zeta_1)}\int_0^{\lambda}(\lambda-qs)^{(\zeta_1-1)}\ d_qs \\
& +\frac{(1-\delta)}{\Gamma_q(\zeta_2)}\int_0^{\mu}(\mu-qs)^{(\zeta_2-1)}\ d_qs+1+\sum_{i=1}^{n}a_i\Big]\Bigg\} \\
& +\|\varphi_2\|\sup_{x\in[0,1]}\Bigg\{\frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^{x}(x-qs)^{(\vartheta_1-\vartheta_2-1)}\ d_qs \\
& +\frac{x^{(\vartheta_1-1)}}{|\Delta|}\Big[\frac{\delta|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1)}\int_0^{\lambda}(\lambda-qs)^{(\vartheta_1-\vartheta_2+\zeta_1-1)}\ d_qs \\
& +\frac{(1-\delta)|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2)}\int_0^{\mu}(\mu-qs)^{(\vartheta_1-\vartheta_2+\zeta_2-1)}\ d_qs \\
& +\frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^{1}(1-qs)^{(\vartheta_1-\vartheta_2-1)}\ d_qs \\
& +\frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\sum_{i=1}^{n}a_i\int_0^{\xi_i}(\xi_i-qs)^{(\vartheta_1-\vartheta_2-1)}\ d_qs\Big]\Bigg\} \\
& +\|\varphi_3\|\sup_{x\in[0,1]}\Bigg\{\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\int_0^{x}(x-qs)^{(\vartheta_1-1)}\ d_qs \\
& +\frac{x^{(\vartheta_1-1)}}{|\Delta|}\Big[\frac{\delta}{\varepsilon\Gamma_q(\vartheta_1+\zeta_1)}\int_0^{\lambda}(\lambda-qs)^{(\vartheta_1+\zeta_1-1)}\ d_qs
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(1-\delta)}{\varepsilon\Gamma_q(\vartheta_1+\zeta_2)}\int_0^{\mu}(\mu-qs)^{(\vartheta_1+\zeta_2-1)}\ d_qs\\
&+\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\int_0^{1}(1-qs)^{(\vartheta_1-1)}\ d_qs\\
&+\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\sum_{i=1}^{n}a_i\int_0^{\xi_i}(\xi_i-qs)^{(\vartheta_1-1)}\ d_qs\Big]\Bigg\}\\
&+\rho\sup_{x\in[0,1]}\Bigg\{\frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^{x}(x-qs)^{(\vartheta_1-\vartheta_2-1)}\ d_qs\\
&+\frac{x^{(\vartheta_1-1)}}{|\Delta|}\Big[\frac{\delta|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1)}\int_0^{\lambda}(\lambda-qs)^{(\vartheta_1-\vartheta_2+\zeta_1-1)}\ d_qs\\
&+\frac{(1-\delta)|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2)}\int_0^{\mu}(\mu-qs)^{(\vartheta_1-\vartheta_2+\zeta_2-1)}\ d_qs\\
&+\frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^{1}(1-qs)^{(\vartheta_1-\vartheta_2-1)}\ d_qs\\
&+\frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\sum_{i=1}^{n}a_i\int_0^{\xi_i}(\xi_i-qs)^{(\vartheta_1-\vartheta_2-1)}\ d_qs\Big]\Bigg\}\\
\leq\ &\|\varphi_1\|\left[1+\frac{\delta\ \lambda^{\zeta_1}}{|\Delta|\Gamma_q(\zeta_1+1)}+\frac{(1-\delta)\ \mu^{\zeta_2}}{|\Delta|\Gamma_q(\zeta_2+1)}+\frac{1}{|\Delta|}+\frac{1}{|\Delta|}\sum_{i=1}^{n}a_i\right]\\
&+\|\varphi_2\|\left[\frac{|\varepsilon-1|}{\varepsilon}\left(\frac{1}{\Gamma_q(\vartheta_1-\vartheta_2+1)}+\frac{\delta\ \lambda^{(\vartheta_1-\vartheta_2+\zeta_1)}}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1+1)}\right.\right.\\
&+\frac{(1-\delta)\ \mu^{(\vartheta_1-\vartheta_2+\zeta_2)}}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2+1)}+\frac{1}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+1)}\\
+\ &\left.\left.\frac{1}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+1)}\sum_{i=1}^{n}a_i\ \xi_i^{\vartheta_1-\vartheta_2}\right)\right]\\
&+\|\varphi_3\|\left[\frac{1}{\varepsilon}\left(\frac{1}{\Gamma_q(\vartheta_1+1)}+\frac{\delta\ \lambda^{(\vartheta_1+\zeta_1)}}{|\Delta|\Gamma_q(\vartheta_1+\zeta_1+1)}\right.\right.\\
+\ &\left.\left.\frac{(1-\delta)\ \mu^{(\vartheta_1+\zeta_2)}}{|\Delta|\Gamma_q(\vartheta_1+\zeta_2+1)}+\frac{1}{|\Delta|\Gamma_q(\vartheta_1+1)}+\frac{1}{|\Delta|\Gamma_q(\vartheta_1+1)}\sum_{i=1}^{n}a_i\ \xi_i^{\vartheta_1}\right)\right]\\
&+\rho\left[\frac{|\varepsilon-1|}{\varepsilon}\left(\frac{1}{\Gamma_q(\vartheta_1-\vartheta_2+1)}\right.\right.\\
&+\frac{\delta\ \lambda^{(\vartheta_1-\vartheta_2+\zeta_1)}}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1+1)}+\frac{(1-\delta)\ \mu^{(\vartheta_1-\vartheta_2+\zeta_2)}}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2+1)}\\
&+\left.\left.\frac{1}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+1)}+\frac{1}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+1)}\sum_{i=1}^{n}a_i\ \xi_i^{\vartheta_1-\vartheta_2}\right)\right]\\
\leq\ &\|\varphi_1\|\ \upsilon_1+\|\varphi_2\|\ \upsilon_3+\|\varphi_3\|\ \upsilon_2+\rho\ \upsilon_3,
\end{aligned}
$$

which implies that $\|\mathcal{W}_1u+\mathcal{W}_2v\| \leq \rho$ by the condition (13) and so $\mathcal{W}_1u+\mathcal{W}_2v \in B_\rho$ for all $u, v \in B_\rho$. Using the continuity of f_1, f_2 and g, it can easily be shown that the operator $\mathcal{W}_2$ is continuous on B_ρ.

