DOI: 10.2478/auom-2024-0024
An. Şt. Univ. Ovidius Constanţa

sciendo
Vol. 32(2),2024, 145–155

Remarks on some connections between ideals and filters in residuated lattices

Dana PICIU, Christina DAN and Florentina BOBOC

Abstract

Ideals and filters are important notions with different meanings in the study of algebraic structures related to logical systems. In this paper we establish new connections between these concepts in residuated lattices.

Keywords: Filter, Ideal, Residuated lattice

1 Introduction

To generalize the lattice of ring ideals, in [13], was defined commutative residuated lattices, linked to multiple valued logic and representing semantics for residuated logics. Thus, the study of algebraic structures (as Boolean, MV or BL-algebras), linked to particular logics was continued.

The class of residuated lattices forms a variety $\mathcal{RL}$, see [4].

In lattice theory, filters and ideals have an important role. For residuated lattices, filters were introduced in [6] as an algebraic notion related to logical provable formulas. Their study is useful to provide completeness with respect to algebraic semantics.

In MV-algebras, the concept of ideal, introduced in [3] as kernel of morphism, is dual to filter. This notion was defined in residuated lattices (see

Key Words: Residuated lattice, filter, ideal, MV-ideal, MV-filter.
2010 Mathematics Subject Classification: Primary 03B50, 06A06; Secondary 03G05, 08A72, 06D35, 06B20, 03C05.
Received: 03.05.2023
Accepted: 21.10.2023

[9]) as a generalization for [7]. Although ideals are useful to study logical systems, in residuated lattices, their study was delayed, due the lack of algebraic addition.

Unlike the classical Boolean lattice theory, ideals and filters are not dual notions in residuated lattices. For this reason, in this paper, we analyse some connections between these notions.

2 Ideals versus filters

An algebra $(A, \vee, \wedge, \odot, \rightarrow, 0, 1)$, such that

- (i) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice with the order $\leq$;
- (ii) $(A, \odot, 1)$ is a commutative monoid;
- (iii) $x \odot z \leq y$ if and only if $x \leq z \rightarrow y$, for $x, y, z \in A$,

is a *residuated lattice, see* [13].

For $A \in \mathcal{RL}$ and $x \in A$ we denote $x^* = x \rightarrow 0$. As usual, $B(A)$ is the Boolean center of A, see [12].

Using the following conditions:

$$x \wedge y = x \odot (x \rightarrow y) \text{ } (div),$$

$$1 = (x \rightarrow y) \vee (y \rightarrow x) \text{ } (prel),$$

$$x = x^{**} \text{ } (DN\text{-}double\ negation) \text{ and}$$
$$(x \wedge y)^* = x^* \vee y^* \text{ } (DM\text{-De Morgan})$$

can be obtained particular residuated lattices. For example, $(div) + (prel)$ generate a BL-algebra and if this satisfies additionally (DN), it is an MV-algebra, see [3], [5] and [12].

For $A \in \mathcal{RL}$, $x, y \in A$, $m \geq 2$, we define $x \boxplus y = x^* \rightarrow y^{**}$ and we denote $mx = \underbrace{x \boxplus ... \boxplus x}_{m \text{ times}}$, $x^m = \underbrace{x \odot ... \odot x}_{m \text{ times}}$.

Also, for $x \in A$, the minimum $m \geq 1$, such that $x^m = 0$ is denoted by $o(x)$; if there is no such m, then $o(x) = \infty$.

We recall some rules of calculus in $\mathcal{RL}$ (see [2] and [12]):

(1) $x \rightarrow y = 1$ if and only if $x \leq y$;

(2) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$, $(x \vee y)^* = x^* \wedge y^*$;

(3) $x \boxplus y = (x^* \odot y^*)^*$, $[(x^*)^m]^* = mx = (mx)^{**}$, for $x, y, z \in A$ and $m \geq 2$.

An ideal in $\mathcal{RL}$ is a generalization of the similar notions from [3], [7].

Definition 1. *([2], [9]) Let $A \in \mathcal{RL}$. An* ideal *is a subset $I \neq \emptyset$ of A such that:*

(*i*) $x \leq i,\ i \in I \Longrightarrow x \in I$;

(*ii*) $i, j \in I \Longrightarrow i \boxplus j \in I$.

