DOI: 10.2478/auom-2024-0023
An. Şt. Univ. Ovidius Constanţa **Vol. 32(2),2024, 127–144**

On Boyd-Wong type multivalued contractions and solvability of $(k-\chi)$-Hilfer fractional differential inclusions

Marija Paunovic, Babak Mohammadi and Vahid Parvaneh

Abstract

In this article, we introduce the Boyd-Wong type multivalued contractions and demonstrate that such mappings have a fixed point. Additionally, we look at the solvability of a few $(k-\chi)$-Hilfer initial value fractional differential inclusions of order $n-1<\alpha<n$ $(n\geq 2)$. To demonstrate the usability of our result, an example is provided.

1 Introduction

Numerous technological sectors have used fractional integro-differential operators to investigate the mathematical description of physical processes. The reader is directed to [13], [14], [15], [23], [24], [25] and [30] to view a number of published articles in this respect. The most useful fractional operators among these efforts are the Riemann-Liouville and Caputo integro-differential operators. The χ-Caputo fractional derivative (c.f.d.), which is the fractional derivative with regard to another strictly rising differentiable function, was recently introduced in [10] and used in [12] and [16].

Then, various scholars applied this operator to a variety of topics (see, for instance, the citations for [10], [18], [19], [20], [21], and [22]).There is no doubt that the Riemann-Liouville, Caputo, Hadamarad, and Erdélyi-Kober

Key Words: Boyd-Wong contraction, multi-valued mapping, $(k-\ell)$-Hilfer initial value fractional differential inclusions.
2010 Mathematics Subject Classification: Primary 47H10; Secondary 54H25.
Received: 12.06.2023
Accepted: 30.10.2023

integro-differential operators are particular instances of the χ-c.f.d.. On the other hand, Hilfer introduced the Hilfer type fractional derivative in reference [9]. Many of the well-known fractional derivative operators are generalized by the $(k-\chi)$-Hilfer operator, which was presented in [3]. Recently, Tariboon et al. employed this operator for multi-point initial value $(k-\chi)$-Hilfer fractional differential equations and inclusions utilizing the Nadler's contraction and Banach contraction principle.

Distinctly, they study the following multi-point initial value fractional differential inclusion of order $1<\rho<2$:

$$\begin{cases} {}^{k}\mathfrak{D}_{a^+}^{\rho,\varrho;\chi}\varsigma(t) \in \mathfrak{F}(t,\varsigma(t)),\ t\in[a,b], \\ \varsigma(a)=0,\ \varsigma(b)=\Sigma_{i=1}^{m}\lambda_i\varsigma(\varepsilon_i), \end{cases} \tag{1}$$

where ${}^{k}\mathfrak{D}_{a^+}^{\rho,\varrho;\chi}$ is the $(k-\chi)$-Hilfer fractional derivative (h.f.c.) operator introduced in [3], $\mathfrak{F}:[a,b]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multi-valued compact valued function, $1<\rho<2$, the increasing function $\chi\in C^1([a,b])$ is such that $\chi'(t)\neq 0$ for all $t\in[a,b]$, $a<\varepsilon_i<b$, $i=1,2,3,\cdots,m$ and $\lambda_i\in\mathbb{R}$.

In this research, we pursue two objectives: First, we introduce the multi-valued contraction mappings of Boyd-Wong (B.-W.) type and demonstrate that they have a fixed point. Second, we demonstrate the solvability of a few $(k-\chi)$-Hilfer fractional differential inclusions of any order $n-1<\rho<n$ using our novel contraction with the following single-point initial value conditions:

$$\begin{cases} {}^{k}\mathfrak{D}_{a^+}^{\rho,\varrho;\chi}\kappa(t) \in \mathfrak{F}(t,\kappa(t));\ t\in[a,b], \\ {}^{k}\mathfrak{I}_{a^+}^{nk-\varsigma_k;\chi}\kappa(t)]_{t=a}=0, \\ \left(\frac{k}{\chi'(t)}\frac{d}{dt}\right)^{n-j}{}^{k}\mathfrak{I}_{a^+}^{nk-\varsigma_k;\chi}\kappa(t)]_{t=a}=d_j, j=1,2,\cdots,n-1,\ \varsigma_k=\rho+\varrho(nk-\rho), \end{cases} \tag{2}$$

when the right hand side function $\mathfrak{F}:[a,b]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ appraises the multi-valued mappings, it does not necessarily appraise the Nadler's contraction, but rather the B.-W. type contraction. We provide an illustration to demonstrate how our new findings are usable.

2 Preliminaries and auxiliary notions

Let (∇,d) be a metric space. Following [17], let $\mathcal{P}_{cb}(\nabla)$ be the class of all nonempty closed bounded subsets of ∇ and $\mathcal{H}$ be the Hausdorff-Pompieu metric on $\mathcal{P}_{cb}(\nabla)$ which is induced by the metric d, that is,

$$\mathcal{H}(\Upsilon_1,\Upsilon_2)=\max\left\{\sup_{\varsigma_1\in\Upsilon_1} d(\varsigma_1,\Upsilon_2),\ \sup_{\varsigma_2\in\Upsilon_2} d(\varsigma_2,\Upsilon_1)\right\},$$

for every $\Upsilon_1, \Upsilon_2 \in \mathcal{P}_{cb}(\nabla)$.

A fixed point of a multi-valued mapping $Ł : \nabla \to \mathcal{P}(\nabla)$ is an element $\varsigma \in \nabla$ such that $\varsigma \in Ł\varsigma$. Also, Ł possesses the approximate valued property whenever for any $x \in \nabla$ there exists $y \in Łx$ such that $d(x, y) = d(x, Łx)$.

Let's review the basic definitions of fractional differential equations (see [26] and [27] for references).

For a continuous function $f : [a, b] \to \mathbb{R}$, the Riemann-Liouville integral (r.l.i.) of fractional order $\rho \geq 0$ is defined by

$$\mathfrak{J}^{\rho}_{a^+} f(\ell) = \frac{1}{\Gamma(\rho)} \int_a^{\ell} (\ell - u)^{\rho-1} f(u) du; \ \rho > 0, \tag{3}$$

and $\mathfrak{J}^0_a f(\ell) = f(\ell)$ for $\rho = 0$ (for agreement). The c.f.d. of order ρ is defined by

$$^{c}\mathfrak{D}^{\rho}_{a^+} f(\ell) = \mathfrak{J}^{n-\rho}_a f^{(n)}(\ell) \ \ (n - 1 < \rho \leq n, n = \lceil \rho \rceil), \tag{4}$$

where $\lceil . \rceil : \mathbb{R} \to \mathbb{Z}$ denotes the ceiling function. The Riemann-Liouville fractional derivative (r.l.f.d.) of order ρ is defined by

$$^{RL}\mathfrak{D}^{\rho}_{a^+} f(\ell) = \left(\frac{d}{d\ell}\right)^n \mathfrak{J}^{n-\rho}_a f(\ell) \ \ (n - 1 < \rho \leq n, n = \lceil \rho \rceil). \tag{5}$$

Definition 1. Let χ be an increasing map so that $\chi'(s) > 0$ for any $s \in [a, b]$. Then, the χ-r.l.i. of order ρ of an integrable function $f : [a, b] \to \mathbb{R}$ with respect to χ is defined as

$$\mathfrak{J}^{\rho;\chi}_{a^+} f(\ell) = \begin{cases} \frac{1}{\Gamma(\rho)} \int_a^{\ell} \chi'(\ell)(\chi(\ell) - \chi(u))^{\rho-1} f(u) du, \ \rho > 0 \\ f(\ell), \ \rho = 0, \end{cases} \tag{6}$$

if the right-hand side of the aforementioned equality has finite values.