Now we establish that the operator $\mathcal{W}_2$ is compact. Firstly, we show that $\mathcal{W}_2$ is uniformly bounded. For each $u \in B_\rho$ and $x \in [0,1]$, we have

$$
\begin{aligned}
\|\mathcal{W}_2u\| \leq \sup_{x\in[0,1]} \Bigg\{ & |f_1(x,u(x))| \\
&+\frac{(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^x (x-qs)^{(\vartheta_1-\vartheta_2-1)}|f_2(s,u(s))|\ d_qs \\
&+\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\int_0^x (x-qs)^{(\vartheta_1-1)}|g(s,u(s))|\ d_qs \\
&+\frac{x^{\vartheta_1-1}}{|\Delta|}\Big[\frac{\delta}{\Gamma_q(\zeta_1)}\int_0^\lambda (\lambda-qs)^{(\zeta_1-1)}|f_1(s,u(s))|\ d_qs \\
&+\frac{\delta(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1)}\int_0^\lambda (\lambda-qs)^{(\vartheta_1-\vartheta_2+\zeta_1-1)}|f_2(s,u(s))|\ d_qs \\
&+\frac{\delta}{\varepsilon\Gamma_q(\vartheta_1+\zeta_1)}\int_0^\lambda (\lambda-qs)^{(\vartheta_1+\zeta_1-1)}|g(s,u(s))|\ d_qs \\
&+\frac{(1-\delta)}{\Gamma_q(\zeta_2)}\int_0^\mu (\mu-qs)^{(\zeta_2-1)}|f_1(s,u(s))|\ d_qs \\
&+\frac{(1-\delta)(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2)}\int_0^\mu (\mu-qs)^{(\vartheta_1-\vartheta_2+\zeta_2-1)}|f_2(s,u(s))|\ d_qs \\
&+\frac{(1-\delta)}{\varepsilon\Gamma_q(\vartheta_1+\zeta_2)}\int_0^\mu (\mu-qs)^{(\vartheta_1+\zeta_2-1)}|g(s,u(s))|\ d_qs \\
&+\frac{(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^1 (1-qs)^{(\vartheta_1-\vartheta_2-1)}|f_2(s,u(s))|\ d_qs \\
&+\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\int_0^1 (1-qs)^{(\vartheta_1-1)}|g(s,u(s))|\ d_qs + |f_1(1,u(1))| \\
&+\frac{(\varepsilon-1)}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\sum_{i=1}^n a_i\int_0^{\xi_i} (\xi_i-qs)^{(\vartheta_1-\vartheta_2-1)}|f_2(s,u(s))|\ d_qs \\
&+\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\sum_{i=1}^n a_i\int_0^{\xi_i} (\xi_i-qs)^{(\vartheta_1-1)}|g(s,u(s))|\ d_qs \\
&+\sum_{i=1}^n a_i|f_1(\xi_i,u(\xi_i))|\Big]\Bigg\}
\end{aligned}
$$

$$\begin{aligned}
&\leq \|\varphi_1\|\left[1+\frac{\delta\ \lambda^{\zeta_1}}{|\Delta|\Gamma_q(\zeta_1+1)}+\frac{(1-\delta)\ \mu^{\zeta_2}}{|\Delta|\Gamma_q(\zeta_2+1)}+\frac{1}{|\Delta|}+\frac{1}{|\Delta|}\sum_{i=1}^{n}a_i\right] \\
&\quad +\|\varphi_2\|\left[\frac{|\varepsilon-1|}{\varepsilon}\left(\frac{1}{\Gamma_q(\vartheta_1-\vartheta_2+1)}+\frac{\delta\ \lambda^{(\vartheta_1-\vartheta_2+\zeta_1)}}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1+1)}\right.\right. \\
&\quad +\frac{(1-\delta)\ \mu^{(\vartheta_1-\vartheta_2+\zeta_2)}}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2+1)}+\frac{1}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+1)} \\
&\quad \left.\left.+\frac{1}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+1)}\sum_{i=1}^{n}a_i\ \xi_i^{\vartheta_1-\vartheta_2}\right)\right] \\
&\quad +\|\varphi_3\|\left[\frac{1}{\varepsilon}\left(\frac{1}{\Gamma_q(\vartheta_1+1)}+\frac{\delta\ \lambda^{(\vartheta_1+\zeta_1)}}{|\Delta|\Gamma_q(\vartheta_1+\zeta_1+1)}+\frac{(1-\delta)\ \mu^{(\vartheta_1+\zeta_2)}}{|\Delta|\Gamma_q(\vartheta_1+\zeta_2+1)}\right.\right. \\
&\quad \left.\left.+\frac{1}{|\Delta|\Gamma_q(\vartheta_1+1)}+\frac{1}{|\Delta|\Gamma_q(\vartheta_1+1)}\sum_{i=1}^{n}a_i\ \xi_i^{\vartheta_1}\right)\right] \\
&= \|\varphi_1\|\upsilon_1+\|\varphi_2\|\ \upsilon_3+\|\varphi_3\|\upsilon_2.
\end{aligned}$$

To establish the equicontinuity of the operator $\mathcal{W}_2$, let $x_1, x_2 \in [0,1]$ with $x_2 > x_1$. It will be verified that $\mathcal{W}_2$ maps bounded sets into equicontinuous sets. For each $u \in B_\rho$, we have

$$\begin{aligned}
&|\mathcal{W}_2u(x_2)-\mathcal{W}_2u(x_1)| \\
\leq\ & \sup_{x\in[0,1]}\Bigg\{|f_1(x_2,u(x_2))-f_1(x_1,u(x_1))| \\
&+\frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^{x_1}\left[(x_2-qs)^{(\vartheta_1-\vartheta_2-1)}-(x_1-qs)^{(\vartheta_1-\vartheta_2-1)}\right]|f_2(s,u(s))|\ d_qs \\
&+\frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_{x_1}^{x_2}(x_2-qs)^{(\vartheta_1-\vartheta_2-1)}|f_2(s,u(s))|\ d_qs \\
&+\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\int_0^{x_1}\left[(x_2-qs)^{(\vartheta_1-1)}-(x_1-qs)^{(\vartheta_1-1)}\right]|g(s,u(s))|\ d_qs \\
&+\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\int_{x_1}^{x_2}(x_2-qs)^{(\vartheta_1-1)}|g(s,u(s))|\ d_qs \\
&+\frac{x_2^{(\vartheta_1-1)}-x_1^{(\vartheta_1-1)}}{|\Delta|}\Big[\frac{\delta}{\Gamma_q(\zeta_1)}\int_0^{\lambda}(\lambda-qs)^{(\zeta_1-1)}|f_1(s,u(s))|\ d_qs \\
&+\frac{\delta|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1)}\int_0^{\lambda}(\lambda-qs)^{(\vartheta_1-\vartheta_2+\zeta_1-1)}|f_2(s,u(s))|\ d_qs
\end{aligned}$$

