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Linear Skew Cyclic Codes over $\mathbb{F}_q$S

Tulay Yildirim

Abstract

In this study, we focus on skew cyclic codes over the family of rings $\mathbb{F}_q$S where q is a power of a prime number and $\mathrm{S} = \mathbb{F}_q + v\mathbb{F}_q$ with $v^2 = v$. Structural properties of these codes are studied in detail. Obtained results lead us to characterize $\mathbb{F}_q$S-linear skew cyclic codes. All the minimal spanning sets and generators of these codes are presented. Furthermore, some good $\mathbb{F}_q$-linear codes are obtained as images of the $\mathbb{F}_q$S-linear skew cyclic codes under the Gray mapping.

1 Introduction

Studies on coding theory were inspired by a paper of Shannon [2], then algebraic structure of linear codes has been interested by many researchers. In the beginning, the problems were based on linear codes over binary fields, later, as a natural extension of the binary field, codes over finite fields were also considered. In 1994, Hammons et al. obtained some binary non-linear codes as images of linear codes over the ring $\mathbb{Z}_4$ [1]. This paper brought a new perspective to mathematical community studying in coding theory. Although, initially many researchers studied on commutative and finite chain rings, recently it has become more trendy to consider more diverse types of rings.

One of the important class of linear codes is known as cyclic codes due to their theoretical and applied properties. A cyclic code viewed as an ideal in a

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particular quotient ring obtained from a polynomial ring. In 2007, Boucher et al. presented cyclic codes over non-commutative polynomial rings [3]. They analyzed cyclic codes by using the skew polynomial ring in the set $\mathbb{F}_q[x;\theta]$, where θ is an automorphism of finite field $\mathbb{F}_q$ with q elements. This new idea of codes is defined as skew cyclic codes over $\mathbb{F}_q$. They also showed some notable examples of these codes having larger minimum distance than the well-known codes. Later on, these codes examined on various finite rings [12], [7]. Abualrub et al. generalized the skew cyclic codes to skew quasi-cyclic codes in [12]. They considered skew cyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2$ with $u^2 = 0$, and constructed some optimal self-dual codes over the this ring in [13].

In the beginning, codes over skew polynomial ring have been studied with certain restrictions on their length, but Siap et. al examined the skew cyclic codes for arbitrary length [6]. In the following years, Jitman and his colleagues studied on skew constacyclic codes over the finite chain ring $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$ where $u^2 = 0$ [17]. By the decomposition theorem, Gursoy et al. studied the structural properties of skew cyclic codes over $\mathbb{F}_q + vF_q$ with $v^2 = v$ [5]. Li et al. studied linear skew constacyclic codes over $\mathbb{F}_q\mathbb{R}$ where $\mathbb{R} = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + uv\mathbb{F}_q$ with $u^2 = u, v^2 = v, uv = vu$ [9]. Bhaintwal introduced a necessary and sufficient condition for the skew cyclic codes over the Galois rings to be free, and determined a distance bound for free skew cyclic codes [14]. Skew cyclic code studies over mixed alphabets have attracted the attention of researchers for last decade. For instance, Benbelkacem et al. considered skew cyclic codes over the ring $\mathbb{F}_4\mathbb{R}$ and constructed a relationship between skew cyclic codes over $\mathbb{F}_4\mathbb{R}$ and DNA codons [16]. In recently, Juan et al. introduced $\mathbb{F}_q\mathbb{R}$-linear skew cyclic codes [8]. For more information, we can refer to see [4, 10, 11, 19, 20].

Even if skew cyclic codes are considered over the ring $\mathbb{Z}_4$, these codes depend on a ring automorphism, and $\mathbb{Z}_4$ has only a trivial automorphism. Similarly, the ring $\mathbb{F}_2 + u\mathbb{F}_2$ with $u^2 = 0$ has identity automorphism. Therefore, we can conclude that there are no skew cyclic codes over the rings $\mathbb{Z}_4$ and $\mathbb{F}_2 + u\mathbb{F}_2$ that are different from the cyclic codes. However, we can consider skew cyclic codes over the ring $\mathrm{S} = \mathbb{F}_q + v\mathbb{F}_q$ where q is power of a prime number with $v^2 = v$ since S has a non-trivial ring automorphism. The aim of this paper is to introduce and study skew cyclic codes over the family of rings $\mathbb{F}_q\mathrm{S}$ where q is a prime power and $\mathrm{S} = \mathbb{F}_q + v\mathbb{F}_q$ with $v^2 = v$. The ring $\mathbb{F}_q\mathrm{S}$ is a finite semi-local and not a finite chain ring.

The paper is organized as follows. Section 2 starts with some basic properties of the ring $\mathrm{S} = \mathbb{F}_q + v\mathbb{F}_q$ and gives a brief description of linear skew cyclic codes over S. Then, we introduce the definition and algebraic structure of the

$\mathbb{F}_q$S-linear skew cyclic codes. Moreover, we determine the necessary condition for the dual codes to be $\mathbb{F}_q$S-linear skew cyclic codes. In section 3, linear skew cyclic codes over the ring $\mathbb{F}_q$S are further generalized to skew generalized quasi-cyclic codes. Also, we introduce a Gray map from $\mathbb{F}_q$S to $\mathbb{F}_q$ and present q-ary images of $\mathbb{F}_q$S-linear skew cyclic codes. In section 4, we describe the generator polynomials of linear skew cyclic codes over the ring $\mathbb{F}_q$S and identify their minimal spanning sets and sizes. Furthermore, we obtain some optimal q-ary codes.

2 Preliminaries and Definitions

In this study, we give our attention to the skew cyclic codes over combined alphabets. Especially, we consider the structure of linear codes over the ring $\mathbb{F}_q$S, where $\mathbb{F}_q$ is a finite filed with q is a power of prime number and $\mathrm{S} = \mathbb{F}_q + v\mathbb{F}_q = \{\xi + v\mu | \xi, \mu \in \mathbb{F}_q,\ with\ v^2 = v\}$ is a commutative ring. Its well known that the ring S is a semi-local ring, with two maximal ideals $< v >= \{\xi v | \xi \in \mathbb{F}_q\}$ and $< 1 - v >= \{\mu - v\mu | \mu \in \mathbb{F}_q\}$ making $\mathrm{S}/ < v >$ and $\mathrm{S}/ < 1-v >$ isomorphic to $\mathbb{F}_q$. The Chinese Remainder Theorem then implies that $\mathrm{S} =< v > \oplus < 1 - v >$ [5]. More concretely, S can be uniquely expressed as $\xi + v\mu = (\xi + \mu)v + \xi(1 - v)$ for all $\xi, \mu \in \mathbb{F}_q$.
Let A and B be codes over S, then $A \oplus B$ and $A \otimes B$ define as $\{\alpha + \beta | \alpha \in A, \beta \in B\}$ and $\{(\alpha, \beta) | \alpha \in A, \beta \in B\}$, respectively. Define $C_1 = \{p + r \in \mathbb{F}_q^n | (p + r)v + p(1 - v) \in C,\ for\ some\ p, r \in \mathbb{F}_q^n\}$ and $C_2 = \{p \in \mathbb{F}_q^n | (p + r)v + p(1 - v) \in C,\ for\ some\ p, r \in \mathbb{F}_q^n\}$. It is clear that C_1 and C_2 are linear codes over $\mathbb{F}_q$. Any linear code C over S can be uniquely expressed as $C = vC_1 \oplus (1 - v)C_2$ [18].

Definition 2.1. (1) The Gray image of S is defined as

$$\Omega \colon \mathrm{S} \to \mathbb{F}_q^2$$
$$\xi + v\mu \mapsto (\xi, \xi + \mu),$$

where $\xi, \mu \in \mathbb{F}_q$.