It should be noted that, if $\chi(u) = u$, then clearly the χ-r.l.i. (6) reduces to the standard r.l.i. (3).

Definition 2. ([12]) Let $n = \lceil \rho \rceil$. For a real mapping $f \in C([a, b], \mathbb{R})$, the χ-r.l.f.d. of order ρ is formulated as

$$^{RL}\mathfrak{D}^{\rho;\chi}_{a^+} f(\ell) = \left(\frac{1}{\chi'(\ell)} \frac{d}{d\ell}\right)^n \mathfrak{J}^{n-\rho;\chi}_{a^+} f(\ell), \tag{7}$$

provided that the right hand side of the above equality is finite-valued.

Similar to this, it is clear that the χ-r.l.f.d. (7) reduces to the traditional r.l.f.d. (5) if $\chi(u) = u$. Almeida presented a novel χ-version of the c.f.d. in the following formulation, which was motivated by these operators:

Definition 3. ([10]) Let $n=\lceil\rho\rceil$ and $\chi\in AC^n([a,b],\mathbb{R})$ be an increasing map with $\chi'(u)>0$ for any $u\in[a,b]$. The χ-c.f.d. of order ρ of f with respect to χ is

$$ {}^{C}\mathfrak{D}^{\rho;\zeta}_{a^+}f(\ell)=\mathfrak{I}^{n-\rho;\zeta}_{a^+}\Big(\frac{1}{\chi'(\ell)}\frac{d}{d\ell}\Big)^n f(\ell), \tag{8}$$

assuming that the right-hand side of this equality has finite values.

It should be understood that, if $\chi(u)=u$, then it is obvious that the χ-c.f.d. of order ρ, in the formula (8) reduces to the standard c.f.d. of order ρ in (4). In the following, some useful specifications of the χ-Caputo and χ-Riemann–Liouville integro-derivative operators can be seen. Let $AC([a,b],\mathbb{R})$ stand for the set of absolutely continuous functions from $[a,b]$ into $\mathbb{R}$. Define $AC^n_\zeta([a,b],\mathbb{R})$ by

$$AC^n_\chi([a,b],\mathbb{R})=\Big\{\ell:[a,b]\to\mathbb{R}|\delta^{n-1}_\chi\ell\in AC([a,b],\mathbb{R}),\delta_\chi=\frac{1}{\chi'(y)}\frac{d}{dy}\Big\}.$$

Lemma 1. *([12]) Let $n=[\rho]+1$. For a real mapping $f\in AC^n([a,b],\mathbb{R})$,*

$$\mathfrak{I}^{\rho;\chi}_{a^+}{}^{c}\mathfrak{D}^{\rho;\chi}_{a^+}f(t)=f(t)-\Sigma^{n-1}_{k=0}\frac{(\delta^k_\chi f)(a)}{k!}(f(t)-f(a))^k, \tag{9}$$

where $\delta^k_\chi=\underbrace{\delta_\chi\delta_\chi\cdots\delta_\chi}_{k\,times}$.

Lemma 2. *([11]) Let $n=[\rho]+1$ and $\rho,\varrho>0$. For a real mapping $f\in C([a,b],\mathbb{R})$ we have:*

(i) $\mathfrak{I}^{\rho;\chi}_{a^+}\mathfrak{I}^{\varrho;\chi}_{a^+}f(t)=\mathfrak{I}^{\rho+\varrho;\zeta}_{a^+}f(t)$,

(ii) ${}^{c}\mathfrak{D}^{\rho;\chi}_{a^+}\mathfrak{I}^{\rho;\chi}_{a^+}f(t)=f(t)$,

(iii) ${}^{c}\mathfrak{D}^{\rho;\chi}_{a^+}(\chi(t)-\chi(a))^{\varrho-1}=\frac{\Gamma(\varrho)}{\Gamma(\varrho-\rho)}(\chi(t)-\chi(a))^{\varrho-\rho-1}$,

(iv) $\mathfrak{I}^{\rho;\chi}_{a^+}(\chi(t)-\chi(a))^{\varrho-1}=\frac{\Gamma(\varrho)}{\Gamma(\varrho+\rho)}(\chi(t)-\chi(a))^{\varrho+\rho-1}$,

(v) ${}^{c}\mathfrak{D}^{\rho;\chi}_{a^+}(\chi(t)-\chi(a))^{k}=0,\ k=0,1,2,...,n-1.$

Mubeen and Habibullah [6] extended the r.l.f.i. operator to k-r.l.f.i. of order ρ as

$$ {}^{k}\mathfrak{I}^{\rho}_{a^+}\mathfrak{h}(\ell)=\frac{1}{k\Gamma_k(\rho)}\int_a^\ell(\ell-u)^{\frac{\rho}{k}-1}\mathfrak{h}(u)du;\ \rho>0, \tag{10}$$

and ${}^k\mathcal{J}_a^0 f(\ell) = f(\ell)$ for $\rho = 0$ (for agreement), where $\mathfrak{h} \in L^1([a,b],\mathbb{R}), k > 0$ and Γ_k is the k-Gamma function which is defined in [5] by

$$\Gamma_k(\mu) = \int_0^\infty t^{\mu-1} e^{-\frac{t^k}{k}} dt$$

where $\mu \in \mathbb{C}$ with $Re(\mu) > 0$, $k \in \mathbb{R}$ and $k > 0$.

The following equalities are met, as is widely known:

(i) $\Gamma_k(\mu) = k^{\frac{\mu}{k}-1}\Gamma(\frac{\mu}{k})$,

(ii) $\Gamma_k(\mu + k) = \mu\Gamma_k(\mu)$,

(iii) $\lim_{k\to 1}\Gamma_k(\mu) = \Gamma(\mu)$.