$$
\begin{aligned}
& +\frac{\delta}{\varepsilon\Gamma_q(\vartheta_1+\zeta_1)}\int_0^{\lambda}(\lambda-qs)^{(\vartheta_1+\zeta_1-1)}|g(s,u(s))|\ d_qs \\
& +\frac{(1-\delta)}{\Gamma_q(\zeta_2)}\int_0^{\mu}(\mu-qs)^{(\zeta_2-1)}|f_1(s,u(s))|\ d_qs \\
& +\frac{(1-\delta)|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2)}\int_0^{\mu}(\mu-qs)^{(\vartheta_1-\vartheta_2+\zeta_2-1)}|f_2(s,u(s))|\ d_qs \\
& +\frac{(1-\delta)}{\varepsilon\Gamma_q(\vartheta_1+\zeta_2)}\int_0^{\mu}(\mu-qs)^{(\vartheta_1+\zeta_2-1)}|g(s,u(s))|\ d_qs \\
& +\frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^{1}(1-qs)^{(\vartheta_1-\vartheta_2-1)}|f_2(s,u(s))|\ d_qs \\
& +\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\int_0^{1}(1-qs)^{(\vartheta_1-1)}|g(s,u(s))|\ d_qs \\
& +\frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\sum_{i=1}^{n}a_i\int_0^{\xi_i}(\xi_i-qs)^{(\vartheta_1-\vartheta_2-1)}|f_2(s,u(s))|\ d_qs \\
& +\frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\sum_{i=1}^{n}a_i\int_0^{\xi_i}(\xi_i-qs)^{(\vartheta_1-1)}|g(s,u(s))|\ d_qs\Big]\Bigg\} \\
\leq\ & |f_1(x_2,u(x_2))-f_1(x_1,u(x_1))| \\
& +\frac{|\varepsilon-1|\|\varphi_2\|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^{x_1}\Big[(x_2-qs)^{(\vartheta_1-\vartheta_2-1)}-(x_1-qs)^{(\vartheta_1-\vartheta_2-1)}\Big]\ d_qs \\
& +\frac{|\varepsilon-1|\|\varphi_2\|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_{x_1}^{x_2}(x_2-qs)^{(\vartheta_1-\vartheta_2-1)}\ d_qs \\
& +\frac{\|\varphi_3\|}{\varepsilon\Gamma_q(\vartheta_1)}\int_0^{x_1}\Big[(x_2-qs)^{(\vartheta_1-1)}-(x_1-qs)^{(\vartheta_1-1)}\Big]\ d_qs \\
& +\frac{\|\varphi_3\|}{\varepsilon\Gamma_q(\vartheta_1)}\int_{x_1}^{x_2}(x_2-qs)^{(\vartheta_1-1)}\ d_qs \\
& +\frac{|x_2^{(\vartheta_1-1)}-x_1^{(\vartheta_1-1)}|}{|\Delta|}\Big[\frac{\|\varphi_1\|\ \delta\ \lambda^{\zeta_1}}{\Gamma_q(\zeta_1+1)}+\frac{\|\varphi_2\|\ \delta\ |\varepsilon-1|\ \lambda^{(\vartheta_1-\vartheta_2+\zeta_1)}}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1+1)} \\
& +\frac{\|\varphi_3\|\ \delta\ \lambda^{(\vartheta_1+\zeta_1)}}{\varepsilon\Gamma_q(\vartheta_1+\zeta_1+1)}+\frac{\|\varphi_1\|\ (1-\delta)\ \mu^{\zeta_2}}{\Gamma_q(\zeta_2+1)} \\
+\ & \frac{\|\varphi_2\|\ (1-\delta)|\varepsilon-1|\ \mu^{(\vartheta_1-\vartheta_2+\zeta_2)}}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2+1)} \\
& +\frac{\|\varphi_3\|\ (1-\delta)\ \mu^{(\vartheta_1+\zeta_2)}}{\varepsilon\Gamma_q(\vartheta_1+\zeta_2+1)}+\frac{\|\varphi_2\||\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+1)}+\frac{\|\varphi_3\|}{\varepsilon\Gamma_q(\vartheta_1+1)} \\
& +\frac{\|\varphi_2\||\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+1)}\sum_{i=1}^{n}a_i\ \xi_i^{\vartheta_1-\vartheta_2}+\frac{\|\varphi_3\|}{\varepsilon\Gamma_q(\vartheta_1+1)}\sum_{i=1}^{n}a_i\ \xi_i^{\vartheta_1}\Big]\to 0,
\end{aligned}
$$

independently of $u \in B_\rho$, when $x_1 \to x_2$. Therefore, $\mathcal{W}_2$ is equicontinuous. Thus, the operator $\mathcal{W}_2$ is relatively compact on B_ρ and hence we deduce by the Arzelá-Ascoli theorem that $\mathcal{W}_2$ is completely continuous. So $\mathcal{W}_2$ is a compact operator on B_ρ.