(2) The automorphism θ over S is defined as

$$\theta \colon \mathrm{S} \to \mathrm{S}$$
$$\xi + v\mu \mapsto \xi + (1 - v)\mu,$$

(3) A subset C of S^n is said to be an S-linear skew cyclic code of length n if

(i) C is an S-submodule of S^n,

(ii) C is closed under the T_θ-cyclic shift, i.e. for $c = (c_0, c_1, \cdots, c_{n-1}) \in C$,

$$T_\theta(c) = (\theta(c_{n-1}), \theta(c_0), \cdots, \theta(c_{n-2})) \in C.$$

The following theories can be obtained by corresponding theorems in [5], with soft modifications on the automorphism θ.

Theorem 2.2. *Let $C = vC_1 \oplus (1-v)C_2$ be an* S*-linear code of length n. Then C is a skew cyclic code if and only if C_1 and C_2 are skew cyclic codes over $\mathbb{F}_q$, with respect to the automorphism θ.*

Theorem 2.3. *If $C = vC_1 \oplus (1-v)C_2$ be a skew cyclic code of length n over* S*, then $C =< vg_1(x) + (1-v)g_2(x) >$ where $g_1(x), g_2(x)$ are generator polynomials of C_1 and C_2, respectively.*

Definition 2.4. (1) Define

$$\mathbb{F}_q\mathrm{S} = \{(\partial, \xi + v\mu) | \partial \in \mathbb{F}_q, (\xi + v\mu) \in \mathrm{S}\}.$$

Let C be a skew cyclic code over $\mathbb{F}_q$S and α (resp. β) be the set of $\mathbb{F}_q$ (resp. S) coordinate positions. Any codeword $c \in C$ has the form

$$c = (\partial_0, \partial_1, \cdots, \partial_{\alpha-1}, e_0, e_1, \cdots, e_{\beta-1}) \in \mathbb{F}_q^\alpha \mathrm{S}^\beta,$$

where $e_i = \xi_i + v\mu_i \in \mathrm{S}^\beta$ for all $i = 0, 1, \cdots, n-1$. Throughout the study, we assume that α and β are odd positive integers.

(2) The ring homomorphism map is defined as

$$\delta \colon \mathrm{S} \to \mathbb{F}_q$$
$$\xi + v\mu \mapsto \xi.$$

For any $r \in \mathrm{S}$, define a scalar multiplication $\star$ by $r \star (\partial, \xi + v\mu) = (\delta(r)\partial, r(\xi + v\mu))$ where $\partial \in \mathbb{F}_q$ and $\xi + v\mu \in \mathrm{S}$. It naturally extends to $\mathbb{F}_q^\alpha \mathrm{S}^\beta$ as follows:

$$r \star x = (\delta(r)\partial_0, \delta(r)\partial_1, \cdots, \delta(r)\partial_{\alpha-1}, re_0, re_1, \cdots re_{\beta-1}), \qquad (1)$$

where $x = (\partial_0, \partial_1, \cdots, \partial_{\alpha-1}, e_0, e_1, \cdots e_{\beta-1}) \in \mathbb{F}_q^\alpha \mathrm{S}^\beta$ for $\alpha, \beta \in \mathbb{N}$.

A nonempty subset C of $\mathbb{F}_q^\alpha \mathrm{S}^\beta$ is called an $\mathbb{F}_q$S-linear code if C is an S-submodule of $\mathbb{F}_q^\alpha \mathrm{S}^\beta$.

Lemma 2.5. *The set $\mathbb{F}_q^\alpha \mathrm{S}^\beta$ is an* S*-module with respect to the addition and scalar multiplication in Equation (1).*

Definition 2.6. [3] Let ψ be an automorphism of the finite field $\mathbb{F}_q$. For any two elements $ax^m, bx^n \in \mathbb{F}_q[x, \psi]$, polynomial set $\mathbb{F}_q[x, \psi]$ is defined as

$$(ax^m)(bx^n) = a\psi^m(b)x^{m+n}.$$

In polynomial representation, a linear code of length n over $\mathbb{F}_q$ is a skew cyclic code if and only if it is a left $\mathbb{F}_q[x;\psi]$-submodule of $\mathbb{F}_q[x;\psi]/(x^n-1)$. Also if C is a left submodule of $\mathbb{F}_q[x;\psi]/(x^n-1)$, then C is generated by a monic polynomial $g(x)$ which is a right divisor of x^n-1 in $\mathbb{F}_q[x;\psi]$ [6] .
For the automorphism θ of S and any two elements ax^m and bx^n in $\mathrm{S}[x;\theta]$, polynomial set $\mathrm{S}[x;\theta]$ is defined as

$$(ax^m)(bx^n) = a\theta^m(b)x^{m+n}.$$

So, $\mathrm{S}[x;\theta]$ is a skew polynomial ring, where addition is the usual polynomial addition and multiplication is defined above.

An element $c = (\partial_0, \partial_1, \cdots, \partial_{\alpha-1}, e_0, e_1, \cdots, e_{\beta-1}) \in \mathbb{F}_q^\alpha \mathrm{S}^\beta$ can be identified with a module element consisting of two polynomials $c(x) = (\partial(x), e(x)) \in \mathbb{F}_q[x;\psi]/(x^\alpha-1) \times \mathrm{S}[x;\theta]/(x^\beta-1)$, where $\partial(x) = \partial_0 + \partial_1 x + \cdots + \partial_{\alpha-1}x^{\alpha-1}$ and $e(x) = e_0 + e_1 x + \cdots + e_{\beta-1}x^{\beta-1}$. This identification gives a one-to-one correspondence between elements in $\mathbb{F}_q^\alpha \times \mathrm{S}^\beta$ and elements $\mathbb{F}_q[x;\psi]/(x^\alpha-1) \times \mathrm{S}[x;\theta]/(x^\beta-1)$.

Let $f(x) = f_0 + f_1 x + \cdots + f_s x^s \in \mathrm{S}[x;\theta]$, $(g(x), h(x)) \in \mathbb{F}_q[x;\psi]/(x^\alpha-1) \times \mathrm{S}[x;\theta]/(x^\beta-1)$, multiplication operation on $\mathbb{F}_q$S is defined as

$$f(x) \star (g(x), h(x)) = (\delta(f(x)).g(x), f(x) * h(x)),$$

where $\delta(f(x)) = \delta(f_0) + \delta(f_1)x + \cdots + \delta(f_s)x^s \in \mathbb{F}_q[x;\psi]/(x^\alpha-1)$. This multiplication is well-defined on $\mathbb{F}_q[x;\psi]/(x^\alpha-1) \times \mathrm{S}[x;\theta]/(x^\beta-1)$. Moreover, $\delta(f(x)).g(x)$ and $f(x)*h(x)$ are defined in $\mathbb{F}_q[x;\psi]/(x^\alpha-1)$ and $\mathrm{S}[x;\theta]/(x^\beta-1)$, respectively. For shortly,

$$\mathrm{S}_{\alpha,\beta} := \mathbb{F}_q[x;\psi]/(x^\alpha-1) \times \mathrm{S}[x;\theta]/(x^\beta-1).$$

Lemma 2.7. $\mathrm{S}_{\alpha,\beta}$ *is a left* $\mathrm{S}[x;\theta]$*-module under the* $\star$ *multiplication.*

Definition 2.8. Let ψ and θ be the automorphisms of $\mathbb{F}_q$ and S, respectively. A code C is an $\mathbb{F}_q$S-linear skew cyclic code of length $\alpha+\beta$ if

(i) C is an S-submodule of $\mathbb{F}_q^\alpha \mathrm{S}^\beta$,

(ii) C is closed under the $T_{\psi\theta}$-cyclic shift, i.e.,

$$T_{\psi\theta}(\partial_i, \xi_i + v\mu_i) = (\psi(\partial_{\alpha-1}), \psi(\partial_0), \cdots, \psi(\partial_{\alpha-2}), \theta(\xi_{\beta-1} + v\mu_{\beta-1}), \theta(\xi_0 + v\mu_0), \cdots, \theta(\xi_{\beta-2} + v\mu_{\beta-2})),$$

where $(\partial_i, \xi_i + v\mu_i) \in C$ with $\partial_i = (\partial_0, \partial_1, \cdots, \partial_{\alpha-1}) \in \mathbb{F}_q^\alpha$ and $(\xi_i + v\mu_i) = (\xi_0 + v\mu_0, \xi_1 + v\mu_1, \cdots, \xi_{\beta-1} + v\mu_{\beta-1}) \in \mathrm{S}^\beta$.