Dorrego in [4] introduced the k-r.l.f.d. for a mapping $\mathfrak{h} \in C([a,b],\mathbb{R})$ of order $n-1 < \rho \le n$ as

$$^{k,RL}\mathfrak{D}_{a^+}^{\rho}\mathfrak{h}(\ell) = \Big(k\frac{d}{d\ell}\Big)^n {}^k\mathfrak{I}_{a^+}^{nk-\rho}\mathfrak{h}(\ell), n = \lceil \frac{\rho}{k} \rceil. \tag{11}$$

Kucche and Mali in [3] introduced the k-c.f.d. as

$$^{k,C}\mathfrak{D}_{a^+}^{\rho}\mathfrak{h}(\ell) = {}^k\mathfrak{I}_{a^+}^{nk-\rho}\Big(k\frac{d}{d\ell}\Big)^n \mathfrak{h}(\ell), n = \lceil \frac{\rho}{k} \rceil. \tag{12}$$

Sousa and Oliveira [22] defined the χ-h.f.c. of the function $\mathfrak{h} \in C([a,b],\mathbb{R})$ of order $n-1 < \rho \le n$ and of type $\varrho \in [0,1]$ for a $\chi \in C^n([a,b],\mathbb{R})$ such that $\chi'(\ell) \neq 0$ ($\ell \in [a,b]$) as

$$^{H}\mathfrak{D}_{a^+}^{\rho,\varrho;\chi}\mathfrak{h}(\ell) = \mathfrak{I}_{a^+}^{\varrho(n-\rho);\chi}\Big(\frac{1}{\chi'(\ell)}\frac{d}{d\ell}\Big)^n \mathfrak{I}_{a^+}^{(1-\varrho)(n-\rho);\chi}\mathfrak{h}(\ell), n = \lceil \frac{\rho}{k} \rceil. \tag{13}$$

In [2], the $(k-\zeta)$-r.l.f.i. of order $\rho \ge 0$ for a function $\mathfrak{h} \in L^1([a,b],\mathbb{R})$ and for a $k > 0$ is defined as

$$^{k}\mathfrak{I}_{a^+}^{\rho;\chi}\mathfrak{h}(\ell) = \begin{cases} \frac{1}{k\Gamma_k(\rho)}\int_a^\ell \chi'(\ell)(\chi(\ell)-\chi(u))^{\frac{\rho}{k}-1}\mathfrak{h}(u)du \\ \mathfrak{h}(\ell), \ \rho = 0. \end{cases} \tag{14}$$

Kucche and Mali in [3] defined the $(k-\chi)$-h.f.c. for a function $\mathfrak{h} \in C([a,b],\mathbb{R})$ of order $\rho \ge 0$ and for a $k > 0$ and of type $\varrho \in [0,1]$ w.r.t. a $\chi \in C^n([a,b],\mathbb{R})$ for which $\chi'(\ell) \neq 0$ ($\ell \in [a,b]$) as

$$^{k,H}\mathfrak{D}_{a^+}^{\rho,\varrho;\chi}\mathfrak{h}(\ell) = {}^k\mathfrak{I}_{a^+}^{\varrho(nk-\rho);\chi}\Big(\frac{k}{\chi'(\ell)}\frac{d}{d\ell}\Big)^n \ {}^k\mathfrak{I}_{a^+}^{(1-\varrho)(nk-\rho);\chi}\mathfrak{h}(\ell), n = \lceil \frac{\rho}{k} \rceil. \tag{15}$$

Remark 1. (i) Taking $\varrho = 0$, (15) reduces to the $(k-\chi)$-r.l.f.d.

$$ {}^{k,RL}\mathfrak{D}^{\rho;\chi}_{a^+}\mathfrak{h}(\ell) = \left(\frac{k}{\chi'(\ell)}\frac{d}{d\ell}\right)^n {}^k\mathfrak{J}^{(nk-\rho);\chi}_{a^+}\mathfrak{h}(\ell). \tag{16} $$

Also, taking $\chi(\ell) = \ell$ in (16), it is reduced to the k-r.l.f.d. (11).

(ii) Taking $\varrho = 1$, (15) reduces to the $(k-\chi)$-c.f.d.

$$ {}^{k,C}\mathfrak{D}^{\rho;\chi}_{a^+}\mathfrak{h}(\ell) = {}^k\mathfrak{J}^{(nk-\rho);\chi}_{a^+}\left(\frac{k}{\chi'(\ell)}\frac{d}{d\ell}\right)^n \mathfrak{h}(\ell) \tag{17} $$

and taking $\chi(\ell) = \ell$ in (17), it reduced to the k-c.f.d. (12).

(iii) Taking $\chi(\ell) = \ell^p$, (15) reduces to the k-Hilfer-Katugampola fractional derivative in [1]:

(a) If $\chi(\ell) = \ell^p$ and $\varrho = 0$, then (15) reduces to the k-Katugampola fractional derivative in [1],

(b) If $\chi(\ell) = \ell^p$ and $\varrho = 1$,then (15) reduces to the k-Caputo-Katugampola fractional derivative in [1];

(iv) Taking $\chi(\ell) = \log \ell$, (15) reduces to the k-Hilfer-Hadamarad fractional derivative in [3]:

(a) If $\chi(\ell) = \log \ell$ and $\varrho = 0$, then (15) reduces to the k-Hadamarad fractional derivative in [3],

(b) If $\chi(\ell) = \log \ell$ and $\varrho = 1$, then (15) reduces to the k-Caputo-Hadamarad fractional derivative in [3].

Remark 2. If $\varsigma_k = \rho + \varrho(nk-\rho)$, then $\varrho(nk-\rho) = \varsigma_k - \rho$ and $(1-\varrho)(nk-\rho) = nk - \varsigma_k$. Therefore, the $(k-\chi)$-Hilfer fractional derivative will get the $(k-\chi)$-r.l.f.d. form as

$$ \begin{aligned} {}^{k,H}\mathfrak{D}^{\rho,\varrho;\chi}_{a^+}\mathfrak{h}(\ell) &= {}^k\mathfrak{J}^{\varsigma_k-\rho;\chi}_{a^+}\left(\tfrac{k}{\chi'(\ell)}\tfrac{d}{d\ell}\right)^n {}^k\mathfrak{J}^{nk-\varsigma_k;\chi}_{a^+}\mathfrak{h}(\ell) \\ &= {}^k\mathfrak{J}^{\varsigma_k-\rho;\chi}_{a^+}\left({}^{k,RL}\mathfrak{D}^{\varsigma_k;\chi}_{a^+}\mathfrak{h}\right)(\ell), \end{aligned} \tag{18} $$

Note that for $n-1 < \frac{\rho}{k} \le n$, we have $n-1 < \frac{\varsigma_k}{k} \le n$.