The set of ideals of A is denoted by $\mathbf{I(A)}$ For $I \in \mathbf{I(A)}$, $i \in I \Longleftrightarrow i^{**} \in I$.

In [9], for each $I \in \mathbf{I(A)}$ is defined a congruence relation and the factor set $A/I = \{x/I : x \in A\} \in \mathcal{RL}$. If A/I is an MV-algebra, then I is called an MV-ideal and its characterization is:

$$(x^{**} \to x)^* \in I, \text{ for every } x \in A,$$

see [14]. Because $\mathcal{RL}$ is a variety we can generalize this definition:

Definition 2. *Let $\mathcal{S}$ be a subvariety of $\mathcal{RL}$. Then $I \in \mathbf{I(A)}$ is called a $\mathcal{S}$ -ideal if $A/I \in \mathcal{S}$.*

$S\_\mathbf{I(A)}$ is the set of $\mathcal{S}$-ideals of A.

Theorem 3. *Let $A \in \mathcal{RL}$ and $\mathcal{S} \subseteq \mathcal{RL}$ be a subvariety. If $I, J \in S\_\mathbf{I(A)}$, then $I \cap J \in S\_\mathbf{I(A)}$.*

Proof. Let $f : A/(I \cap J) \to (A/I)\Pi(A/J)$,

$$f(x/(I \cap J)) = (x/I, x/J), \text{ for } x \in A.$$

Obviously, f is correctly defined and one to one. Since $I, J \in S\_\mathbf{I(A)}$ we obtain that A/I, $A/J \in \mathcal{S}$. Because $\mathcal{S}$ is a variety, $(A/I)\Pi(A/J) \in \mathcal{S}$.

Since f is an one to one morphism in $\mathcal{RL}$, $A/(I \cap J)$ is isomorphic with $f(A/(I \cap J))$ which is a subalgebra of $(A/I)\Pi(A/J)$.

Finally, $f(A/(I \cap J)) \in \mathcal{S}$, thus $A/(I \cap J) \in \mathcal{S}$, so $I \cap J \in S\_\mathbf{I(A)}$. $\square$

In particular, if $\mathcal{S} = \mathcal{MV}$, the subvariety of MV-algebras, we obtain a result from [14].

For $A \in \mathcal{RL}$, let $\mathbf{IP(A)} = \{I \in \mathbf{I(A)} \backslash \{A\} : i^{**} \wedge j^{**} \in I \Longrightarrow i \in I$ or $j \in I\}$ and $\mathbf{IM(A)} = \{I \in \mathbf{I(A)} \backslash \{A\} :\ x \notin I \Longleftrightarrow (mx)^* \in I$, for some $m \geq 1\}$.

We say that $I \in \mathbf{IP(A)}$ is *prime and* $I \in \mathbf{IM(A)}$ is *maximal, see [2], [11].*

Definition 4. *$A \in \mathcal{RL}$ with the property that for any $x \in A$ there is $m \geq 1$ such that $mx \in B(A)$ is called* i-Hyperarchimedean*.*

Moreover, a residuated lattice A verifying (DM) is i-Hyperarchimedean iff $\mathbf{IP(A)} = \mathbf{IM(A)}$, see [11].

In residuated lattices, ideals and the dual of filters (deductive systems) are differently.

Definition 5. *([6], [12]) Let $A \in \mathcal{RL}$. A* filter *is a subset $F \neq \emptyset$ of A such that :*

(i) $f \leq x,\ f \in F \Longrightarrow x \in F$;

(ii) $f,\ g \in F \Longrightarrow f \odot g \in F$.

We denote by $\mathbf{F(A)}$ the set of filters of A. For each $F \in \mathbf{F(A)}$ is defined a congruence relation and the factor set $A/F \in \mathcal{RL}$, see [12].

Also, let $\mathbf{FP(A)} = \{F \in \mathbf{F(A)} \backslash \{A\} : f \vee g \in F \Longrightarrow f \in F$ or $g \in F\}$ and $\mathbf{FM(A)} = \{F \in \mathbf{F(A)} \backslash \{A\} : F$ is maximal in $(\mathbf{F(A)}, \subseteq)\}$.