Lemma 2.9. *The code C is an $\mathbb{F}_q$S-linear skew cyclic code of length $\alpha+\beta$ if and only if C is a left $\mathrm{S}[x;\theta]$-submodule of $\mathrm{S}_{\alpha,\beta}$.*

Proof. Let $c = (\partial_0, \partial_1, \cdots, \partial_{\alpha-1}, \xi_0 + v\mu_0, \xi_1 + v\mu_1, \cdots, \xi_{\beta-1} + v\mu_{\beta-1}) \in C$ and $c(x) = (g(x), h(x))$ be a codeword of $\mathbb{F}_q$S-linear skew cyclic code C, where $(g(x), h(x)) \in \mathrm{S}_{\alpha,\beta}$. If C is a $\mathbb{F}_q$S-linear skew cyclic code, then

$$(\psi(\partial_{\alpha-1}), \psi(\partial_0), \cdots, \psi(\partial_{\alpha-2}), \theta(\xi_{\beta-1} + v\mu_{\beta-1}), \theta(\xi_0 + v\mu_0), \cdots, \theta(\xi_{\beta-2} + v\mu_{\beta-2})) \in C,$$

and its polynomial representation is

$$x \star c(x) = (\psi(\partial_{\alpha-1}) + \psi(\partial_0)x + \cdots + \psi(\partial_{\alpha-2})x^{\alpha-1}, \theta(\xi_{\beta-1} + v\mu_{\beta-1}) + \theta(\xi_0 + v\mu_0)x + \cdots + \theta(\xi_{\beta-2} + v\mu_{\beta-2})x^{\beta-1}) \in C.$$

Furthermore, $x^2 \star c(x) \in C$, $x^3 \star c(x) \in C$, so on. Since C is a linear code, one gets $f(x) \star c(x) \in C$, for any $f(x) \in \mathrm{S}[x;\theta]$. Hence, C is a left $\mathrm{S}[x;\theta]$-submodule of $\mathrm{S}_{\alpha,\beta}$.
On the other hand, let C be a left $\mathrm{S}[x;\theta]$-submodule of the left $\mathrm{S}[x;\theta]$-module of $\mathrm{S}_{\alpha,\beta}$. For any $c(x) \in C$, one gets $x^i \star c(x) \in C$, for each $i \in \mathbb{N}$. Therefore, C is an $\mathbb{F}_q$S-linear skew cyclic code. □

The Euclidean inner product on the ring $\mathbb{F}_q^\alpha \mathrm{S}^\beta$ is defined as

$$< x, y >= v \sum_{i=0}^{\alpha-1} x_i y_i + \sum_{j=0}^{\beta-1} x'_j y'_j,$$

where
$x = (x_0, x_1, \cdots, x_{\alpha-1}, x'_0, x'_1 \cdots, x'_{\beta-1})$, $y = (y_0, y_1, \cdots, y_{\alpha-1}, y'_0, y'_1, \cdots, y'_{\beta-1})$ in $\mathbb{F}_q^\alpha \mathrm{S}^\beta$.

The dual code of C, denoted by $C^\perp$, is also $\mathbb{F}_q$S linear code and defined by

$$C^\perp = \{y \in \mathbb{F}_q^\alpha \mathrm{S}^\beta | < x, y >= 0 \; for \; all \; x \in C\}.$$

Theorem 2.10. *If C is an $\mathbb{F}_q$S-linear skew code of length $n = \alpha+\beta$ with $\beta \in 2\mathbb{Z}$, then $C^\perp$ is an $\mathbb{F}_q$S-linear skew cyclic code of length n.*

Proof. Let $x = (\partial_0, \partial_1, \cdots, \partial_{\alpha-1}, \xi_0 + v\mu_0, \xi_1 + v\mu_1, \cdots, \xi_{\beta-1} + v\mu_{\beta-1}) \in C$ and $y = (e_0, e_1, \cdots, e_{\alpha-1}, f_0 + vg_0, f_1 + vg_1, \cdots, f_{\beta-1} + vg_{\beta-1}) \in C^\perp$. Our aim is to prove that $T_{\psi\theta}(y) \in C^{\perp}$, so it is enough to show that $< T_{\psi\theta}(y), x >= 0$. Let $\gamma = lcm(\alpha, \beta)$ be an even integer since $\beta \in 2\mathbb{Z}$. By the hypothesis, one gets $T_{\psi\theta}^{\gamma}(x) = x$ and $T_{\psi\theta}^{\gamma-1}(x) \in C$. Thus, $< y, T_{\psi\theta}^{\gamma-1}(x) >= 0$ for $T_{\psi\theta}^{\gamma-1}(x) = (\psi(\partial_1), \cdots, \psi(\partial_{\alpha-1}), \psi(\partial_0), \theta(\xi_1+v\mu_1), \cdots, \theta(\xi_{\beta-1}+v\mu_{\beta-1}), \theta(\xi_0+v\mu_0))$. According to the definition of the Euclidean inner product,

$$v\sum_{i=0}^{\alpha-1} e_i\psi(\partial_{i+1}) + \sum_{j=0}^{\beta-1}(f_j + vg_j)\theta(\xi_{j+1} + v\mu_{j+1}) = 0,$$

where first and second sums are taken module α and β, respectively. So we have

$$\begin{aligned}
&e_0\psi(\partial_1) + \cdots + e_{\alpha-2}\psi(\partial_{\alpha-1}) + e_{\alpha-1}\psi(\partial_0) = 0,\\
&(f_0 + vg_0)\theta(\xi_1 + v\mu_1) + \cdots + (f_{\beta-2} + vg_{\beta-2})\theta(\xi_{\beta-1} + v\mu_{\beta-1})\\
&\ + (f_{\beta-1} + vg_{\beta-1})\theta(\xi_0 + v\mu_0) = 0.
\end{aligned}$$

Applying ψ and θ to the above equalities, respectively;

$$\begin{aligned}
&\psi(e_0)(\partial_1) + \cdots + \psi(e_{\alpha-2})(\partial_{\alpha-1}) + \psi(e_{\alpha-1})(\partial_0) = 0,\\
&\theta(f_0 + vg_0)(\xi_1 + v\mu_1) + \cdots + \theta(f_{\beta-2} + vg_{\beta-2})(\xi_{\beta-1} + v\mu_{\beta-1})\\
&\ + \theta(f_{\beta-1} + vg_{\beta-1})(\xi_0 + v\mu_0) = 0.
\end{aligned}$$

This exactly yields that $< T_{\psi\theta}(y), x >= 0$ for

$$\begin{aligned}
T_{\psi\theta}(y) =&(\psi(e_{\alpha-1}), \psi(e_0), \cdots, \psi(e_{\alpha-2}), \theta(f_{\beta-1} + vg_{\beta-1}), \theta(f_0 + vg_0),\\
&\cdots, \theta(f_{\beta-2} + vg_{\beta-2})).
\end{aligned}$$

□

3 The Gray Map

In this chapter, we study on Gray map over the ring $\mathbb{F}_q$S. The Gray mapping on S is defined in Definition 2.1 (1). This map can be extended to S^n in a natural way, for $c = (c_0, c_1, \cdots, c_{n-1}) \in \mathrm{S}^n$ where $c_i = \xi_i + v\mu_i$, $i = 0, 1 \cdots, n-1$.

$$\begin{aligned}
\Omega\colon \mathrm{S}^n &\to \mathbb{F}_q^{2n}\\
c &\mapsto (\xi(c), \xi(c) + \mu(c)),
\end{aligned}$$

where $\xi(c) = (\xi_0, \xi_1, \cdots, \xi_{n-1})$ and $\mu(c) = (\mu_0, \mu_1, \cdots, \mu_{n-1})$ and they are unique. For $\xi + v\mu \in \mathrm{S}$, $\xi, \mu \in \mathbb{F}_q$, the Lee weight of $(\xi + v\mu)$ can be defined as

$$w_L(\xi + v\mu) = w_H(\xi) + w_H(\xi + \mu),$$

where w_H represents the Hamming weight of a codeword. The Lee weight of a codeword is the rational sum of the Lee weights of its components.