Lemma 3. *([8]) Let $\rho, k \in \mathbb{R}^+ = (0,\infty)$ and $n = \lceil \frac{\rho}{k} \rceil$. Assume that $\mathfrak{h} \in C^n([a,b],\mathbb{R})$ and ${}^k\mathfrak{J}^{nk-\rho;\chi}_{a^+}\mathfrak{h} \in C^n([a,b],\mathbb{R})$. Then*

$$ \begin{aligned} &{}^k\mathfrak{J}^{\rho;\chi}_{a^+}\left({}^{k,RL}\mathfrak{D}^{\rho;\chi}_{a^+}\mathfrak{h}\right)(\ell) \\ &= \mathfrak{h}(\ell) - \textstyle\sum_{j=1}^{n-1} \frac{(\chi(\ell)-\chi(a))^{\frac{\rho}{k}-j}}{\Gamma_k(\rho-jk+k)} \left[\left(\tfrac{k}{\chi'(\ell)}\tfrac{d}{d\ell}\right)^{n-j} {}^k\mathfrak{J}^{nk-\rho;\chi}_{a^+}\mathfrak{h}(\ell)\right]_{\ell=a} \end{aligned} \tag{19} $$

Lemma 4. *([8]) Let $\rho, k \in \mathbb{R}^+ = (0,\infty)$ with $\rho < k$ and $\varrho \in [0,1]$. Assume that $\varsigma_k = \rho + \varrho(k-\rho)$ and $\mathfrak{h} \in C^n([a,b],\mathbb{R})$. Then*

$$ {}^k\mathfrak{J}_{a^+}^{\rho;\chi}({}^{k,H}\mathfrak{D}_{a^+}^{\rho,\varrho;\chi}\mathfrak{h})(\ell) \quad = {}^k\mathfrak{J}_{a^+}^{\varsigma_k;\chi}\left({}^{k,RL}\mathfrak{D}_{a^+}^{\varsigma_k;\chi}\mathfrak{h}\right)(\ell), \tag{20} $$

For some relations on k-r.l.f.d.s we refer the reader to [28, 29, 30, 31, 32, 33] and references therein. Some properties of $(k-\chi)$-h.f.c.s are investigated in [3]. Moreover, by applying the famous Banach contraction principle, they studied the solvability of the following initial value problem involving $(k-\chi)$-h.f.c.

$$ \begin{cases} {}^{k,H}\mathfrak{D}_{a^+}^{\rho,\varrho;\chi}\kappa(t) = f(t,\kappa(t)),\ t \in [a,b] \\ {}^k\mathfrak{J}^{k-\varsigma_k;\chi}\kappa(a) = x_a \in \mathbb{R}, \varsigma_k = \rho + \varrho(k-\rho) \end{cases} \tag{21} $$

where ${}^k\mathfrak{D}_{a^+}^{\rho,\varrho;\chi}$ is the $(k-\chi)$-h.f.c. operator of order $0 < \rho \leq 1$ and of type $\varrho \in [0,1]$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and the increasing function $\chi \in C^1([a,b])$ is such that $\chi'(t) \neq 0$ $(t \in [a,b])$.

Recently, Tariboon et al. [8] used this operators to multi-point initial value $(k-\chi)$-Hilfer fractional differential equations and inclusions at form (1) using the Banach contraction principle and the Nadler contraction principle.

This paper is currently in the following state: In Section 3, we provide the B.-W. type contraction for multivalued mappings and demonstrate that such mappings have a fixed point. In Section 4, we use our new contraction to demonstrate that the $(k-\chi)$-Hilfer fractional differential inclusion of any order $n-1 < \rho \leq< n$ with the single-point initial value condition (2) is solvable. However, the right-hand side function $\mathfrak{F} : [a,b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ does not always teach the Nadler contraction for multi-valued mappings. We provide an illustration to demonstrate how our new findings are usable.

3 Main results

The set of all functions $\eta : [0,\infty) \to [0,\infty)$ so that

(δ_1) η is continuous,

(δ_2) $\lim_{n\to\infty} \eta(t_n) = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} t_n = 0$, for all $(t_n) \subseteq [0,\infty)$,

is denoted by Ψ.

Some examples of elements of Ψ are the following functions defined on $[0,\infty)$:

(i) $\eta_1(t) = a_1 t + a_2 t^2 + \cdots + a_n t^n$, $n \in \mathbb{N}, a_i \geq 0$, $i = 1, 2, \cdots n$,

(ii) $\eta_2(t) = \ln(t+1)$

(iii) $\eta_3(t) = \frac{t}{t+1}$,

(iv) $\eta_4(t) = \sin t + t$,

(v) $\eta_5(t) = \sqrt[m]{t^n}$, $m, n \in \mathbb{N}$.

The collection of all functions $\zeta : [0,\infty) \to [0,\infty)$ such that

(δ_1) ζ is nondecreasing and upper semi continuous,

(δ_2) $\zeta(t) < t$ for all $t > 0$,

is denoted by Φ.

Some examples of elements of Φ are the following functions defined on $[0,\infty)$:

(i) $\zeta_1(t) = \frac{t}{t+1}$,

(ii) $\zeta_2(t) = kt\,(0 \leq k < 1)$,

(iii) $\zeta_3(t) = \frac{t^2}{t+1}$,

(iv) $\zeta_4(t) = \frac{t^3}{1+t^2}$.

Definition 4. Let (∇, d) be a metric space and $Ł : \nabla \to \mathcal{P}_{cb}(\nabla)$ be a multivalued mapping. We say that Ł is a $(\eta-\zeta)$-B.-W. type multi-valued contraction if there exist $\eta \in \Psi$ and $\zeta \in \Phi$ such that

$$\eta(\mathcal{H}(Łx, Ły)) \leq \zeta(\eta(d(x,y))), \tag{22}$$

for all $x, y \in \nabla$.

Theorem 1. *Let (∇, d) be a complete metric space and $Ł : \nabla \to \mathcal{P}_{cb}(\nabla)$ is a $(\eta-\zeta)$-B.-W. type multi-valued contraction satisfying comparable approximate valued property. Moreover, let $\lim_{n\to\infty} \mathcal{H}(\{x_n\}, Łx_n) = 0$, for any sequence (x_n) with $x_{n+1} \in Łx_n$. Then Ł has at least one fixed point.*

Proof. Choose a fixed element $\varsigma_0 \in \nabla$. If $\varsigma_0 \in Ł\varsigma_0$, then we have nothing to prove. Suppose that $\varsigma_0 \notin Ł\varsigma_0$. Since Ł has comparable approximative valued property, there exists $\varsigma_1 \in Ł\varsigma_0$ such that $d(\varsigma_0, Ł\varsigma_0) = d(\varsigma_0, \varsigma_1)$. It is clear that $\varsigma_1 \neq \varsigma_0$. If $\varsigma_1 \in Ł\varsigma_1$, then ς_1 is a fixed point of Ł. Suppose that $\varsigma_1 \notin Ł\varsigma_1$. Then, there exists $\varsigma_2 \in Ł\varsigma_1$ such that $d(\varsigma_1, Ł\varsigma_1) = d(\varsigma_1, \varsigma_2)$. It is clear that $\varsigma_2 \neq \varsigma_1$. By continuing this process, we obtain a sequence $\{\varsigma_n\}$ in ∇ such that $\varsigma_n \in Ł\varsigma_{n-1}$, $\varsigma_n \neq \varsigma_{n-1}$ and $d(\varsigma_{n-1}, \varsigma_n) = d(\varsigma_{n-1}, Ł\varsigma_{n-1})$ for all $n \in \mathbb{N}$.

In view of (22), we obtain that

$$\begin{aligned}\eta(d(\varsigma_{n+1},\varsigma_{n+2})) &= \eta(d(\varsigma_{n+1},\text{Ł}\varsigma_{n+1}))\\ &\leq \eta(\mathcal{H}(\text{Ł}\varsigma_n,\text{Ł}\varsigma_{n+1}))\\ &\leq \zeta(\eta(d(\varsigma_n,\varsigma_{n+1})))\end{aligned} \tag{23}$$

for each $n \geq 0$. Put $t_n := \eta(d(\varsigma_n,\varsigma_{n+1}))$. From (23), we have

$$t_{n+1} \leq \zeta(t_n) \leq t_n, \quad for\ each\ n \geq 0. \tag{24}$$

So, (t_n) is a nonincreasing sequence in $[0,\infty)$ and so there is $r \geq 0$ so that $t_n \to r^+$.