In the last step, it will be shown that the operator $\mathcal{W}_1$ is a contraction. For any $u, v \in B_\rho$ and $x \in [0,1]$, we have

$$\begin{aligned}
&\|\mathcal{W}_1 u - \mathcal{W}_1 v\| \\
\leq\ & \sup_{x\in[0,1]} \Bigg\{ \frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)} \int_0^x (x-qs)^{(\vartheta_1-\vartheta_2-1)} |u(s)-v(s)|\ d_q s \\
& + \frac{x^{\vartheta_1-1}}{|\Delta|} \Big[\frac{\delta|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1)} \int_0^\lambda (\lambda-qs)^{(\vartheta_1-\vartheta_2+\zeta_1-1)} |u(s)-v(s)|\ d_q s \\
& + \frac{(1-\delta)|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2)} \int_0^\mu (\mu-qs)^{(\vartheta_1-\vartheta_2+\zeta_2-1)} |u(s)-v(s)|\ d_q s \\
& + \frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)} \int_0^1 (1-qs)^{(\vartheta_1-\vartheta_2-1)} |u(s)-v(s)|\ d_q s \\
& + \frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)} \sum_{i=1}^n a_i \int_0^{\xi_i} (\xi_i-qs)^{(\vartheta_1-\vartheta_2-1)} |u(s)-v(s)|\ d_q s \Big] \Bigg\} \\
\leq\ & \upsilon_3 \|u-v\|,
\end{aligned}$$

which shows that $\mathcal{W}_1$ is a contraction as $\upsilon_3 < 1$. Thus all the hypotheses of Krasnoselskii's fixed point theorem [27] are verified. Therefore, there exists at least one solution to the problem (1)-(2) on $[0,1]$. □

Theorem 3.2. Assume that

(A_2) there exist continuous nondecreasing functions $\varpi_1, \varpi_2, \varpi_3 : [0,\infty) \to (0,\infty)$ and functions $\psi_1, \psi_2, \psi_3 \in C([0,1],\mathbb{R}^+)$ such that $|f_1(x,u)| \leq \psi_1(x)\varpi_1(|u|)$, $|f_2(x,u)| \leq \psi_2(x)\varpi_2(|u|)$ and $|g(x,u)| \leq \psi_3(x)\varpi_3(|u|)$ for each $(x,u) \in [0,1]\times\mathbb{R}$;

(A_3) there exists a constant $G > 0$ such that

$$\frac{(1-\upsilon_3)G}{\|\psi_1\|\varpi_1(G)\ \upsilon_1 + \|\psi_2\|\varpi_2(G)\ \upsilon_3 + \|\psi_3\|\varpi_3(G)\ \upsilon_2} > 1, \quad \upsilon_3 < 1,$$

where $\upsilon_1,\ \upsilon_2$ and υ_3 are defined by (10).

Then the fractional hybrid q-difference equation (1) with q-integral nonlocal boundary conditions (2) has at least one solution on $[0,1]$.

Proof. We verify the hypothesis of Leray-Schauder's nonlinear alternative [28] in several steps. Let us first show that the operator $\mathcal{W}$, defined by (9), *maps bounded sets (balls) into bounded sets in* E. For a positive number ω, let $B_\omega = \{u \in \mathcal{E} : \|u\| \leq \omega\}$ be a bounded ball in $\mathcal{E}$. Then, arguing as in the proof of the last result for $x \in [0,1]$, we have

$$\begin{aligned}
\|\mathcal{W}u\| \quad \leq \quad & \|\psi_1\|\varpi_1(\omega) \sup_{x\in[0,1]} \Bigg\{ 1 + \frac{x^{(\vartheta_1-1)}}{|\Delta|}\Big[\frac{\delta}{\Gamma_q(\zeta_1)}\int_0^\lambda (\lambda - qs)^{(\zeta_1-1)}\ d_qs \\
& + \frac{(1-\delta)}{\Gamma_q(\zeta_2)}\int_0^\mu (\mu - qs)^{(\zeta_2-1)}\ d_qs + 1 + \sum_{i=1}^n a_i\Big]\Bigg\} \\
& + \|\psi_3\|\varpi_3(\omega) \sup_{x\in[0,1]} \Bigg\{ \frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\int_0^x (x-qs)^{(\vartheta_1-1)}\ d_qs \\
& + \frac{x^{(\vartheta_1-1)}}{|\Delta|}\Big[\frac{\delta}{\varepsilon\Gamma_q(\vartheta_1+\zeta_1)}\int_0^\lambda (\lambda-qs)^{(\vartheta_1+\zeta_1-1)}\ d_qs \\
& + \frac{(1-\delta)}{\varepsilon\Gamma_q(\vartheta_1+\zeta_2)}\int_0^\mu (\mu-qs)^{(\vartheta_1+\zeta_2-1)}\ d_qs \\
& + \frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\int_0^1 (1-qs)^{(\vartheta_1-1)}\ d_qs \\
& + \frac{1}{\varepsilon\Gamma_q(\vartheta_1)}\sum_{i=1}^n a_i \int_0^{\xi_i} (\xi_i - qs)^{(\vartheta_1-1)}\ d_qs\Big]\Bigg\} \\
& + (\omega + \|\psi_2\|\varpi_2(\omega)) \sup_{x\in[0,1]} \Bigg\{ \frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^x (x-qs)^{(\vartheta_1-\vartheta_2-1)}\ d_qs \\
& + \frac{x^{(\vartheta_1-1)}}{|\Delta|}\Big[\frac{\delta|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1)}\int_0^\lambda (\lambda-qs)^{(\vartheta_1-\vartheta_2+\zeta_1-1)}\ d_qs \\
& + \frac{(1-\delta)|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2)}\int_0^\mu (\mu-qs)^{(\vartheta_1-\vartheta_2+\zeta_2-1)}\ d_qs \\
& + \frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\int_0^1 (1-qs)^{(\vartheta_1-\vartheta_2-1)}\ d_qs \\
& + \frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)}\sum_{i=1}^n a_i \int_0^{\xi_i} (\xi_i-qs)^{(\vartheta_1-\vartheta_2-1)}\ d_qs\Big]\Bigg\} \\
\leq \quad & \|\psi_1\|\varpi_1(\omega)\ \upsilon_1 + \|\psi_3\|\varpi_3(\omega)\ \upsilon_2 + \|\psi_2\|\varpi_2(\omega)\ \upsilon_3 + \omega\ \upsilon_3.
\end{aligned}$$