Definition 3.1. The Gray map over $\mathbb{F}_q$S is defined as

$$\Phi\colon \mathbb{F}_q\mathrm{S} \to \mathbb{F}_q^3$$
$$(\partial, \xi + v\mu) \mapsto (\partial, \Omega(\xi + v\mu)) = (\partial, \xi, \xi + \mu),$$

and it can be extended componentwise from $\mathbb{F}_q^{\alpha}\mathrm{S}^{\beta}$ to $\mathbb{F}_q^{\alpha+2\beta}$ as

$$\begin{aligned}\Phi(&\partial_0,\partial_1, \cdots, \partial_{\alpha-1}, \xi_0 + v\mu_0, \xi_1 + v\mu_1, \cdots, \xi_{\beta-1} + v\mu_{\beta-1})\\ &= (\partial_0, \partial_1, \cdots, \partial_{\alpha-1}, \Omega(\xi_0 + v\mu_0), \Omega(\xi_1 + v\mu_1), \cdots, \Omega(\xi_{\beta-1} + v\mu_{\beta-1})),\end{aligned}$$

for all $(\partial_0, \partial_1, \cdots, \partial_{\alpha-1}) \in \mathbb{F}_q^{\alpha}$ and $(\xi_0 + v\mu_0, \xi_1 + v\mu_1, \cdots, \xi_{\beta-1} + v\mu_{\beta-1}) \in \mathrm{S}^{\beta}$. If C is an $\mathbb{F}_q$S-linear code, then $\Phi(C)$ is also $\mathbb{F}_q$-linear. The Lee weight of $(\partial, \xi + v\mu) \in \mathbb{F}_q^{\alpha}\mathrm{S}^{\beta}$ is defined as

$$w_L((\partial, \xi + v\mu)) = w_H(\partial) + w_H(\xi) + w_H(\xi + \mu).$$

The Lee distance between two codewords r and s in $\mathbb{F}_q^{\alpha}\mathrm{S}^{\beta}$ is defined as

$$d_L(r, s) = w_L(r - s).$$

Proposition 3.2. *Φ is an $\mathbb{F}_q$ linear distance preserving map from $\mathbb{F}_q^{\alpha}\mathrm{S}^{\beta}$ (Lee distance) to $\mathbb{F}_q^{\alpha+2\beta}$ (Hamming distance).*

Proof. Let
$c_1 = (\partial_0, \partial_1, \cdots, \partial_{\alpha-1}, e_0, e_1, \cdots, e_{\beta-1})$ and $c_2 = (\partial'_0, \partial'_1, \cdots, \partial'_{\alpha-1}, e_0,' e'_1, \cdots, e'_{\beta-1})$ be two codewords in $\mathbb{F}_q^{\alpha}\mathrm{S}^{\beta}$, where $e_i = \xi_i + v\mu_i$ and $e'_i = \xi'_i + v\mu'_i$ are in S^{β} for $i = 0, 1, \cdots, \beta - 1$. So,

$$\begin{aligned}\Phi(c_1 + c_2) = (&\partial_0 + \partial'_0, \partial_1 + \partial'_1, \cdots, \partial'_{\alpha-1} + \partial'_{\alpha-1}, \xi_0 + \xi'_0, \xi_1 + \xi'_1, \cdots, \xi_{\beta-1} + \xi'_{\beta-1},\\ &\xi_0 + \xi'_0 + \mu_0 + \mu'_0, \xi_1 + \xi'_1 + \mu_1 + \mu'_1, \cdots, \xi_{\beta-1} + \xi'_{\beta-1} + \mu_{\beta-1} + \mu'_{\beta-1})\\ &= (\partial_0, \partial_1, \cdots, \partial_{\alpha-1}, \xi_0, \xi_1, \cdots, \xi_{\beta-1}, \xi_0 + \mu_0, \xi_1 + \mu_1, \cdots, \xi_{\beta-1} + \mu_{\beta-1}) +\\ &(\partial'_0, \partial'_1, \cdots, \partial'_{\alpha-1}, \xi_0,' \xi_1,' \cdots, \xi'_{\beta-1}, \xi'_0 + \mu'_0, \xi'_1 + \mu'_1, \cdots, \xi'_{\beta-1} + \mu'_{\beta-1})\\ &= \Phi(c_1) + \Phi(c_2).\end{aligned}$$

Moreover for $r \in \mathbb{F}_q$, one gets

$$\begin{aligned}\Phi(rc_1) &= \Phi(r\partial_0, r\partial_1, \cdots, r\partial_{\alpha-1}, re_0, re_1, \cdots, re_{\beta-1})\\ &= (r\partial_0, r\partial_1, \cdots, r\partial_{\alpha-1}, r\xi_0, r\xi_1, \cdots r\xi_{\beta-1}, r(\xi_0 + \mu_0), r(\xi_1 + \mu_1), r(\xi_{\beta-1} + \mu_{\beta-1}))\\ &= r(\partial_0, \partial_1, \cdots, \partial_{\alpha-1}, \xi_0, \xi_1, \cdots \xi_{\beta-1}, (\xi_0 + \mu_0), (\xi_1 + \mu_1), (\xi_{\beta-1} + \mu_{\beta-1}))\\ &= r\Phi(c_1).\end{aligned}$$

Thus, the map Φ is an $\mathbb{F}_q$-linear map. Let $e = (\xi_0 + v\mu_0, \xi_1 + v\mu_1, \cdots, \xi_{\beta-1} + v\mu_{\beta-1})$, and $e' = (\xi'_0 + v\mu'_0, \xi'_1 + v\mu'_1, \cdots, \xi'_{\beta-1} + v\mu'_{\beta-1})$ be two elements in S^β. Then

$$\begin{aligned}\Omega(e-e') =&(\xi_0 - \xi'_0, \xi_1 - \xi'_1, \cdots, \xi_{\beta-1} - \xi'_{\beta-1}, \xi_0 - \xi'_0 + \mu_0 - \mu'_0, \xi_1 - \xi'_1 + \mu_1 - \mu'_1,\\ &\cdots, \xi_{\beta-1} - \xi'_{\beta-1} + \mu_{\beta-1} - \mu'_{\beta-1}) = \Omega(e) - \Omega(e').\end{aligned}$$

Thus,

$$\begin{aligned}d_L(c_1, c_2) &= w_L(c_1 - c_2)\\ &= w_H(\partial - \partial') + w_H(\Omega(e - e')\\ &= w_H(\partial - \partial') + w_H(\Omega(e) - \Omega(e'))\\ &= d_H(\partial, \partial') + d_H(\Omega(e), \Omega(e'))\\ &= d_H((\partial, \Omega(e)), (\partial', \Omega(e'))).\end{aligned}$$

This shows that Φ is a distance preserving map. □

Corollary 3.3. *If C is an $\mathbb{F}_q$S-linear code with parameters $(\alpha + \beta, M, d_L)$, then $\Phi(C)$ is a q-ary linear code with parameters $[\alpha + 2\beta, log_q M, d_L]$, where M denotes the number of codewords in C.*