Thus, we say that $F \in \mathbf{FP(A)}$ is *prime and* $F \in \mathbf{FM(A)}$ is *maximal, see* [10], [12].

If $\mathbf{FP(A)} = \mathbf{FM(A)}$, then the residuated lattice A is called *Hyperarchimedean.* Moreover, $A \in \mathcal{RL}$ is Hyperarchimedean iff for every $x \in A$ there exists a natural number $m \geq 1$ such that $x^m \in B(A)$, see [10].

3 Connections between filters and ideals

As we mentioned, ideals and filters are not dual notions in residuated lattices.

The purpose of this section is to prove that these concepts, as algebraic structures, have different meanings and generate various constructions.

We recall that, in an MV-algebra, F is a filter iff the set of its complement elements $F^* = \{f^* : f \in F\}$ is an ideal and conversely, I is an ideal iff $I^* = \{i^* : i \in I\}$ is a filter.

Based on the following examples, we remark that, in residuated lattices, this statement does not hold. Hence, the notions of ideals and filters are not dual under complementation.

Example 6. *For $A \in \mathcal{RL}$ from [7], Example 3.5, we remark that*

$$I = \{0, x, y, z\} \in \mathbf{I(A)} \text{ but } I^* = \{1, t\} \notin \mathbf{F(A)}.$$

Also,

$$F = \{1, t, u, v\} \in \mathbf{F(A)} \text{ but } F^* = \{0, z\} \notin \mathbf{I(A)}.$$

Example 7. *Moreover, if we consider $A \in \mathcal{RL}$ from [10], Example 1.6, we have*

$$\mathbf{I}(\mathbf{A}) = \{\{0\}, \{0, x\}, \{0, y\}, A\},$$

$$\mathbf{IP}(\mathbf{A}) = \mathbf{IM}(\mathbf{A}) = \{\{0, x\}, \{0, y\}\},$$

$$\mathbf{F}(\mathbf{A}) = \{\{1\}, \{1, z\}, \{1, x, z\}, \{1, y, z\}, A\},$$

$$\mathbf{FP}(\mathbf{A}) = \mathbf{F}(\mathbf{A}) \backslash \{\mathbf{A}\} \textit{ and } \mathbf{FM}(\mathbf{A}) = \mathbf{FP}(\mathbf{A}) \backslash \{\{1\}\}.$$

Then

$$\mathbf{IP}(\mathbf{A}) = \mathbf{IM}(\mathbf{A}) \textit{ but } \mathbf{FP}(\mathbf{A}) \neq \mathbf{FM}(\mathbf{A}),$$

$$F = \{1\} \in \mathbf{FP}(\mathbf{A}) \textit{ but } F^* = \{0\} \notin \mathbf{IP}(\mathbf{A}),$$

$$P = \{0, x\} \in \mathbf{IP}(\mathbf{A}) \textit{ but } P^* = \{1, y\} \notin \mathbf{F}(\mathbf{A}),$$

$$M = \{0, y\} \in \mathbf{IM}(\mathbf{A}) \textit{ but } M^* = \{1, x\} \notin \mathbf{F}(\mathbf{A}).$$

In a residuated lattice A, for $x \in A$, we denote by $(x]$ and $< x >$ the principal ideal, respectively filter, of A generated by x, see [2] and [10].

In fact, $< x > \neq A \iff o(x) = \infty$ and $(x] \neq A \iff o(x^*) = \infty$, see [10] and [11].

Using this result we deduce that:

Proposition 8. *Let $A \in \mathcal{RL}$ and $x \in A$. Then*

(*i*) *$(x]$ is proper iff $< x^* >$ is proper;*

(*ii*) *If $|FM(A)| = 1$ and $(x]$ is proper then $< x >$ is not proper.*

Proof. (*i*). $(x]$ is proper iff $o(x^*) = \infty$ iff $< x^* >$ is proper.