Next, it will be shown that $\mathcal{W}$ maps bounded sets into equicontinuous sets of $\mathcal{E}$. Taking $x_1, x_2 \in [0,1]$ with $x_1 < x_2$ and $u \in B_\omega$, we obtain

$$\begin{aligned}
& |\mathcal{W}u(x_2) - \mathcal{W}u(x_1)| \\
\leq \; & |f_1(x_2, u(x_2)) - f_1(x_1, u(x_1))| + \frac{(\omega + \|\psi_2\|\varpi_2(\omega))|\varepsilon - 1|}{\varepsilon \Gamma_q(\vartheta_1 - \vartheta_2)} \\
& \times \int_0^{x_1} \Big[(x_2 - qs)^{(\vartheta_1 - \vartheta_2 - 1)} - (x_1 - qs)^{(\vartheta_1 - \vartheta_2 - 1)} \Big] \, d_q s \\
& + \frac{(\omega + \|\psi_2\|\varpi_2(\omega))|\varepsilon - 1|}{\varepsilon \Gamma_q(\vartheta_1 - \vartheta_2)} \int_{x_1}^{x_2} \Big[(x_2 - qs)^{(\vartheta_1 - \vartheta_2 - 1)} \, d_q s \\
& + \frac{\|\psi_3\|\varpi_3(\omega)}{\varepsilon \Gamma_q(\vartheta_1)} \int_0^{x_1} \Big[(x_2 - qs)^{(\vartheta_1 - 1)} - (x_1 - qs)^{(\vartheta_1 - 1)} \Big] \, d_q s \\
& + \frac{\|\psi_3\|\varpi_3(\omega)}{\varepsilon \Gamma_q(\vartheta_1)} \int_{x_1}^{x_2} (x_2 - qs)^{(\vartheta_1 - 1)} \, d_q s \\
& + \frac{|x_2^{(\vartheta_1 - 1)} - x_1^{(\vartheta_1 - 1)}|}{|\Delta|} \Big[\frac{\|\psi_1\|\varpi_1(\omega) \; \delta \; \lambda^{\zeta_1}}{\Gamma_q(\zeta_1 + 1)} \\
& + \frac{\delta \; (\omega + \|\psi_2\|\varpi_2(\omega)) \; |\varepsilon - 1| \; \lambda^{\vartheta_1 - \vartheta_2 + \zeta_1}}{\varepsilon \Gamma_q(\vartheta_1 - \vartheta_2 + \zeta_1 + 1)} \\
& + \frac{\delta \; \|\psi_3\|\varpi_3(\omega) \; \lambda^{(\vartheta_1 + \zeta_1)}}{\varepsilon \Gamma_q(\vartheta_1 + \zeta_1 + 1)} + \frac{\|\psi_1\|\varpi_1(\omega) \; (1 - \delta) \; \mu^{\zeta_2}}{\Gamma_q(\zeta_2 + 1)} \\
& + \frac{(\omega + \|\psi_2\|\varpi_2(\omega)) \; (1 - \delta) \; |\varepsilon - 1| \; \mu^{(\vartheta_1 - \vartheta_2 + \zeta_2)}}{\varepsilon \Gamma_q(\vartheta_1 - \vartheta_2 + \zeta_2 + 1)} \\
& + \frac{\|\psi_3\|\varpi_3(\omega) \; (1 - \delta) \; \mu^{(\vartheta_1 + \zeta_2)}}{\varepsilon \Gamma_q(\vartheta_1 + \zeta_2 + 1)} + \frac{(\omega + \|\psi_2\|\varpi_2(\omega)) \; |\varepsilon - 1|}{\varepsilon \Gamma_q(\vartheta_1 - \vartheta_2 + 1)} + \frac{\|\psi_3\|\varpi_3(\omega)}{\varepsilon \Gamma_q(\vartheta_1 + 1)} \\
& + \frac{(\omega + \|\psi_2\|\varpi_2(\omega)) \; |\varepsilon - 1|}{\varepsilon \Gamma_q(\vartheta_1 - \vartheta_2 + 1)} \sum_{i=1}^{n} a_i \; \xi_i^{\vartheta_1 - \vartheta_2} + \frac{\|\psi_3\|\varpi_3(\omega)}{\varepsilon \Gamma_q(\vartheta_1 + 1)} \sum_{i=1}^{n} a_i \; \xi_i^{\vartheta_1} \Big] \to 0,
\end{aligned}$$

independently of $u \in B_\omega$, as $x_2 \to x_1$. Therefore, it follows by the Arzelá-Ascoli theorem that $\mathcal{W} : \mathcal{E} \to \mathcal{E}$ is completely continuous.

The conclusion of the Leray-Schauder nonlinear alternative [28] will be applicable once it is shown that the set of all solutions to the equation $u = \Theta \mathcal{W} u$ is bounded for $\Theta \in [0,1]$. For that, let u be a solution of the equation $u = \Theta \mathcal{W} u$ for $\Theta \in [0,1]$ and $x \in [0,1]$. Then, as in the first step, we obtain

$$\|u\| \leq \upsilon_1 \|\psi_1\|\varpi_1(\|u\|) + \upsilon_2 \|\psi_3\|\varpi_3(\|u\|) + \upsilon_3 \|\psi_2\|\varpi_2(\|u\|) + \upsilon_3 \|u\|,$$

which can be expressed as

$$\frac{(1 - \upsilon_3)\|u\|}{\|\psi_1\|\varpi_1(\|u\|) \; \upsilon_1 + \|\psi_3\|\varpi_3(\|u\|) \; \upsilon_2 + \|\psi_2\|\varpi_2(\|u\|) \; \upsilon_3} \leq 1.$$

By the condition (A_3), there exists a positive number G such that $\|u\| \neq G$. Define a set

$$V = \{u \in \mathcal{E} : \|u\| < G\}, \tag{14}$$

such that the operator $\mathcal{W} : \overline{V} \to \mathcal{E}$ is continuous and completely continuous ($\overline{V}$ is closure of V). By the given choice of V, it is not possible to find $u \in \partial V$ (∂V is boundary of V) satisfying $u = \Theta \mathcal{W} u$. Therefore, we deduce by the nonlinear alternative of Leray-Schauder type [28] that the operator $\mathcal{W}$ has a fixed point in $\overline{V}$, which corresponds to a solution of the problem (1)-(2) on $[0, 1]$. □

Finally, we establish the existence of a unique solution to the problem (1)-(2) by applying Banach's fixed point theorem [29].

Theorem 3.3. Assume that

(A_4) there exists positive constants M_1, M_2 such that, for each pair of elements $u, v \in \mathbb{R}$,

$$|f_1(x,u) - f_1(x,v)| \leq M_1|u - v|, \quad x \in [0,1];$$

$$|f_2(x,u) - f_2(x,v)| \leq M_2|u - v|, \quad x \in [0,1];$$

(A_5) there exists a positive constant M_3 such that, for each pair of elements $u, v \in \mathbb{R}$,

$$|g(x,u) - g(x,v)| \leq M_3|u - v|, \qquad x \in [0,1].$$

Then there exists a unique solution to the problem (1)-(2) on $[0, 1]$, provided that

$$\upsilon := M_2\ \upsilon_3 + M_1\ \upsilon_1 + M_3\ \upsilon_2 + \upsilon_3 < 1, \tag{15}$$

where υ_1, υ_2 and υ_3 are given in (10).