Theorem 3.4. *Let C be a linear self-orthogonal code over $\mathbb{F}_q$S. Then $\Phi(C)$ is a self-orthogonal code over $\mathbb{F}_q$.*

Proof. Let C be a self-orthogonal $\mathbb{F}_q$S-linear code of length $\alpha + \beta$. Let $c_1 = (\partial_1, \xi_1 + v\mu_1)$ and $c_2 = (\partial_2, \xi_2 + v\mu_2)$ be codewords of C over $\mathbb{F}_q$S, where $\partial_i \in \mathbb{F}_q^\alpha$ and $\xi_i, \mu_i \in \mathbb{F}_q^\beta$ for $i = 1, 2$. Then

$$< c_1, c_2 >= \xi_1\xi_2 + v(\partial_1\partial_2 + \xi_1\mu_2 + \xi_2\mu_1 + \mu_1\mu_2) = 0 + v0 \in \mathrm{S}.$$

So, one gets $\xi_1\xi_2 = 0$ and $\partial_1\partial_2 + \xi_1\mu_2 + \xi_2\mu_1 + \mu_1\mu_2 = 0$. By the $\Phi(c_1) = (\partial_1, \xi_1, \xi_1 + \mu_1)$ and $\Phi(c_2) = (\partial_2, \xi_2, \xi_2 + \mu_2)$, we have $\Phi(c_1)\Phi(c_2) = 0$. Thus, $\Phi(C)$ is self-orthogonal. □

Theorem 3.5. *Let C be an $\mathbb{F}_q$S-linear skew cyclic code of length $\alpha + \beta$. Then $\Phi(C) = C_1 \otimes C_2 \otimes C_3$, where C_1 is a skew cyclic code of length α in $\mathbb{F}_q[x]/(x^\alpha - 1)$ and C_2, C_3 are skew cyclic codes of length β in $\mathrm{S}[x;\theta]/(x^\beta - 1)$.*

Proof. Let $c = (\partial_0, \partial_1, \cdots, \partial_{\alpha-1}, \xi_0 + v\mu_0, \xi_1 + v\mu_1, \cdots, \xi_{\beta-1} + v\mu_{\beta-1})$ be a codeword in C and

$$\begin{aligned}C_1 &= (\partial_0, \partial_1, \cdots, \partial_{\alpha-1}),\\ C_2 &= (\xi_0, \xi_1, \cdots, \xi_{\beta-1}),\\ C_3 &= (\xi_0 + \mu_0, \xi_1 + \mu_1, \cdots, \xi_{\beta-1} + \mu_{\beta-1}).\end{aligned}$$

A codeword of C_1 corresponds to a codeword of C. By the hypothesis, one gets

$$T_{\psi\theta}(c) = (\psi(\partial_{\alpha-1}), \psi(\partial_0), \cdots, \psi(\partial_{\alpha-2}), \theta(\xi_{\beta-1} + v\mu_{\beta-1}), \theta(\xi_0 + v\mu_0), \\ \cdots, \theta(\xi_{\beta-2} + v\mu_{\beta-2})) \in C.$$

Thus we have $(\psi(\partial_{\alpha-1}), \psi(\partial_0), \cdots, \psi(\partial_{\alpha-2})) \in C_1$. So, C_1 is a cyclic code of length α in $\mathbb{F}_q[x]/(x^\alpha - 1)$. The proof follows by the same argument to show that C_2 and C_3 are skew cyclic codes of length β in $\mathrm{S}[x;\theta]/(x^\beta - 1)$.

□

Let C be a linear code over S of length n and σ a map from S^n to S^n given by $\sigma(c_0, c_1, \cdots, c_{n-1}) = (c_{n-1}, c_0, \cdots, c_{n-2})$. If $\sigma(C) = C$, then C is called a cyclic code.
In [8], the authors studied the structural properties of skew generalized quasi-cyclic codes (GQC) over finite fields. In the following definition, we present skew GQC over $\mathbb{F}_q$.

Definition 3.6. [7] Let C be a linear code over $\mathbb{F}_q$ of length $n = \sum_{i=1}^{l} m_i$ with $m_i \in \mathbb{N}$. Denote $\mathcal{F}_i = \mathbb{F}_q[x,\psi]/(x^{m_i} - 1)$ for $i = 1, 2, \cdots, l$. If C is an $\mathbb{F}_q[x;\psi]$-submodule of $\mathbb{F}_q[x;\psi]$-module $\mathcal{F} = \prod_{i=1}^{l} \mathcal{F}_i$, then C is called a skew generalized quasi-cyclic code over $\mathbb{F}_q$ of length n.
Let λ be a skew generalized quasi-cyclic shift defined as

$$\begin{aligned} \lambda\colon \mathbb{F}_q^{\alpha+2\beta} &\to \mathbb{F}_q^{\alpha+2\beta} \\ (p, r, s) &\mapsto (\sigma(\psi(p)), \sigma(\theta(s)), \sigma(\theta(r))), \end{aligned}$$

where σ is a cyclic shift, $p = (p_0, p_1, \cdots, p_{\alpha-1}) \in \mathbb{F}_q^\alpha$ as well as $r = (r_0, r_1, \cdots, r_{\beta-1})$ and $s = (s_0, s_1, \cdots, s_{\beta-1})$ are in $\mathbb{F}_q^\beta$.

Proposition 3.7. *Let Φ be the Gray map over $\mathbb{F}_q$S. Then $\Phi T_{\psi\theta} = \lambda\Phi$.*

Proof. Let $c = (\partial_0, \partial_1, \cdots, \partial_{\alpha-1}, \xi_0 + v\mu_0, \xi_1 + v\mu_1 \cdots, \xi_{\beta-1} + v\mu_{\beta-1})$. Then

$$\begin{aligned} \Phi T_{\psi\theta}(c) &= \Phi(\psi(\partial_{\alpha-1}), \psi(\partial_0), \cdots, \psi(\partial_{\alpha-2}), \theta(\xi_{\beta-1} + v\mu_{\beta-1}), \theta(\xi_0 + v\mu_0), \\ &\cdots, \theta(\xi_{\beta-2} + v\mu_{\beta-2})) \\ &= \Phi(\psi(\partial_{\alpha-1}), \psi(\partial_0), \cdots, \psi(\partial_{\alpha-2}), \xi_{\beta-1} + (1-v)\mu_{\beta-1}, \xi_0 + (1-v)\mu_0, \\ &\cdots, \xi_{\beta-2} + (1-v)\mu_{\beta-2}) \\ &= \psi(\partial_{\alpha-1}), \psi(\partial_0), \cdots, \psi(\partial_{\alpha-2}), \xi_{\beta-1} + \mu_{\beta-1}, \xi_0 + \mu_0, \cdots, \xi_{\beta-2} + \mu_{\beta-2}, \\ &\xi_{\beta-1}, \xi_0 \cdots, \xi_{\beta-2}. \end{aligned}$$

On the other hand,

$$\begin{aligned}\lambda\Phi(c) &= (\partial_0, \partial_1, \cdots, \partial_{\alpha-1}, \xi_0 + v\mu_0, \xi_1 + v\mu_1 \cdots, \xi_{\beta-1} + v\mu_{\beta-1})\\ &= \lambda(\partial_0, \partial_1, \cdots, \partial_{\alpha-1}, \xi_0, \xi_1, \cdots, \xi_{\beta-1}, \xi_0 + \mu_0, \xi_1 + \mu_1, \xi_{\beta-1} + \mu_{\beta-1})\\ &= (\psi(\partial_{\alpha-1}), \psi(\partial_0), \cdots, \psi(\partial_{\alpha-2}), \theta(\xi_{\beta-1} + \mu_{\beta-1}), \theta(\xi_0 + \mu_0),\\ &\cdots, \theta(\xi_{\beta-2} + \mu_{\beta-2}), \theta(\xi_{\beta-1}), \theta(\xi_0), \cdots, \theta(\xi_{\beta-2}),\\ &= \psi(\partial_{\alpha-1}), \psi(\partial_0), \cdots, \psi(\partial_{\alpha-2}), \xi_{\beta-1} + \mu_{\beta-1}, \xi_0 + \mu_0, \cdots, \xi_{\beta-2} + \mu_{\beta-2},\\ &\xi_{\beta-1}, \xi_0 \cdots, \xi_{\beta-2}.\end{aligned}$$