(*ii*). From $(x]$ proper, we deduce that $o(x^*) = \infty$. Since A has an unique maximal filter, using [10], $o(x) < \infty$ or $o(x^*) < \infty$. Then, by hypothesis, $o(x) < \infty$, so, $< x >$ is not proper. □

The characterization of residuated lattices in which prime and maximal spectrum coincide (for filters, respectively ideals) are known and in the following, we show that the notions of Hyperarchimedean and i-Hyperarchimedean do not coincide in $\mathcal{RL}$ (as it happens in $\mathcal{MV}$, for example):

Corollary 9. *Let $A \in \mathcal{RL}$. Then*

Hyperarchimedean $\Rightarrow$ *i-Hyperarchimedean*

i-Hyperarchimedean $\not\Rightarrow$ *Hyperarchimedean.*

Proof. Suppose that A is Hyperarchimedean and let $x \in A$. Then $x^* \in A$, so there is a natural number $m \geq 1$ such that $(x^*)^m \in B(A)$. Hence $mx = [(x^*)^m]^* \in B(A)$, so A is i-Hyperarchimedean.

To show that the converse implication is not true, we consider $A \in \mathcal{RL}$ from [5], Example 2, p.271, which satisfies (DM). We remark that $kx = 1$, if $x \neq 0$ and $kx = 0$, if $x = 0$, for every $k \geq 2$. Since $0, 1 \in B(A)$ we deduce that A is i-Hyperarchimedean. But $p^k = p$ for every $k \geq 1$ and $p \notin B(A)$ since $p \vee p^* = p \vee 0 = p \neq 1$. We conclude that A is not Hyperarchimedean. □

Let $A \in \mathcal{RL}$ and $S \subseteq A$. The set of complement elements (with respect to S) is denoted by

$$C(S) = \{x \in A : x^* \in S\}.$$

We recall the following result:

Proposition 10. *([2], [7], [14]) Let $A \in \mathcal{RL}$, $F \in \mathbf{F}(\mathbf{A})$ and $I \in \mathbf{I}(\mathbf{A})$. Then:*

(i) $C(F) \in \mathbf{I}(\mathbf{A})$, $C(I) \in \mathbf{F}(\mathbf{A})$, $F \subseteq C(C(F)), I = C(C(I))$;

(ii) *A/I and $A/C(I)$ coincide;*

(iii) *If $I \in \mathbf{IP}(\mathbf{A})$ then $C(I) \in \mathbf{FP}(\mathbf{A})$; Also, if A satisfies (DM) and $F \in \mathbf{FP}(\mathbf{A})$ then $C(F) \in \mathbf{IP}(\mathbf{A})$;*

(iv) *If $I \in \mathbf{IM}(\mathbf{A})$ then $C(I) \in \mathbf{FM}(\mathbf{A})$; Also, if $F \in \mathbf{FM}(\mathbf{A})$ then $C(F) \in \mathbf{IM}(\mathbf{A})$.*

The previous result proves that complement elements can establish connections between ideals and filters in residuated lattices.

Also, we can characterize proper ideals în residuated lattices verifying (div):

Proposition 11. *Let $A \in \mathcal{RL}$ satisfying (div) and $I \in \mathbf{I}(\mathbf{A})$. Then $(x^{**} \to x)^* \in I$, for every $x \in L$.*

Proof. Using Proposition 10 and [2], Corollary 3.1, $I \in \mathcal{MV}\_\mathbf{I}(\mathbf{A})$, so $C(I) \in \mathbf{F}(\mathbf{A})$ and $A/I = A/C(I)$ is an MV-algebra. Thus, $C(I)$ is a fantastic filter of A, see [1]. We deduce that $x^{**} \to x \in C(I)$, so $(x^{**} \to x)^* \in I$, for every $x \in A$. □

As for ideals, if $\mathcal{S} \subseteq \mathcal{RL}$ is a subvariety, $F \in \mathbf{F}(\mathbf{A})$ is called a $\mathcal{S}$ *-filter* if $A/F \in \mathcal{S}$. Also, if $F, G \in \mathbf{F}(\mathbf{A})$ and F is a $\mathcal{S}$-filter such that $F \subseteq G$, then G is a $\mathcal{S}$-filter, see [1]. In particular, the set of MV-filters is denoted by $\mathcal{MV}\_\mathbf{F}(\mathbf{A})$.