We now demonstrate that $r = 0$. On the contrary, suppose that $r > 0$. Taking the limit through (24), $r \leq \zeta(r)$, which is a contradiction. So, $\lim_{n\to\infty} \eta(d(\varsigma_n,\varsigma_{n+1})) = \lim_{n\to\infty} t_n = r = 0$. Therefore, $\lim_{n\to\infty} d(\varsigma_n,\varsigma_{n+1})) = 0$. We claim that $\{\varsigma_n\}$ is a Cauchy sequence. If $\{\varsigma_n\}$ is not Cauchy, then there are $\varepsilon > 0$ and subsequences $\{\varsigma_{m_i}\}$ and $\{\varsigma_{n_i}\}$ of $\{\varsigma_n\}$ so that $n_i > m_i > i$,

$$d(\varsigma_{m_i},\varsigma_{n_i}) \geq \varepsilon \tag{25}$$

and

$$d(\varsigma_{m_i},\varsigma_{n_i-1}) < \varepsilon. \tag{26}$$

Using (25), we get

$$\varepsilon \leq d(\varsigma_{m_i},\varsigma_{n_i}) \leq d(\varsigma_{m_i},\varsigma_{n_i-1}) + d(\varsigma_{n_i-1},\varsigma_{n_i}) < \varepsilon + d(\varsigma_{n_i-1},\varsigma_{n_i}). \tag{27}$$

As $i \to \infty$, we find

$$\lim_{i\to\infty} d(\varsigma_{m_i},\varsigma_{n_i}) = \varepsilon. \tag{28}$$

Also, we have

$$\begin{aligned}&d(\varsigma_{m_i},\varsigma_{n_i}) - d(\varsigma_{m_i},\varsigma_{m_i+1}) - d(\varsigma_{n_i},\varsigma_{n_i+1})\\ &\leq d(\varsigma_{m_i+1},\varsigma_{n_i+1})\\ &\leq d(\varsigma_{m_i},\varsigma_{m_i+1}) + d(\varsigma_{m_i},\varsigma_{n_i}) + d(\varsigma_{n_i},\varsigma_{n_i+1}).\end{aligned}$$

As $i \to \infty$, we find

$$\lim_{i\to\infty} d(\varsigma_{m_i+1},\varsigma_{n_i+1}) = \varepsilon. \tag{29}$$

Also,

$$\begin{aligned}d(\varsigma_{m_i+1},\varsigma_{n_i+1}) &\leq H(\varsigma_{m_i+1},\text{Ł}\varsigma_{m_i}) + \mathcal{H}(\text{Ł}\varsigma_{m_i},\text{Ł}\varsigma_{n_i}) + H(\varsigma_{n_i+1},\text{Ł}\varsigma_{n_i})\\ &\leq d(\varsigma_{m_i},\varsigma_{m_i+1}) + H(\varsigma_{m_i},\text{Ł}\varsigma_{m_i}) + \mathcal{H}(\text{Ł}\varsigma_{m_i},\text{Ł}\varsigma_{n_i})\\ &\quad + H(\varsigma_{n_i},\text{Ł}\varsigma_{n_i}) + d(\varsigma_{n_i},\varsigma_{n_i+1}).\end{aligned} \tag{30}$$

By (22) and (30), we find

$$\begin{aligned}\eta(\varepsilon) &= \lim_{i\to\infty} \eta(d(\varsigma_{m_i+1}, \varsigma_{n_i+1})) \\ &\leq \lim_{i\to\infty} \eta(\mathcal{H}(\text{Ł}\varsigma_{m_i}, \text{Ł}\varsigma_{n_i})) \\ &\leq \lim_{i\to\infty} \zeta(\eta(d(\varsigma_{m_i}, \varsigma_{n_i}))) \\ &\leq \zeta(\eta(\varepsilon))\end{aligned} \tag{31}$$

a contradiction.

Thus, $\{\varsigma_n\}$ is a Cauchy sequence in the complete metric space (∇, d) and hence there exists $z \in \nabla$ so that

$$\lim_{n\longrightarrow\infty} \varsigma_n = z. \tag{32}$$

We claim that $d(z, \text{Ł}z) = 0$. On the contrary, suppose that $d(z, \text{Ł}z) \neq 0$.

We have

$$\eta(d(\varsigma_{n+1}, \text{Ł}z)) \leq \eta(\mathcal{H}(\text{Ł}\varsigma_n, \text{Ł}z)) \leq \zeta(\eta(d(\varsigma_n, z))). \tag{33}$$

Taking the limit through (33), we obtain $\eta(d(z, \text{Ł}z)) \leq \zeta(\eta(d(z, \text{Ł}z)))$, which is a contradiction. Thus, $d(z, \text{Ł}z) = 0$. Now since Ł has comparable approximate valued property, there exists $u \in \nabla$ such that $u \in \text{Ł}z$ and $d(z, u) = d(z, \text{Ł}z)$. Consequently, $d(z, u) = 0$ and so $z = u \in \text{Ł}z$. The proof is completed. □

Let $\mathcal{P}_{cp}(\nabla)$ be the family of all nonempty compact subsets of ∇.

Corollary 1. *Let (∇, d) be a complete metric space and $\text{Ł} : \nabla \to \mathcal{P}_{cp}(\nabla)$ be a $(\eta-\zeta)$-B.-W. type multi-valued contraction satisfying comparable approximate valued property. Moreover, let $\lim_{n\to\infty} \mathcal{H}(\{x_n\}, \text{Ł}x_n) = 0$, for any sequence (x_n) with $x_{n+1} \in \text{Ł}x_n$. Then Ł possesses at least one fixed point.*

Using the identity function η, we get the following outcome:

Corollary 2. *Let (∇, d) be a complete metric space and $\text{Ł} : \nabla \to \mathcal{P}_{cp}(\nabla)$ be a B.-W. type multi-valued contraction, i.e., there exists $\zeta \in \Phi$ such that*

$$\mathcal{H}(\text{Ł}x, \text{Ł}y)) \leq \zeta(d(x, y)), \tag{34}$$

for all $x, y \in \nabla$. Moreover, let $\lim_{n\to\infty} \mathcal{H}(\{x_n\}, \text{Ł}x_n) = 0$, for any sequence (x_n) such that $x_{n+1} \in \text{Ł}x_n$. Then there exists at least one fixed point for Ł.

4 Application to fractional differential equations

From now on, assume that $\nabla = C([a,b],[0,\infty))$ is the Banach space of continuous nonnegative real valued functions $z : [a,b] \to [0,\infty)$ endowed with the norm

$$\|z\| = \sup_{t\in[a,b]} |z(t)|.$$

Define $d(z_1,z_2) = \|z_1 - z_2\|$ for all $z_1, z_2 \in \nabla$. Then (∇, d) is a complete metric space.