Therefore, $\Phi T_{\psi\theta} = \lambda\Phi$. □

Theorem 3.8. *Let C be a linear code of length $\alpha + \beta$ over $\mathbb{F}_q$S. C is an $\mathbb{F}_q$S-linear skew cyclic code if and only if $\Phi(C)$ is a generalized skew quasi-cyclic code over $\mathbb{F}_q$ of length $\alpha + 2\beta$.*

Proof. Let C be an $\mathbb{F}_q$S-linear skew cyclic code, then we have $\Phi(T_{\psi\theta}(C)) = \Phi(C)$. By Proposition 3.7, $\lambda(\Phi(C)) = \Phi T_{\psi\theta}(C) = \Phi(C)$. Thus, $\Phi(C)$ is a generalized skew quasi-cyclic code over $\mathbb{F}_q$.
On the other hand, according to Proposition 3.7, one gets $\Phi(T_{\psi\theta}(C)) = \lambda(\Phi(C)) = \Phi(C)$. Therefore, C is an $\mathbb{F}_q$S-linear skew cyclic code. □

4 Generator Polynomials

In this section, we study the generator polynomials of $\mathbb{F}_q$S-linear skew cyclic codes in $\mathrm{S}_{\alpha,\beta}$. Then, we introduce the minimal spanning sets of these codes and construct some optimal q-ary linear codes with good parameters. Throughout the section, we denote the zero vectors or zero polynomials by $\mathbf{0}$ and $f(x)|_r(x^\alpha - 1)(f(x)|_l(x^\alpha - 1))$ indicates that $f(x)$ is a right(left) divisor of $x^\alpha - 1$, respectively. We first introduce ideals of $\mathbb{F}_q[x;\psi]/(x^\alpha - 1)$ and $\mathrm{S}[x;\theta]/(x^\beta - 1)$ with their useful identities in the followings:

Proposition 4.1. *Let $I_1 := \{g(x) \in \mathbb{F}_q[x;\psi]/(x^\alpha - 1)|(g(x), 0) \in C\}$ be an ideal of $\mathbb{F}_q[x;\psi]/(x^\alpha - 1)$. Then I_1 is generated by a divisor of $x^\alpha - 1$.*

Proof. Let $g_1(x)$ and $g_2(x)$ be polynomials in I_1, then $(g_1(x), \mathbf{0})$ and $(g_2(x), \mathbf{0})$ are in C. Thus,

$$(g_1(x), \mathbf{0}) + (g_2(x), \mathbf{0}) = (g_1(x) + g_2(x), \mathbf{0}) \in C,$$

which implies that $g_1(x) + g_2(x) \in I_1$. Moreover, let $f(x) \in \mathbb{F}_q[x;\psi]/(x^\alpha - 1)$ and $g(x) \in I_1$, and so $(g(x), \mathbf{0}) \in C$. Since C is a left $\mathrm{S}[x;\theta]$-module,

$$f(x) \star (g(x), \mathbf{0}) = (f(x)g(x), \mathbf{0}) \in C.$$

Hence $(f(x)g(x)(mod(x^\alpha - 1))) \in I_1$. □

Proposition 4.2. *Let* $I_2 := \{h(x) \in \mathrm{S}[x;\theta]/(x^\beta - 1) \mid (j(x), h(x)) \in C, \exists j(x) \in \mathbb{F}_q[x;\psi]/(x^\alpha - 1))\}$ *be an ideal of* $\mathrm{S}[x;\theta]/(x^\beta - 1)$*. Then* I_2 *is principally generated by* $\mathrm{S}[x;\theta]$*-submodule of* $\mathrm{S}[x;\theta]/(x^\beta - 1)$*.*

Proof. Let $h_1(x)$ and $h_2(x)$ be polynomials in I_2, then there exists $j_1(x)$ and $j_2(x)$ in $\mathbb{F}_q[x;\psi]/(x^\alpha - 1)$ such that $(j_1(x), h_1(x))$ and $(j_2(x), h_2(x))$ are in C. Thus,

$$(j_1(x), h_1(x)) + (j_2(x), h_2(x)) = (j_1(x) + j_2(x), h_1(x) + h_2(x)) \in C,$$

this ensures that $h_1(x) + h_2(x) \in I_2$. Now, let $r(x) \in \mathrm{S}[x;\theta]/(x^\beta - 1)$ and $(j(x), h(x)) \in C$. Since C is a left $\mathrm{S}[x;\theta]$-module of $\mathrm{S}_{\alpha,\beta}$,

$$r(x) \star (j(x), h(x)) = ((\delta(r(x)).j(x) \; mod(x^\alpha - 1)), (r(x) * h(x)) \; mod(x^\beta - 1))) \in C.$$

Thus, $(r(x) * h(x) \; mod(x^\beta - 1)) \in I_2$. Moreover, by Theorem 2.3, $I_2 =< h(x) >=< vh_1(x) + (1 - v)h_2(x) >$. □

Theorem 4.3. *Let* C *be an* $\mathbb{F}_q$S*-linear skew cyclic code of length* $\alpha + \beta$*. Then*

$$C =< (g(x), 0), (j(x), h(x)) >,$$

where $j(x) \in \mathbb{F}_q[x;\psi]/(x^\alpha - 1)$*,* $g(x)|_r(x^\alpha - 1)$ *and* $h(x) = vh_1(x) + (1 - v)h_2(x)$*. Moreover,* $deg(j(x)) < deg(g(x))$*.*

Proof. Let $(c_1(x), c_2(x)) \in C$ with $c_2(x) \in I_2$. By Proposition 4.2, we have $c_2(x) = p(x)h(x)$ for some $p(x) \in \mathrm{S}[x;\theta]/(x^\beta - 1)$. Also, there exists $j(x) \in \mathbb{F}_q[x;\psi]/(x^\alpha - 1)$ such that $(j(x), h(x)) \in C$. So,

$$\begin{aligned}
(c_1(x), c_2(x)) &= (c_1(x), 0) + (0, c_2(x)) \\
&= (c_1(x), 0) + (0, p(x)h(x)) \\
&= (c_1(x), 0) + (q(\delta(p(x)).j(x)), p(x)h(x)) \\
&= (c_1(x), 0) + (\delta(p(x))j(x), p(x)h(x)) \\
&\quad + (\delta(p(x))j(x), 0) + \cdots + (\delta(p(x))j(x), 0) \\
&= (c_1(x), 0) + p(x) \star [(j(x), h(x)) + (j(x), 0) + \cdots + (j(x), 0)].
\end{aligned}$$

Thus $\delta(p(x))j(x) + c_1(x) \in I_1$. By Proposition 4.1, there exists $r(x) \in I_1$ such that $\delta(p(x))j(x) + c_1(x) = r(x)g(x)$. Hence $(c_1(x), c_2(x)) = (r(x)g(x), 0) + p(x) \star (j(x), h(x))$.
Now, we need to show that $deg(j(x)) < deg(g(x))$. Assume that $deg(j(x)) \geq deg(g(x))$ and $deg(j(x) - g(x)) = m \in \mathbb{N}$. Let

$$D =< (g(x), 0), (j(x) - x^m g(x), h(x)) > .$$

Then, one gets $D \subseteq C$. Furthermore, we have

$$(j(x), h(x)) = (j(x) - x^m g(x), h(x)) + x^m \star (g(x), 0).$$

Thus, $C \subseteq D$ and so $C = D$. This yields a contradiction. Hence $deg(j(x)) < deg(g(x))$. □

In the following Corollary, we deduce the properties and generators of several classes of $\mathbb{F}_q$S-linear skew cyclic codes.