Proof. In the first step, we show that the operator $\mathcal{W} : \mathcal{E} \to \mathcal{E}$ (defined in (9)) satisfies $\mathcal{W} B_r \subset B_r$, where $B_r = \{u \in E : \|u\| \leq r\}$ with

$$r \geq (K_1\upsilon_1 + K_3\upsilon_2 + K_2\upsilon_3)/(1 - \upsilon), \ \sup_{x\in[0,1]} |f_1(x,0)| = K_1 < +\infty,$$

$\sup_{x\in[0,1]} |f_2(x,0)| = K_2 < +\infty$ and $\sup_{x\in[0,1]} |g(x,0)| = K_3 < +\infty$. Let u be an arbitrary element in B_r and $x \in [0, 1]$. Then, by the conditions (A_4) and (A_5), we have

$$|f_1(x,u(x))| \leq |f_1(x,u(x)) - f_1(x,0)| + |f_1(x,0)| \leq M_1\ r + K_1,$$

$$|f_2(x,u(x))| \leq |f_2(x,u(x)) - f_2(x,0)| + |f_2(x,0)| \leq M_2\ r + K_2,$$

and

$$|g(x,u(x))| \leq |g(x,u(x)) - g(x,0)| + |g(x,0)| \leq M_3\ r + K_3.$$

Consequently, for any $u \in B_r,\ x \in [0,1]$, we have

$$\begin{aligned}
\|\mathcal{W}u\| \leq\ & (M_1 r + K_1) \sup_{x\in[0,1]} \Bigg\{ 1 + \frac{x^{(\vartheta_1-1)}}{|\Delta|} \Big[\frac{\delta}{\Gamma_q(\zeta_1)} \int_0^\lambda (\lambda - qs)^{(\zeta_1-1)}\ d_q s \\
& + \frac{(1-\delta)}{\Gamma_q(\zeta_2)} \int_0^\mu (\mu - qs)^{(\zeta_2-1)}\ d_q s + 1 + \sum_{i=1}^n a_i \Big] \Bigg\} \\
& + (M_3 r + K_3) \sup_{x\in[0,1]} \Bigg\{ \frac{1}{\varepsilon\Gamma_q(\vartheta_1)} \int_0^x (x - qs)^{(\vartheta_1-1)}\ d_q s \\
& + \frac{x^{(\vartheta_1-1)}}{|\Delta|} \Big[\frac{\delta}{\varepsilon\Gamma_q(\vartheta_1+\zeta_1)} \int_0^\lambda (\lambda - qs)^{(\vartheta_1+\zeta_1-1)}\ d_q s \\
& + \frac{(1-\delta)}{\varepsilon\Gamma_q(\vartheta_1+\zeta_2)} \int_0^\mu (\mu - qs)^{(\vartheta_1+\zeta_2-1)}\ d_q s \\
& + \frac{1}{\varepsilon\Gamma_q(\vartheta_1)} \int_0^1 (1 - qs)^{(\vartheta_1-1)}\ d_q s \\
& + \frac{1}{\varepsilon\Gamma_q(\vartheta_1)} \sum_{i=1}^n a_i \int_0^{\xi_i} (\xi_i - qs)^{(\vartheta_1-1)}\ d_q s \Big] \Bigg\} \\
& + ((1+M_2) r + K_2) \sup_{x\in[0,1]} \Bigg\{ \frac{|\varepsilon - 1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)} \int_0^x (x - qs)^{(\vartheta_1-\vartheta_2-1)}\ d_q s \\
& + \frac{x^{(\vartheta_1-1)}}{|\Delta|} \Big[\frac{\delta|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1)} \int_0^\lambda (\lambda - qs)^{(\vartheta_1-\vartheta_2+\zeta_1-1)}\ d_q s \\
& + \frac{(1-\delta)|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2)} \int_0^\mu (\mu - qs)^{(\vartheta_1-\vartheta_2+\zeta_2-1)}\ d_q s \\
& + \frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)} \int_0^1 (1 - qs)^{(\vartheta_1-\vartheta_2-1)}\ d_q s \\
& + \frac{|\varepsilon-1|}{\varepsilon\Gamma_q(\vartheta_1-\vartheta_2)} \sum_{i=1}^n a_i \int_0^{\xi_i} (\xi_i - qs)^{(\vartheta_1-\vartheta_2-1)}\ d_q s \Big] \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
\leq\ & (M_1 r + K_1)\bigg[1 + \frac{\delta\ \lambda^{\zeta_1}}{|\Delta|\Gamma_q(\zeta_1+1)} + \frac{(1-\delta)\ \mu^{\zeta_2}}{|\Delta|\Gamma_q(\zeta_2+1)} + \frac{1}{|\Delta|} + \frac{1}{|\Delta|}\sum_{i=1}^{n} a_i\bigg] \\
& + (M_3 r + K_3)\bigg[\frac{1}{\varepsilon}\bigg(\frac{1}{\Gamma_q(\vartheta_1+1)} + \frac{\delta\ \lambda^{(\vartheta_1+\zeta_1)}}{|\Delta|\Gamma_q(\vartheta_1+\zeta_1+1)} \\
& + \frac{(1-\delta)\ \mu^{(\vartheta_1+\zeta_2)}}{|\Delta|\Gamma_q(\vartheta_1+\zeta_2+1)} + \frac{1}{|\Delta|\Gamma_q(\vartheta_1+1)} + \frac{1}{|\Delta|\Gamma_q(\vartheta_1+1)}\sum_{i=1}^{n} a_i\ \xi_i^{\vartheta_1}\bigg)\bigg] \\
& + ((1+M_2) r + K_2)\bigg[\frac{|\varepsilon-1|}{\varepsilon}\bigg(\frac{1}{\Gamma_q(\vartheta_1-\vartheta_2+1)} \\
& + \frac{\delta\ \lambda^{(\vartheta_1-\vartheta_2+\zeta_1)}}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1+1)} + \frac{(1-\delta)\ \mu^{(\vartheta_1-\vartheta_2+\zeta_2)}}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2+1)} \\
& + \frac{1}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+1)} + \frac{1}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+1)}\sum_{i=1}^{n} a_i\ \xi_i^{\vartheta_1-\vartheta_2}\bigg)\bigg],
\end{aligned}
$$