Lemma 5. *For a function $g \in L^1([a,b],[0,\infty))$, function $\kappa \in C([a,b],[0,\infty))$ is a solution of the equation*

$$\begin{cases} {}^{k,H}\mathfrak{D}^{\rho,\varrho;\chi}_{a^+}\kappa(\ell) = g(\ell);\ \ell \in [a,b], \\ {}^{k}\mathfrak{J}^{nk-\varsigma_k;\chi}_{a^+}\kappa(\ell)]_{\ell=a} = 0, \\ \left(\frac{k}{\chi'(\ell)}\frac{d}{d\ell}\right)^{n-j} {}^{k}\mathfrak{J}^{nk-\varsigma_k;\chi}_{a^+}\kappa(\ell)]_{\ell=a} = d_j,\ j = 1,2,\cdots,n-1, \\ \varsigma_k = \rho + \varrho(nk-\rho) \end{cases} \tag{35}$$

if and only if

$$\kappa(\ell) = \sum_{j=1}^{n-1} d_j \frac{(\chi(\ell)-\chi(a))^{\frac{\varsigma_k}{k}-j}}{\Gamma_k(\varsigma_k - jk + k)} + \frac{1}{k\Gamma_k(\rho)}\int_a^\ell \zeta'(t)(\chi(\ell)-\chi(t))^{\frac{\rho}{k}-1} g(t)dt,$$

for all $\ell \in [a,b]$.

Proof. Asssume that $\kappa \in \nabla$ is a solution of the equation (35). Then, from Remark 2,

$${}^{k,H}\mathfrak{D}^{\rho,\varrho;\chi}_{a^+}\kappa(\ell) \quad = {}^{k}\mathfrak{J}^{\varsigma_k-\rho;\chi}_{a^+}\left({}^{k,RL}\mathfrak{D}^{\varsigma_k;\chi}_{a^+}\kappa\right)(\ell) = g(\ell), \tag{36}$$

Appling ${}^{k}\mathfrak{J}^{\rho;\chi}_{a^+}$ on both sides of the above equality, we get

$${}^{k}\mathfrak{J}^{\varsigma_k;\chi}_{a^+}\left({}^{k,RL}\mathfrak{D}^{\varsigma_k;\chi}_{a^+}\kappa\right)(\ell) = {}^{k}\mathfrak{J}^{\rho;\chi}_{a^+} g(\ell). \tag{37}$$

From Lemma 3, we get

$$\kappa(\ell) - \sum_{j=1}^{n-1} \frac{(\chi(\ell)-\chi(a))^{\frac{\varsigma_k}{k}-j}}{\Gamma_k(\varsigma_k - jk + k)}\left[\left(\frac{k}{\chi'(\ell)}\frac{d}{d\ell}\right)^{n-j} {}^{k}\mathfrak{J}^{nk-\varsigma_k;\chi}_{a^+}\kappa(\ell)\right]_{\ell=a} = {}^{k}\mathfrak{J}^{\rho;\chi}_{a^+} g(\ell).$$

Thus

$$\kappa(\ell)=\sum_{j=1}^{n-1} d_j \frac{(\chi(\ell)-\chi(a))^{\frac{\varsigma_k}{k}-j}}{\Gamma_k(\varsigma_k-jk+k)}+\frac{1}{k\Gamma_k(\rho)}\int_a^\ell \chi'(t)(\chi(\ell)-\chi(t))^{\frac{\rho}{k}-1}g(t)dt,$$

where

$$d_j=\left[\Big(\frac{k}{\chi'(\ell)}\frac{d}{d\ell}\Big)^{n-j}{}_k\mathfrak{I}_{a^+}^{nk-\varsigma_k;\chi}\kappa(\ell)\right]_{\ell=a}.$$

An inverse direct calculation can be used to quickly find the proof's opposite. □

Definition 5. A function $\varsigma\in\mathcal{C}:=C([a,b],[0,\infty))$ is a solution of the system (2) if and only if it satisfies the initial conditions and there is $\mathfrak{z}\in L^1([a,b],[0,\infty))$ such that $\mathfrak{z}(\ell)\in\mathfrak{F}(\ell,z(\ell))$ for almost all $\ell\in[a,b]$ and

$$\kappa(\ell)=\sum_{j=1}^{n-1} d_j \frac{(\chi(\ell)-\chi(a))^{\frac{\varsigma_k}{k}-j}}{\Gamma_k(\varsigma_k-jk+k)}+\frac{1}{k\Gamma_k(\rho)}\int_a^\ell \zeta'(t)(\zeta(\ell)-\zeta(t))^{\frac{\rho}{k}-1}\mathfrak{z}(t)dt,$$

for all $\ell\in[a,b]$.

For each $\kappa\in\mathcal{C}$, we define the set of selections of the operator $\mathfrak{F}$ as follows

$$\mathfrak{S}_{\mathfrak{F},\kappa}=\Big\{\mathfrak{z}\in L^1([a,b],[0,\infty)):\mathfrak{z}(\ell)\in\mathfrak{F}(\ell,\kappa(\ell)),\forall\ell\in[a,b]\Big\}.$$

Define the operator $\mathfrak{U}:\mathcal{C}\to\mathcal{P}(\mathcal{C})$ by

$$\mathfrak{U}(z)=\Big\{\mathfrak{p}\in\mathcal{C}:\text{there exists }\mathfrak{z}\in\mathfrak{S}_{\mathfrak{F},z}\text{ such that }\mathfrak{p}(\ell)=\Upsilon(\ell),\forall\ell\in[a,b]\Big\},\tag{38}$$

where

$$\Upsilon(\ell)=\sum_{j=1}^{n-1} d_j \frac{(\chi(\ell)-\chi(a))^{\frac{\varsigma_k}{k}-j}}{\Gamma_k(\varsigma_k-jk+k)}+\frac{1}{k\Gamma_k(\rho)}\int_a^\ell \chi'(t)(\chi(\ell)-\chi(t))^{\frac{\rho}{k}-1}\mathfrak{z}(t)dt.$$

Theorem 2. *Let $\mathfrak{F}:[a,b]\times\mathbb{R}\to\mathcal{P}_{cp}(\mathbb{R})$ be a multi-valued mapping. Suppose that the following conditions are satisfied:*

(i) The multi-valued mapping $\mathfrak{F}$ is integrable, and $\mathfrak{F}(.,u):[a,b]\to\mathcal{P}_{cp}(\mathbb{R})$ is measurable for all $u\in\mathbb{R}$,

(ii) There exist $\zeta\in\Phi$ such that

$$\mathcal{H}(\mathfrak{F}(\ell,u),\mathfrak{F}(\ell,v))\le\frac{\Gamma_k(\rho+k)}{(\chi(b)-\chi(a))^{\frac{\rho}{k}}}\zeta(|v-v|),\tag{39}$$

for all $\ell\in[a,b]$ and $u,v\in[0,\infty)$.