Corollary 4.4. *Let C be an $\mathbb{F}_q$S-linear skew cyclic codes of length $\alpha + \beta$ and polynomials $g(x), j(x), h_1(x)$ and $h_2(x)$ are in $\mathbb{F}_q[x;\psi]/(x^\alpha - 1)$. Then*

(i) $C =< (g(x), 0) >$, *where* $g(x)|_r(x^\alpha - 1)$ *in* $\mathbb{F}_q[x;\psi]$.

(ii) $C =< (j(x), h(x)) >=< (j(x), vh_1(x)+(1-v)h_2(x)) >$ *where* $h(x)|_r(x^\beta - 1)$ *in* $\mathrm{S}[x;\theta]$.

(iii) $C =< (g(x), 0), (j(x), h(x)) >$ *where* $deg(j(x)) < deg(g(x))$ *and* $g(x)|_r(x^\alpha - 1)$ *in* $\mathbb{F}_q[x;\psi]$, $h(x)|_r(x^\beta - 1)$ *in* $\mathrm{S}[x;\theta]$.

Corollary 4.5. *Let C be an $\mathbb{F}_q$S-linear skew cyclic codes of length $\alpha + \beta$ and generated by $< (g(x), 0), (0, h(x)) >$. Then $C = C_1 \otimes C_2$, where C_1 is a skew cyclic code over $\mathbb{F}_q$ and C_2 is a skew cyclic code over* S.

Example 4.6. Let C be an $\mathbb{F}_4$S-linear skew cyclic code of length $\alpha+\beta$, where $\mathbb{F}_4 = \mathbb{F}_2[w] = \{0, 1, w, w^2\}$, and $\mathrm{S} = \{\xi + v\mu | \xi, \mu \in \mathbb{F}_4\}$ is a commutative ring with 16 elements with $v^2 = v$, and also $\alpha = \beta = 6$. Suppose that ψ is an identity automorhism of $\mathbb{F}_4$ and θ be an automorphism of S, then

$$C =< (x^4 + w^2x^2 + w, 0), (x^3 + wx^2 + (1 + w), x^3 + (w^2 + v)) >,$$

where $h_1(x) = x^3 + w$, $h_2(x) = x^3 + (w^2 + v)$, and satisfying

$$x^6 - 1 = (x^2 + w^2)(x^4 + w^2x^2 + w),$$
$$x^6 - 1 = (x^3 + (v + w))(x^3 + (w^2 + v)).$$

Example 4.7. Let C be an $\mathbb{F}_4$S-linear skew cyclic code of length $\alpha+\beta$, where $\mathbb{F}_4 = \mathbb{F}_2[w] = \{0, 1, w, w^2\}$, and $\mathrm{S} = \{\xi + v\mu | \xi, \mu \in \mathbb{F}_4\}$ is a commutative ring with 16 elements with $v^2 = v$, and also $\alpha = 4$ and $\beta = 6$. Suppose that $\psi(\partial) = \partial^2$ and θ be an automorphism of S, then

$$C =< (x^3 + w^2x^2 + x + w^2, 0), (x^2 + w, x^4 + (w^2 + v)x^2 + (w + v)) >,$$

where $h_1(x) = x^4 + wx^2 + w^2$, $h_2(x) = x^4 + w^2x^2 + w$, and satisfying

$$x^4 - 1 = (x + w)(x^3 + w^2x^2 + x + w^2),$$
$$x^6 - 1 = (x^2 + w^2 + v)(x^4 + (w^2 + v)x^2 + (w + v)).$$

Theorem 4.8. *Let C be an $\mathbb{F}_q$S-linear skew cyclic code of length $\alpha+\beta$ and*

$$C =< (g(x),0),(j(x),h(x)) >,$$

where $j(x) \in \mathbb{F}_q[x;\psi]/(x^\alpha-1)$, $g'(x)g(x)=(x^\alpha-1)$ and $h'(x)h(x)=(x^\beta-1)$. Suppose that

$$S_g = \bigcup_{i=0}^{deg(g'(x))-1} \{x^i \star (g(x),0)\},$$

and

$$S_h = \bigcup_{i=0}^{deg(h'(x))-1} \{x^i \star (j(x),h(x))\}.$$

Then the set $S = S_g \cup S_h$ forms a minimal spanning set for C and $|C| = q^{deg(g'(x))+2deg(h'(x))}$.

Proof. Let

$$\delta(a_1(x)) \star (g(x),0) + a_2(x) \star (j(x),h(x)) \in C,$$

where the polynomials $a_1(x)$ and $a_2(x)$ are in $\mathrm{S}[x;\theta]/(x^\beta-1)$.
First of all, we claim that $\delta(a_1(x)) \star (g(x),0) \in Span(S_g)$. If $deg(\delta(a_1(x))) < deg(g'(x))$, then we are done. On the other hand, let $q_1(x)$ and $r_1(x)$ be two polynomials in $\mathrm{S}[x;\theta]/(x^\beta-1)$ so that $\delta(a_1(x)) = \delta(q_1(x)).g'(x) + \delta(r_1(x))$ where $\delta(r_1(x)) = 0$ or $deg(\delta(r_1(x))) < deg(g'(x))$. So,

$$\begin{aligned}\delta(a_1(x)) \star (g(x),0) &= (\delta(q_1(x)).g'(x) + \delta(r_1(x))) \star (g(x),0)\\ &= \delta(q_1(x)) \star (g'(x).g(x),0) + \delta(r_1(x)) \star (g(x),0)\\ &= \delta(r_1(x)) \star (g(x),0).\end{aligned}$$

Thus, $\delta(a_1(x)) \star (g(x),0)$ belongs to $Span(S_g)$.
Now, we prove that $a_2(x)\star(j(x),h(x)) \in Span(S)$. If $deg(a_2(x)) < deg(h'(x))$, then $a_2(x) \star (j(x),h(x)) \in Span(S_h)$. On the other hand, let $q_2(x)$ and $r_2(x)$ be two polynomials in $\mathrm{S}[x;\theta]/(x^\beta-1)$ so that $a_2(x) = q_2(x) * h'(x) + r_2(x)$, where $r_2(x) = 0$ or $deg(r_2(x))) < deg(h'(x))$. So,

$$\begin{aligned}a_2(x) \star (j(x),h(x)) &= (q_2(x) * h'(x) + r_2(x)) \star (j(x),h(x))\\ &= q_2(x) \star (\delta(h'(x)).j(x), h'(x)h(x)) + r_2(x) \star (j(x),h(x)).\end{aligned}$$

Clearly, $r_2(x)\star(j(x),h(x)) \in Span(S_h)$. Therefore, we have $a_2(x)\star(j(x),h(x)) \in Span(S_g \cup S_h)$. □

Corollary 4.9. *(i) Let C be an $\mathbb{F}_q$S-linear skew cyclic code generated by (i) in Corollary 4.4 and*

$$S = \bigcup_{i=0}^{deg(g'(x))-1} \{x^i \star (g(x), 0)\},$$

then the set S forms a minimal spanning set for C.
(ii) Let C be an $\mathbb{F}_q$S-linear skew cyclic code generated by (ii) in Corollary 4.4 and

$$S = \bigcup_{i=0}^{deg(h'(x))-1} \{x^i \star (j(x), h(x))\},$$

then the set S forms a minimal spanning set for C.
(iii) Let C be an $\mathbb{F}_q$S-linear skew cyclic code generated by (iii) in Corollary 4.4 and

$$S_g = \bigcup_{i=0}^{deg(g'(x))-1} \{x^i \star (g(x), 0)\}$$

and

$$S_h = \bigcup_{i=0}^{deg(h'(x))-1} \{x^i \star (j(x), h(x))\},$$

then the set $S_g \cup S_h$ forms a minimal spanning set for C and $|C| = q^{deg(g'(x)+2deg(h'(x))}$.