which, by (15) and definition of r, yields $\|\mathcal{W}u\| \leq r$. Therefore, $\mathcal{W}B_r \subset B_r$. Next we establish that the operator $\mathcal{W} : \mathcal{E} \to \mathcal{E}$ is a contraction. For any $x \in [0,1]$ and $u, v \in \mathbb{R}$, we get

$$
\begin{aligned}
& \|\mathcal{W}u - \mathcal{W}v\| \\
\leq\ & \Big[M_1\Big\{1 + \frac{\delta\ \lambda^{\zeta_1}}{|\Delta|\Gamma_q(\zeta_1+1)} + \frac{(1-\delta)\ \mu^{\zeta_2}}{|\Delta|\Gamma_q(\zeta_2+1)} + \frac{1}{|\Delta|} + \frac{1}{|\Delta|}\sum_{i=1}^{n} a_i\Big\} \\
& + M_3\Big\{\frac{1}{\varepsilon}\Big(\frac{1}{\Gamma_q(\vartheta_1+1)} + \frac{\delta\ \lambda^{(\vartheta_1+\zeta_1)}}{|\Delta|\Gamma_q(\vartheta_1+\zeta_1+1)} + \frac{(1-\delta)\ \mu^{(\vartheta_1+\zeta_2)}}{|\Delta|\Gamma_q(\vartheta_1+\zeta_2+1)} \\
& + \frac{1}{|\Delta|\Gamma_q(\vartheta_1+1)} + \frac{1}{|\Delta|\Gamma_q(\vartheta_1+1)}\sum_{i=1}^{n} a_i \xi_i^{\vartheta_1}\Big)\Big\} \\
& + \Big\{\frac{|\varepsilon-1|}{\varepsilon}\Big(\frac{1}{\Gamma_q(\vartheta_1-\vartheta_2+1)} + \frac{\delta\ \lambda^{(\vartheta_1-\vartheta_2+\zeta_1)}}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1+1)} \\
& + \frac{(1-\delta)\ \mu^{(\vartheta_1-\vartheta_2+\zeta_2)}}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2+1)} + \frac{1}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+1)} \\
& + \frac{1}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+1)}\sum_{i=1}^{n} a_i \xi_i^{\vartheta_1-\vartheta_2}\Big)\Big\} \\
& + M_2\Big\{\frac{|\varepsilon-1|}{\varepsilon}\Big(\frac{1}{\Gamma_q(\vartheta_1-\vartheta_2+1)} + \frac{\delta\ \lambda^{(\vartheta_1-\vartheta_2+\zeta_1)}}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+\zeta_1+1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(1-\delta)\ \mu^{(\vartheta_1-\vartheta_2+\zeta_2)}}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+\zeta_2+1)} \\
&+\frac{1}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+1)}+\frac{1}{|\Delta|\Gamma_q(\vartheta_1-\vartheta_2+1)}\sum_{i=1}^{n}a_i\xi_i^{\vartheta_1-\vartheta_2}\Big)\Big\}\Big]\|u-v\| \\
\leq\ & \upsilon\|u-v\|.
\end{aligned}
$$

where υ is given in (15). From the foregoing inequality together with the condition (15), we deduce that $\mathcal{W}$ is a contraction. Therefore, by the conclusion of Banach's fixed point theorem [29], the operator $\mathcal{W}$ has a unique fixed point, which is indeed a unique solution to the problem (1)-(2). □

4 Examples

Here, we present the illustrative examples for the results obtained in the last section.

Example 4.1. Consider the fractional hybrid q-difference equation

$$
\frac{8}{9}D_{2/5}^{2/3}\Big[u(x)-f_1(x,u)\Big]+\frac{1}{9}D_{2/5}^{1/3}\Big[u(x)-f_2(x,u)\Big]=g(x,u), \quad (16)
$$

subject to q-integral nonlocal boundary conditions

$$
\begin{aligned}
u(0)=0,\quad u(1)\ &=\ \frac{1}{5}\int_0^{1/4}\frac{(1/4-qs)^{(\frac{1}{9}-1)}}{\Gamma_q(\frac{1}{9})}u(s)d_qs \\
&+\frac{4}{5}\int_0^{1/2}\frac{(1/2-qs)^{(\frac{1}{8}-1)}}{\Gamma_q(\frac{1}{8})}u(s)d_qs+\sum_{i=1}^{3}a_iu(\xi_i). \quad (17)
\end{aligned}
$$

Here $\vartheta_1=\frac{2}{3}$, $q=\frac{2}{5}$, $\vartheta_2=\frac{1}{3}$, $\lambda=\frac{1}{4}$, $\mu=\frac{1}{2}$, $\varepsilon=\frac{8}{9}$, $\delta=\frac{1}{5}$, $\zeta_1=\frac{1}{9}$, $\zeta_2=\frac{1}{8}$, $n=3$, $a_1=\frac{1}{4}$, $a_2=\frac{1}{2}$, $a_3=\frac{3}{4}$, $\xi_1=\frac{5}{8}$, $\xi_2=\frac{6}{8}$, $\xi_3=\frac{7}{8}$, $x\in[0,1]$ and

$$
\begin{aligned}
&f_1(x,u)=\frac{1}{9}\Big(\frac{x^2|u(x)|}{1+|u(x)|}+4\Big),\ \ f_2(x,u)=\frac{1}{2}\Big(\frac{x|\sin u|}{1+|\sin u|}+\frac{1}{4}\Big), \\
&g(x,u)=\frac{1}{5}\Big(\frac{x|u(x)|}{1+|u(x)|}+7\Big).
\end{aligned}
$$

Observe that $|f_1(x,u)|\leq\dfrac{x^2+4}{9}=\varphi_1(x)$, $|f_2(x,u)|\leq\dfrac{4x+1}{8}=\varphi_2(x)$ and $|f_3(x,u)|\leq\dfrac{x+7}{5}=\varphi_3(x)$. Moreover, $\|\varphi_1\|=\sup_{x\in[0,1]}\varphi_1(x)\simeq 0.5556$,

$\|\varphi_2\| = \sup_{x\in[0,1]} \varphi_2(x) \simeq 0.625$ and $\|\varphi_3\| = \sup_{x\in[0,1]} \varphi_3(x) \simeq 1.6$. Using these values, it is found that $\upsilon_3 \simeq 0.329511 < 1$ (υ_3 is given in (10)). Clearly all the assumptions of Theorem 3.1 are satisfied. Therefore, the conclusion of Theorem 3 applies and hence the fractional q-integro-difference equation (16) with q-integral nonlocal boundary conditions (17) has at least one solution on $[0, 1]$.