Moreover, let $\lim_{n\to\infty} \mathcal{H}(\{v_n(\ell)\}, \mathfrak{F}(\ell, v_n(\ell)) = 0$*, for any sequence* (v_n) *with* $v_{n+1}(\ell) \in \mathfrak{F}(\ell, v_n(\ell))$*. Thus, there is at least one answer to the inclusion problem (2).*

Proof. We shall show that the multi-valued mapping $\mathfrak{U}$, defined in (38), has a fixed point. Let $z_1, z_2 \in \mathcal{C}$ and $\hbar_1^* \in \mathfrak{U}(z_1)$ and choose $\mathfrak{z}_1 \in \mathfrak{S}_{\mathfrak{F},z_1}$ such that

$$\hbar_1^*(\ell) = \sum_{j=1}^{n-1} d_j \frac{(\chi(\ell)-\chi(a))^{\frac{\varsigma_k}{k}-j}}{\Gamma_k(\varsigma_k - jk + k)} + \frac{1}{k\Gamma_k(\rho)} \int_a^\ell \chi'(t)(\chi(\ell)-\chi(t))^{\frac{\rho}{k}-1} \mathfrak{z}_1(t)dt$$

for all $\ell \in [a, b]$. From (39), we have

$$\mathcal{H}(\mathfrak{F}(\ell, z(\ell)), \mathfrak{F}(\ell, z'(\ell))) \le \frac{\Gamma_k(\rho+k)}{(\chi(b)-\chi(a))^{\frac{\rho}{k}}} \zeta(|z_1(\ell) - z_2(\ell)|).$$

Thus, there exists $\Upsilon \in \mathfrak{F}(\ell, z'(\ell))$ such that

$$|\mathfrak{z}_1(\ell) - \Upsilon(\ell)| \le \frac{\Gamma_k(\rho+k)}{(\chi(b)-\chi(a))^{\frac{\rho}{k}}} \zeta(|z(\ell) - z'(\ell)|).$$

Now, define a multi-valued mapping $\mathfrak{N} : [a, b] \to \mathcal{P}(\mathcal{C})$ as

$$\mathfrak{N}(\ell) = \left\{ \Upsilon \in \mathcal{C} : |\mathfrak{z}_1(\ell) - \Upsilon(\ell)| \le \frac{\Gamma_k(\rho+k)}{(\chi(b)-\chi(a))^{\frac{\rho}{k}}} \zeta(|z_1(\ell) - z_2(\ell)|) \right\},$$

for all $\ell \in [a, b]$. As $\mathfrak{z}_1$ and

$$\frac{\Gamma_k(\rho+k)}{(\chi(b)-\chi(a))^{\frac{\rho}{k}}} \zeta(|z_1(\ell) - z_2(\ell)|)$$

are measurable, so is $\mathfrak{N}(.) \cap \mathfrak{F}(., z'(.))$. Now, let $\mathfrak{z}_2(\ell) \in \mathfrak{F}(\ell, z_2(\ell))$ be such that

$$|\mathfrak{z}_1(\ell) - \mathfrak{z}_2(\ell)| \le \frac{\Gamma_k(\rho+k)}{(\chi(b)-\chi(a))^{\frac{\rho}{k}}} \zeta(|z_1(\ell) - z_2(\ell)|).$$

Now, we define $\hbar_2^* \in \mathfrak{U}(z_2)$ as

$$\hbar_2^*(\ell) = \sum_{j=1}^{n-1} d_j \frac{(\chi(\ell)-\chi(a))^{\frac{\varsigma_k}{k}-j}}{\Gamma_k(\varsigma_k - jk + k)} + \frac{1}{k\Gamma_k(\rho)} \int_a^\ell \chi'(t)(\chi(\ell)-\chi(t))^{\frac{\rho}{k}-1} \mathfrak{z}_2(t)dt$$

for all $\ell \in [a, b]$. Then

$|\hbar_1^*(\ell)-\hbar_2^*(\ell)| \leq \frac{1}{k\Gamma_k(\rho)}\int_a^\ell \chi'(t)(\chi(\ell)-\chi(t))^{\frac{\rho}{k}-1}(|\mathfrak{z}_1(\ell)-\mathfrak{z}_2(\ell)|)dt$
$\leq \frac{1}{k\Gamma_k(\rho)}\int_a^\ell \chi'(t)(\chi(\ell)-\chi(t))^{\frac{\rho}{k}-1}\frac{\Gamma_k(\rho+k)}{(\chi(b)-\chi(a))^{\frac{\rho}{k}}}\zeta(|z_1(\ell)-z_2(\ell)|)dt$
$\leq \frac{1}{(\chi(b)-\chi(a))^{\frac{\rho}{k}}}\zeta(\|z_1-z_2\|)(\chi(\ell)-\chi(a))^{\frac{\rho}{k}} \leq \zeta(\|z_1-z_2\|)$.

Therefore

$$\|\hbar_1^*-\hbar_2^*\| \leq \zeta(\|z_1-z_2\|)$$

Thus

$$\mathcal{H}(\mathfrak{U}(z),\mathfrak{U}(z')) \leq \zeta(\|z_1-z_2\|).$$

All of the requirements of Corollary (2) have now been met. Thus, there is a fixed point for $\mathfrak{U}:\mathcal{C}\to\mathcal{P}_{cp}(\mathcal{C})$ and so the problem (2) possesses a solution. □

Example 1. Consider the fractional differential inclusion

$$\begin{cases} {}^{\frac{4}{3},H}D_{2^+}^{\frac{10}{3},\frac{4}{5};\ell^2}\kappa(\ell) \in \left[0, e^\ell + \frac{4\sqrt{\pi}}{15\sqrt{15}}\frac{\kappa(\ell)}{\kappa(\ell)+1}\right], \quad \ell\in[2,3], \\ {}_{\frac{4}{3}}\mathfrak{J}_{2^+}^{\frac{2}{15};\ell^2}\kappa(\ell)]_{\ell=2} = 0, \\ \left(\frac{4}{6\ell}\frac{d}{d\ell}\right)^1 {}_{\frac{4}{3}}\mathfrak{J}_{2^+}^{\frac{2}{15};\ell^2}\kappa(\ell)]_{\ell=2} = 5, \\ \left(\frac{4}{6\ell}\frac{d}{d\ell}\right)^2 {}_{\frac{4}{3}}\mathfrak{J}_{2^+}^{\frac{2}{15};\ell^2}\kappa(\ell)]_{\ell=2} = 7. \end{cases} \tag{40}$$

Note that,

$$\mathfrak{F}(\ell,u) = \left[0, e^\ell + \frac{4\sqrt{\pi}}{15\sqrt{15}}\frac{u}{u+1}\right].$$