For MV-ideals we have similar results:

Proposition 12. *Let $A \in \mathcal{RL}$ and $I, J \in \mathbf{I}(\mathbf{A})$ proper ideals such that $I \subseteq J$. Then $I \in \mathcal{MV}\_\mathbf{I}(\mathbf{A}) \Longrightarrow J \in MV\_\mathbf{I}(\mathbf{A})$.*

Proof. Since $I \in \mathcal{MV}\_\mathbf{I}(\mathbf{A})$ we deduce that A/I is an MV-algebra. Thus, using Proposition 10, (ii), $A/C(I)$ is also MV-algebra, so $C(I) \in \mathcal{MV}\_\mathbf{F}(\mathbf{A})$. Obviously, $I \subseteq J$ implies $C(I) \subseteq C(J)$. We obtain that $C(J) \in \mathcal{MV}\_\mathbf{F}(\mathbf{A})$, so $A/C(J) = A/J$ is an MV-algebra. Thus, $J \in MV\_\mathbf{I}(\mathbf{A})$. □

Using this result we deduce that:

Theorem 13. *In a residuated lattice* A,

$$\{0\} \in \mathcal{MV}\_\mathbf{I}(\mathbf{A}) \textit{ iff any proper ideal is an MV-ideal.}$$

Proposition 14. *Let* $A \in \mathcal{RL}$ *and* $I \in \mathbf{I}(\mathbf{A})$. *Then*

$$I \in \mathcal{MV}\_\mathbf{I}(\mathbf{A}) \textit{ iff } C(I) \in \mathcal{MV}\_\mathbf{F}(\mathbf{A}).$$

Proof. Using Proposition 10, $I \in \mathcal{MV}\_\mathbf{I}(\mathbf{A})$ iff A/I is an MV-algebra iff $A/C(I)$ is an MV-algebra iff $C(I) \in \mathcal{MV}\_\mathbf{F}(\mathbf{A})$. □

As for lattices, in $A \in \mathcal{RL}$, a particular set involving complement elements is $D(A) = \{d \in A : d^* = 0\}$.

Remark 15. $d \in D(A)$ *iff* $d \to r = r$, *for every* $r \in A$ *with* $r = r^{**}$.

Next, we establish new characterizations for MV-ideals in residuated lattices:

Proposition 16. *Let* $A \in \mathcal{RL}$. *Then*

$$D(A) \in \mathcal{MV}\_\mathbf{F}(\mathbf{A}) \text{ iff any proper ideal is an MV-ideal.}$$

Proof. Since $D(A) \in \mathbf{F}(\mathbf{A})$, the equivalences follow by Theorem 13 and Proposition 14 using the fact that $\{0\} \in \mathcal{MV}\_\mathbf{I}(\mathbf{A})$ iff $C(\{0\}) = \{x \in A : x^* = 0\} = D(A) \in \mathcal{MV}\_\mathbf{F}(\mathbf{A})$. □

Remark 17. *In particular, if* $A \in \mathcal{RL}$ *is an MV-algebra, then every ideal is an MV-ideal.*

Finally, we study the transport of ideals and filters through morphisms of residuated lattices.

Let $A_1, A_2 \in \mathcal{RL}$ and $f : A_1 \to A_2$ be a *morphism of residuated lattices,* see [12]. Then, $Ker(f) = f^{-1}(1) \in \mathbf{F}(\mathbf{A}_1)\backslash\{A_1\}$ and $i - Ker(f) = f^{-1}(0) \in \mathbf{I}(\mathbf{A}_1)\backslash\{A_1\}$, see [10] and [11].

Proposition 18. *If* $f : A_1 \to A_2$ *is a morphism in* $\mathcal{RL}$, *then:*

(*i*) *If* F_2 *is a proper (prime, maximal) filter in* A_2 *then* $f^{-1}(F_2)$ *is a proper (prime, maximal) filter in* A_1;

(*ii*) *If* f *is surjective and* $F_1 \in \mathbf{F}(\mathbf{A}_1)$ *then* $f(F_1) \in \mathbf{F}(\mathbf{A}_2)$;

(*iii*) *If* f *is surjective and* $F_1 \in \mathbf{FM}(\mathbf{A}_1)$ *such that* $f(F_1)$ *is proper, then* $f(F_1) \in \mathbf{FM}(\mathbf{A}_2)$.