Example 4.2. Consider the fractional hybrid q-difference equation

$$\begin{aligned} &\frac{8}{9} D_{2/5}^{2/3}\left[u(x) - \frac{|\cos u(x)|}{(3+x)^2(1+|\cos u(x)|)}\right] \\ &+ \quad \frac{1}{9} D_{2/5}^{1/3}\left[u(x) - \frac{|u(x)|}{(5+x)^2(1+|u(x)|)}\right] = \frac{|\sin u(x)|}{(x+4)^2(1+|\sin u(x)|}, \end{aligned} \tag{18}$$

$(x \in [0, 1])$ subject to q-integral nonlocal boundary conditions (17). With the given data, we find that $\upsilon_1 \simeq 2.63041$, $\upsilon_2 \simeq 2.78009$, $\upsilon_3 \simeq 0.329511$ and

$$\begin{aligned} |f_1(x,u)| &\le \frac{1}{(3+x)^2}\frac{|\cos u(x)|}{(1+|\cos u(x)|)} = \psi_1(x)\varpi_1(|u|), \\ |f_2(x,u)| &\le \frac{|u(x)|}{(5+x)^2(1+|u(x)|)} = \psi_2(x)\varpi_2(|u|), \\ |g(x,u)| &\le \frac{|\sin u(x)|}{(x+4)^2(1+|\sin u(x)|} = \psi_3(x)\varpi_3(|u|), \end{aligned}$$

with $\psi_1(x) = \dfrac{1}{(3+x)^2}$, $\psi_2(x) = \dfrac{1}{(5+x)^2}$, $\psi_3(x) = \dfrac{1}{(4+x)^2}$, $\varpi_1(||u||) = \varpi_2(||u||) = \varpi_3(||u||) = ||u||$. By the condition (H_5), we find that $G > 5.22778$. Thus the hypotheses of Theorem 3.2 are satisfied. Therefore, the conclusion of Theorem 3.2 implies that the hybrid q-difference equation (18) with nonlocal q-integral boundary conditions (17) has at least one solution on $[0, 1]$.

Example 4.3 Let us consider the fractional hybrid q-difference equation

$$\frac{8}{9} D_{2/5}^{2/3}\Big[u(x) - f_1(x,u(x))\Big] + \frac{1}{9} D_{2/5}^{1/3}\Big[u(x) - f_2(x,u(x))\Big] = g(x,u(x)), \tag{19}$$

$(x \in [0, 1])$ equipped with q-integral nonlocal boundary conditions (17). Here

$$\begin{aligned} &f_1(x,u(x)) = 0.09(1+\sin(u(x))), \ f_2(x,u(x)) = \frac{8|u(x)|}{100\ (1+|u(x)|)}, \\ &g(x,u(x)) = 0.07\tan^{-1} u(x). \end{aligned}$$

Clearly $f_1(x, u(x)$ and $f_2(x, u(x)$ satisfy the condition (A_4) with $M_1 = 0.09$ and $M_2 = 0.08$ respectively, while (A_5) is satisfied by $g(x, u(x))$ with $M_3 = 0.07$. With the given data, it is found that $\upsilon_1 \simeq 2.63041, \upsilon_2 \simeq 2.78009, \upsilon_3 \simeq 0.329511$ and $\upsilon \simeq 0.787215 < 1$, that is, the condition (15) is satisfied. As all the assumptions of Theorem 3.3 are satisfied, therefore the equation (19) with q-integral nonlocal boundary conditions (17) has a unique solution on $[0, 1]$ by the conclusion of Theorem 3.3.

5 Conclusions

We have discussed the existence and uniqueness of solutions to a fractional q-difference equation involving dual hybrid terms and equipped with nonlocal multipoint and Riemann-Liouville fractional q-integral boundary conditions. The desired results are derived with the aid of standard fixed point theorems. Our results are novel in the given configuration and enhance the literature on boundary value problems of fractional q-difference equations. As a special case, for $\varepsilon = \delta = 1$, our results correspond to the following problem:

$$D_q^{\vartheta_1}[u(x) - f_1(x, u(x))] = g(x, u(x)), \qquad 0 < x < 1,$$

$$u(0) = 0, \quad u(1) = \int_0^{\lambda} \frac{(\lambda - qs)^{(\zeta_1 - 1)}}{\Gamma_q(\zeta_1)} u(s) d_q s + \sum_{i=1}^{n} a_i u(\xi_i), \quad \zeta_1 > 0,$$

which are indeed new. By taking $a_i = 0$ for all $i = 1, \ldots, n$, our results become the ones for the problem:

$$\varepsilon D_q^{\vartheta_1}[u(x) - f_1(x, u(x))] + (1-\varepsilon) D_q^{\vartheta_2}[u(x) - f_2(x, u(x))] = g(x, u(x)), \quad 0 < x < 1,$$

$$u(0) = 0, \;\; u(1) = \delta \int_0^{\lambda} \frac{(\lambda - qs)^{(\zeta_1 - 1)}}{\Gamma_q(\zeta_1)} u(s) d_q s + (1-\delta) \int_0^{\mu} \frac{(\mu - qs)^{(\zeta_2 - 1)}}{\Gamma_q(\zeta_2)} u(s) d_q s.$$

Conflict of Interest

All authors declare no conflicts of interest in this paper.

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Boshra ALHARBI
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group,
Department of Mathematics, Faculty of Science,
King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia.
Email: balharbi0383@stu.kau.edu.sa

Ahmed ALSAEDI
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group,
Department of Mathematics, Faculty of Science,
King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia.
Email: aalsaedi@hotmail.com

Ravi P. AGARWAL
Department of Mathematics,
Texas A&M University,
700 University Blvd., Kingsville,
Texas 78363-8202, USA
Email: ravi.agarwal@tamuk.edu

Bashir AHMAD
Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group,
Department of Mathematics, Faculty of Science,
King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia.
Email: bashirahmad_qau@yahoo.com