Example 4.10. Let C be an $\mathbb{F}_4$S-linear skew cyclic codes of length 12 given in Example 4.6. Then we have

$$g'(x).g(x) = x^6 - 1 \Rightarrow g'(x) = x^2 + w^2,$$
$$h'(x) * h(x) = x^6 - 1 \Rightarrow h'(x) = x^3 + (v + w).$$

Therefore, the minimal generating set of C has the form

$S_g = \{(x^4 + w^2x^2 + w, 0), (x^5 + w^2x^3 + wx, 0)\}$,
$S_h = \{(x^3 + wx^2 + (1 + w), x^3 + (w^2 + v)), (x^4 + wx^3 + (1 + w)x, x^4 + (w + v)x)$,
$(x^5 + wx^4 + (1 + w)x^2, x^5 + (1 + w + v)x^2)\}$.

Hence, by Corollary 4.9 (iii), the generator matrix of C can be obtained as follows:

$$\begin{bmatrix} w & 0 & w^2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & w & 0 & w^2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ (1+w) & 0 & w & 1 & 0 & 0 & (w^2+v) & 0 & 0 & 1 & 0 & 0 \\ 0 & (1+w) & 0 & w & 1 & 0 & 0 & (w+v) & 0 & 0 & 1 & 0 \\ 0 & 0 & (1+w) & 0 & w & 1 & 0 & 0 & (w^2+v) & 0 & 0 & 1 \end{bmatrix}.$$

Example 4.11. Let C be an $\mathbb{F}_4$S-linear skew cyclic code of length 10 given in Example 4.7. Then we have

$$g'(x).g(x) = x^4 - 1 \Rightarrow g'(x) = x + w,$$
$$h'(x) * h(x) = x^6 - 1 \Rightarrow h'(x) = x^2 + w^2 + v.$$

Therefore, the minimal generating set of C has the form

$S_g = \{(x^3 + w^2x^2 + x + w^2, 0)\}$,
$S_h = \{(x^2 + w, x^4 + (w^2 + v)x^2 + (w + v)), (x^3 + w^2x, x^5 + (w + v)x^3 + (w^2 + v)x)\}$.

Hence, by Corollary 4.9 (iii), the generator matrix of C can be obtained as follows:

$$\begin{bmatrix} w^2 & 1 & w^2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ w & 0 & 1 & 0 & (w+v) & 0 & (w^2+v) & 0 & 1 & 0 \\ 0 & w^2 & 0 & 1 & 0 & (w^2+v) & 0 & (w+v) & 0 & 1 \end{bmatrix}.$$

In the following examples, we introduce some optimal linear codes with good parameters which are some of the comparable well-known linear codes in [15].

Example 4.12. Let $C =< (x^3-1, 0), (x^2+x+1, x-1) >$ be a $\mathbb{F}_4\mathbb{F}_4[v]$- linear skew cyclic code in $\mathbb{F}_4[x;\psi]/(x^3 - 1) \times \mathbb{F}_4[x;\theta]/(x^2 - 1)$, where $h'(x) = x + 1$ and $j(x) = x^2 + x + 1$. By Corollary 4.9 (iii), the generator matrix of C can be written as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Moreover, by Corollary 3.3, $\Phi(C)$ is a quasi-cyclic code with parameters $[7, 2, 5]$.

Example 4.13. Let $C =< (x^8-1, 0), (x^7+x^6+x^5+x^4+x^3+x^2+x+1, x+2) >$ be a $\mathbb{F}_9\mathbb{F}_9[v]$-linear skew cyclic code in $\mathbb{F}_9[x;\psi]/(x^8 - 1) \times \mathbb{F}_9[x;\theta]/(x^2 - 1)$, where $h'(x) = x + 1$. By Corollary 4.9 (iii), the generator matrix of C can be written as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \end{bmatrix}.$$

Moreover, by Corollary 3.3, $\Phi(C)$ is a quasi-cyclic code with parameters $[12, 2, 10]$.

Example 4.14. Let $C =< (x^{10}-1, 0), (x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1, x-1)$ be a $\mathbb{F}_9\mathbb{F}_9[v]$-linear skew cyclic code in $\mathbb{F}_9[x;\psi]/(x^{10}-1) \times \mathbb{F}_9[x;\theta]/(x^2-1)$, where $h'(x) = x-2$. By Corollary 4.9 (iii), the generator matrix of C can be written as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Moreover, by Corollary 3.3, $\Phi(C)$ is a quasi-cyclic code with parameters $[14, 2, 12]$.

5 Conclusion

In this paper, we studied skew cyclic codes over the ring $\mathbb{F}_q$S with power of prime number p and $\mathrm{S} = \mathbb{F}_q + v\mathbb{F}_q$, $v^2 = v$. We constructed algebraic structure of these codes as a left submodule of a skew polynomial ring. Then, we introduced all generators and also minimal spanning sets of these codes. Finally, we obtained some $\mathbb{F}_q$-linear codes with good parameters which are some of the comparable well-known linear codes in [15]. Moreover, some optimal $\mathbb{F}_4\mathbb{F}_4[v]$-linear skew cyclic codes are given in Table 1 with automorphisms of $\mathbb{F}_4$ and $\mathbb{F}_4[v]$ are defined as $\psi(\partial) = \partial$ and θ, respectively.

Table 1: Optimal $\mathbb{F}_4\mathbb{F}_4[v]$-linear skew cyclic codes

(α, β)	Generators	$[n, k, d]_4$
$(4, 2)$	$C_1 =< (x^4-1, 0), (x^3+x^2+x+1, x+(w+v)) >$	$[8, 2, 6]$
$(6, 2)$	$C_2 =< (x^6 - 1, 0), (x^5 + x^4 + x^3 + x^2 + x + 1, x + (w^2 + v)) >$	$[10, 2, 8]$
$(3, 4)$	$C_3 =< (x^3 - 1, 0), (x^2 + x + 1, w^2x + w^2) >$	$[11, 6, 5]$
$(4, 4)$	$C_4 =< (x^3 + x^2 + x + 1, 0), (x^2 - 1, w^2x + w^2) >$	$[12, 7, 4]$
$(5, 6)$	$C_5 =< (x^4+x^3+x^2+x+1, 0), (x^2+w^2x + 1, x + 1) >$	$[17, 11, 5]$
$(6, 5)$	$C_6 =< (x^5 + x^4 + x^3 + x^2 + x + 1, 0), (x^3 + w^2x^2 + w^2x + 1, x + 1) >$	$[16, 9, 6]$
$(9, 5)$	$C_7 =< (x^8 + x^7 + x^6 + x^3 + x^2 + x + 1, 0), (x^5 + wx^4 + w^2x^3 + w^2x^2 + x + w, x + 1) >$	$[19, 9, 8]$
$(10, 4)$	$C_8 =< (x^{10} - 1, 0), (x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1, wx^3 + wx^2 + wx + w) >$	$[18, 2, 14]$
$(14, 4)$	$C_9 =< (x^7 + x^6 + x^3 + x^2 + x + 1, 0), (x^6 + x^5 + x^4 + x^3 + x^2 + x + 1, (w + 1)x + (1 + w)) >$	$[22, 13, 6]$
$(18, 6)$	$C_{10} =< (x^{18}-1, 0), (x^{17}+x^{16}+x^{15}+x^{14}+x^{13}+x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1, x^5+(w^2+wv)x^4+w^2x^3+(w+v)x^2+wx+(1 + w^2v) >$	$[30, 2, 24]$

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Tulay YILDIRIM,
Eskipazar Vocational School,
Karabuk University, Karabuk, Turkey
Email: tulayturan@karabuk.edu.tr