Proof. As in the case of BL-algebras, see [8]. □

Proposition 19. *Let* A_1, $A_2 \in \mathcal{RL}$ *and* $f : A_1 \to A_2$ *be a morphism of residuated lattices. Then:*

(*i*) *If* $I_2 \in \mathbf{I}(\mathbf{A}_2)\backslash\{A_2\}$ *then* $f^{-1}(I_2) \in \mathbf{I}(\mathbf{A}_1)\backslash\{A_1\}$;

(*ii*) $I_2 \in \mathcal{MV}\_\mathbf{I}(\mathbf{A}_2) \Longrightarrow f^{-1}(I_2) \in \mathcal{MV}\_\mathbf{I}(\mathbf{A}_1)$;

(*iii*) *If* f *is surjective,* $I_1 \in \mathbf{I}(\mathbf{A}_1) \Longrightarrow f(I_1) \in \mathbf{I}(\mathbf{A}_2)$;

(*iv*) *If* f *is surjective and* $I_1 \in \mathcal{MV}\_\mathbf{I}(\mathbf{A}_1) \Longrightarrow f(I_1) \in \mathcal{MV}\_\mathbf{I}(\mathbf{A}_2)$.

Proof. (*i*). Clearly, $f^{-1}(I_2) \in \mathbf{I}(\mathbf{A}_1)$. Moreover, if I_2 is proper and $f^{-1}(I_2)$ $= A_1$, then $1 \in f^{-1}(I_2)$ and $1 = f(1) \in I_2$. Obviously, $I_2 = A_2$, a contradiction. We conclude that $f^{-1}(I_2)$ is proper.

(*ii*). If $x \in A_1$, then $f(x) \in A_2$, so $(x^{**} \to x)^* \in f^{-1}(I_2)$. Thus, $f^{-1}(I_2) \in \mathcal{MV}\_\mathbf{I}(\mathbf{A}_1)$.

(*iii*). As for MV-algebras, see [3].

(*iv*). Let $y \in A_2$. Then $y = f(x)$ with $x \in A_1$ and $(y^{**} \to y)^* = f((x^{**} \to x)^*) \in f(I_{1.})$. □

Proposition 20. *Let* $f : A_1 \to A_2$ *be a morphism in* $\mathcal{RL}$. *Then:*

(*i*) $Ker(f) \subseteq C(i - Ker(f))$; *If* A_2 *verifies (DN), then* $Ker(f) = C(i - Ker(f))$;

(*ii*) $C(Ker(f)) = i - Ker(f)$.

Proof. (*i*). $x \in Ker(f) \Rightarrow x^* \in i - Ker(f) \Rightarrow x \in C(i - Ker(f))$. If A_2 verifies (DN) then $x \in C(i - Ker(f)) \Rightarrow x^* \in i - Ker(f) \Rightarrow x \in Ker(f)$.

(*ii*). $x \in C(Ker(f))$ iff $x^* \in Ker(f)$ iff $x \in i - Ker(f)$. □

Proposition 21. *Let* $A_1, A_2 \in \mathcal{RL}$, $S_1 \subseteq A_1, S_2 \subseteq A_2$ *and* $f : A_1 \to A_2$ *be a morphism of residuated lattices. Then:*

(*i*) $C(f^{-1}(S_2)) = f^{-1}(C(S_2))$;

(ii) *If* $F_2 \in \mathbf{F}(\mathbf{A}_2)$, $I_2 \in \mathbf{I}(\mathbf{A}_2)$ *then*

$$C(f^{-1}(F_2)) = f^{-1}(C(F_2)) \text{ and } C(f^{-1}(I_2)) = f^{-1}(C(I_2));$$

(iii) $f(C(S_1)) \subseteq C(f(S_1))$;

(iv) *If* f *is surjective then*

$$f(C(F_1)) = C(f(F_1)) \text{ and } f(I_1) = C(f(C(I_1))), \text{ for } F_1 \in \mathbf{F}(\mathbf{A}_1),\ I_1 \in \mathbf{I}(\mathbf{A}_1).$$

Proof. (i). $x \in f^{-1}(C(S_2))$ iff $f(x)^* = f(x^*) \in S_2$ iff $x^* \in f^{-1}(S_2)$ iff $x \in C(f^{-1}(S_2))$.

(ii). Using (i).

(iii). Let $y \in f(C(S_1))$. Then there exists $x \in C(S_1)$ such that $y = f(x)$. Thus, $y^* = f(x^*)$, with $x^* \in S_1$. We deduce that $y^* \in f(S_1)$, so $y \in C(f(S_1))$.

(iv). Using (iii), $f(C(F_1)) \subseteq C(f(F_1))$. Let $y \in C(f(F_1))$. Then $y^* \in f(F_1)$, so there exists $x \in F_1$ such that $y^* = f(x)$.

Using Proposition 10, $F_1 \subseteq C(C(F_1))$, so $x \in C(C(F_1))$. Thus, $y^{**} = f(x^*)$, with $x^* \in C(F_1)$, so $y^{**} \in f(C(F_1))$. Since by Proposition 10, $C(F_1) \in \mathbf{I}(\mathbf{A}_1)$ using Proposition 19, (iii), we have that $f(C(F_1)) \in \mathbf{I}(\mathbf{A}_2)$. Thus, $y \in f(C(F_1))$, so $C(f(F_1)) \subseteq f(C(F_1))$. We conclude that, $f(C(F_1)) = C(f(F_1))$.

Also, using Proposition 10, $C(I_1) \in \mathbf{F}(\mathbf{A}_1)$, so, $f(I_1) = f(C(C(I_1))) = C(f(C(I_1)))$. □

Theorem 22. *Let* $f : A_1 \to A_2$ *be a morphism in* $\mathcal{RL}$. *Then:*

(i) *If* $I_2 \in \mathbf{IM}(A_2)$ *then* $f^{-1}(I_2) \in \mathbf{IM}(A_1)$ *;*

(ii) *If* A_1 *satisfies (DM) and* $I_2 \in \mathbf{IP}(A_2)$ *then* $f^{-1}(I_2) \in \mathbf{IP}(A_1)$*;*

(iii) *If* f *is surjective and* $I_1 \in \mathbf{IM}(A_1)$ *such that* $f(C(I_1)) \neq A_2$ *then* $f(I_1) \in \mathbf{IM}(A_2)$ *.*

Proof. (i). Using Proposition 19, (i), $f^{-1}(I_2) \in \mathbf{I}(\mathbf{A}_1)\backslash\{A_1\}$. From Proposition 10, (iv), $I_2 \in \mathbf{IM}(A_2)$ implies that $C(I_2) \in \mathbf{FM}(A_2)$. Using Proposition 18, (i), we have $f^{-1}(C(I_2)) \in \mathbf{FM}(A_1)$. From Proposition 21, (ii), we obtain $C(f^{-1}(I_2)) = f^{-1}(C(I_2))$, so we deduce that $C(f^{-1}(I_2)) \in \mathbf{FM}(A_1)$. From Proposition 10, (iv), we obtain that $C(C(f^{-1}(I_2))) \in \mathbf{IM}(A_1)$. But $C(C(f^{-1}(I_2))) = f^{-1}(I_2)$, since $f^{-1}(I_2) \in \mathbf{I}(\mathbf{A}_1)$, see Proposition 10. We conclude that $f^{-1}(I_2) \in \mathbf{IM}(A_1)$.

(ii). Obviously, by Proposition 19, (i), $f^{-1}(I_2)$ is a proper ideal in A_1. Using Proposition 10, (iii), $I_2 \in \mathbf{IP}(A_2)$ implies that $C(I_2) \in \mathbf{FP}(A_2)$. From Proposition 18, (i), we deduce that $f^{-1}(C(I_2)) \in \mathbf{FP}(A_1)$. Since from Proposition

21, (ii), $C(f^{-1}(I_2)) = f^{-1}(C(I_2))$, we have that $C(f^{-1}(I_2)) \in \mathbf{FP}(A_1)$. Using Proposition 10, (iii), since A_1 satisfies (DM) , we obtain that $C(C(f^{-1}(I_2))) \in \mathbf{IP}(A_1)$. But $C(C(f^{-1}(I_2))) = f^{-1}(I_2)$, so $f^{-1}(I_2) \in \mathbf{IP}(A_1)$.

(iii). From Proposition 10, (iv), $I_1 \in \mathbf{IM}(A_1)$ implies that $C(I_1) \in \mathbf{FM}(A_1)$. From Proposition 18, (iii), we deduce that $f(\ C(I_1)) \in \mathbf{FM}(A_2)$. Using Proposition 10, (iv), we have that $C(f(C(I_1))) \in \mathbf{IM}(A_2)$. But, $f(I_1) = C(f(C(I_1)))$, see Proposition 21, (iv), thus $f(I_1) \in \mathbf{IM}(A_2)$. $\square$

4 Conclusions

In this paper we develop an algebraic analysis between filters and ideals in residuated lattices.

In a future work, we intend to use the operation $\boxplus$ to translate some results from MV-algebras for this more general case and we will search for new properties of the operator C which will help us establish other connections between filters and ideals.

References

[1] Buşneag, D., Piciu, D., *A new approach for classification of filters in residuated lattices,* Fuzzy Sets and Systems, **260** (2015), 121-130.

[2] Buşneag, D., Piciu, D., Dina, A., *Ideals in residuated lattices,* Carpathian J. Math., **37** (1) (2021), 53-63.

[3] Cignoli, R., D'Ottaviano, I.M.L., Mundici, D., *Algebraic Foundations of many-valued Reasoning*, Trends in Logic-Studia Logica Library 7, Dordrecht: Kluwer Academic Publishers, 2000.

[4] Idziak, P. M. , *Lattice operations in BCK-algebras*, Math. Japonica, **29** (1984), 839-846.

[5] Iorgulescu, A., *Algebras of logic as BCK algebras,* A.S.E., Bucharest, 2009.

[6] Kondo, M., *States on bounded commutative residuated lattices*, Math. Slovaca, **64** (2014), 1093-1104.

[7] Lele, C., Nganou, J.B., *MV-algebras derived from ideals in BL-algebras*, Fuzzy Sets and Systems **218** (2013), 103-113.

[8] Leustean, L.,*The prime and maximal spectra and the reticulation of BL-algebras*, Centr. Eur. J. Math., **1** (3) (2003), 382-397.

[9] Liu, Y., Qin, Y., Qin, X., Xu, Y., *Ideals and fuzzy ideals in residuated lattices*, Int. J. Math. Learn & Cyber, **8** (2017), 239-253.

[10] Piciu, D., *Algebras of fuzzy logic*, Ed. Universitaria, Craiova, 2007.

[11] Piciu, D., *Prime, minimal prime and maximal ideals spaces in residuated lattices*, Fuzzy Sets and Systems, **405** (2021), 47-64.

[12] Turunen, E., *Mathematics Behind Fuzzy Logic*, Physica-Verlag, 1999.

[13] Ward, M., Dilworth, R. P., *Residuated lattices*, Trans. Amer. Math. Soc. **45**(3) (1939), 335-354.

[14] Woumfo, F., Koguep Njionou, B., Alomo, E., Lele, C., *Ideals and Bosbach states on residuated lattices,* New Mathematics and Natural Computation, **16**(3) (2020), 551-571.

Dana PICIU,
Department of Mathematics,
Faculty of Science,
University of Craiova,
A. I. Cuza Street 13, 200585, Craiova, Romania.
Email: dana.piciu@edu.ucv.ro; piciudanamarina@yahoo.com

Christina DAN,
Department of Mathematics,
Faculty of Science,
University of Craiova,
A. I. Cuza Street 13, 200585, Craiova, Romania.
Email: christina.dan@edu.ucv.ro; christi_the_d@yahoo.com

Florentina BOBOC,
Department of Mathematics,
Faculty of Science,
University of Craiova,
A. I. Cuza Street 13, 200585, Craiova, Romania.
Email: florentina.boboc@edu.ucv.ro; florentinaclaudiaboboc@yahoo.com