Obviously, $\mathfrak{F}$ is continuous and compact valued. Here,

$$k=\frac{4}{3}, \rho=\frac{10}{3}, \varrho=\frac{4}{5}, a=2, b=3, d_1=7, d_2=5, \chi(\ell)=\ell^2$$

and $\varsigma_k=\rho+\varrho(nk-\rho)=\frac{174}{45}$, $n=\lceil\frac{\varsigma_k}{k}\rceil=\lceil\frac{174}{60}\rceil=3$, $nk-\varsigma_k=4-\frac{174}{45}=\frac{6}{45}=\frac{2}{15}$. Thus

$$\frac{\Gamma_k(\rho+k)}{(\chi(b)-\chi(a))^{\frac{\rho}{k}}} = \frac{k^{\frac{\rho}{k}}\Gamma(\frac{\rho+k}{k})}{5^{\frac{5}{2}}} = \frac{(\frac{4}{3})^{\frac{5}{2}}\Gamma(\frac{7}{2})}{5^{\frac{5}{2}}} = \frac{4\sqrt{\pi}}{15\sqrt{15}},$$

and

$$\mathcal{H}\Big(\mathfrak{F}(\ell,u),\mathfrak{F}(\ell,v)\Big) = |\frac{4\sqrt{\pi}}{15\sqrt{15}}\frac{u}{u+1} - \frac{4\sqrt{\pi}}{15\sqrt{15}}\frac{v}{v+1}|$$
$$= \frac{4\sqrt{\pi}}{15\sqrt{15}}|\frac{u}{u+1} - \frac{v}{v+1}| \leq \frac{4\sqrt{\pi}}{15\sqrt{15}}|\frac{u-v}{(u+1)(v+1)}|$$
$$\leq \frac{4\sqrt{\pi}}{15\sqrt{15}}\frac{|u-v|}{|u-v|+1} = \frac{\Gamma_k(\rho+k)}{(\chi(b)-\chi(a))^{\frac{\rho}{k}}}\zeta(|v-v|),$$

where $\zeta(t) = \frac{t}{t+1}$.

As a result, the condition (ii) in the Theorem 2 is met. The additional prerequisites for the theorem 2 are clear. Therefore, the problem (40) has a solution according to this theorem.

5 Conclusion

In this article, we introduce a multi-valued contraction of the B.-W. type and demonstrate that such mappings have a fixed point. We examine various $(k-\chi)$-Hilfer initial value fractional differential inclusions of arbitrary order $n-1<\rho<n$ $(n\geq 2)$ for solvability. Our conclusion is based on certain findings concerning the presence of $(k-\chi)$-Hilfer fractional differential inclusions (h.f.d.i.s). To demonstrate the usability of our primary result, an example is provided. We suggest developing such fixed point theorems for multivalued mappings of two variables that fulfill a multivalued contraction of the B.-W. type in the future. Next, we suggest looking into whether the $(k-\chi)$-Hilfer fractional differential systems of inclusions are solvable when the right hand functions reveal a multi-valued contraction of the B.-W. type.

Acknowledgement

This research was supported by the Science Fund of the Republic of Serbia, # 7632, Mathematical Methods in Image Processing under Uncertainty – MaMIPU.

References

[1] S. Naz, M.N. Naeem, On the generalization of κ-fractional Hilfer-Katugampola derivative with Cauchy problem, *Turk. J. Math.,* 45, (2021), 110–124.

[2] Y.C. Kwun, G. Farid, W. Nazeer; S. Ullah, S.M. Kang, Generalized Riemann-Liouville k-fractional integrals associated with Ostrowski type

inequalities and error bounds of Hadamard inequalities, *IEEE Access,* 6, (2018), 64946–64953.

[3] K.D. Kucche, A.D. Mali, On the nonlinear (k, ψ)-Hilfer fractional differential equations, *Chaos Solitons Fractals,* 152, (2021), 111335.

[4] G.A. Dorrego, An alternative definition for the k-Riemann-Liouville fractional derivative, *Appl. Math. Sci.,* 9, (2015), 481–491.

[5] R. Diaz, E. Pariguan, On hypergeometric functions and Pochhammer k-symbol, *Divulg. Mat.,* 2, (2007), 179–192.

[6] S. Mubeen, G.M. Habibullah, kfractional integrals and applications, *Int. J. Contemp. Math. Sci.,* 7, (2012), 89–94.

[7] D. W. Boyd , J. S. W. Wong, On nonlinear contractions, *Proc. Amer. Math. Soc.,* 20, (1969), 458–464.

[8] Tariboon, J., Samadi, A., Ntouyas, S.K. Multi-point boundary value problems for (k, ϕ)-Hilfer fractional differential equations and inclusions, *Axioms,* 11, (2022), 110.

[9] R. Hilfer, Applications of Fractional Calculus in Physics, *World Scientific*, Singapore, (2000).

[10] R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci.Numer. Simul.,* 44, (2017), 460–481.

[11] R. Almeida, A.B. Malinowska, M. Teresa, T. Monteiro, Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications, *Math. Methods Appl. Sci.,* 41(1), (2018), 336–352.

[12] F. Jarad, T. Abdeljawad, Generalized fractional derivatives and Laplace transform, *Discrete Contin. Dyn. Syst.,* 13(3), (2020), 709–722.

[13] S. Hamani, M. Benchohra, John R. Graef, Existence results for boundary value problems with nonlinear fractional inclusions and integral conditions,*Electron. J. Diff. Equ.,* 2010(20), (2010), 1–16.

[14] M. Benchohra, J. R. Graef, S. Hamani, Existence results for boundary value problems with nonlinear fractional differential equations, *Appl. Anal.,* 87, (2008), 851–863.

[15] S. Belmor, C. Ravichandran, F. Jarad, Nonlinear generalized fractional differential equations with generalized fractional integral conditions, *J. Taibah Univ. Sci.,* 14(1), (2020), 114–123.

[16] S. Belmor, F. Jarad, T. Abdeljawad, M.A. Alqudah, On fractional differential inclusion problems involving fractional order derivative with respect to another function. *Fractals,* 20(8),(2020), 2040002.

[17] S. B. Nadler, Multivalued contraction mappings, *Pacific J. Math.,* 30, (1969) 475–88.

[18] T. Abdeljawad, F. Madjidi, F. Jarad, N. Sene, On dynamic systems in the frame of singular function dependent kernel fractional derivatives,*Mathematics*, 7(10), (2019), 946.

[19] R. Ameen, F. Jarad, T. Abdeljawad, Ulam stability for delay fractional differential equations with a generalized Caputo derivative, *Filomat,* 32(15), (2018), 5265–5274.

[20] F. Jarad, S. Harikrishnan, K. Shah, K. Kanagarajan, Existence and stability results to a class of fractional random implicit differential equations involving a generalized Hilfer fractional derivative, *Discrete Contin. Dyn. Syst.,* 13(3), (2020) 723–739.

[21] B. Samet, H. Aydi, Lyapunov-type inequalities for an anti-periodic fractional boundary value problem involving ψ-Caputo fractional derivative, *J.Inequal. Appl.,* 2018(286), (2018), 9.

[22] J. Sousa, C. Vanterler da, K. D. Kucche, E. C. De Oliveira, Stability of ψ-Hilfer impulsive fractional differential equations, *Appl. Math. Lett.*, 88, (2019), 73–80.

[23] J. Vanterler da C. Sousa, E. Capelas de Oliveira, Leibniz type rule: ψ-Hilfer fractional operator, *Communications in Nonlinear Science and Numerical Simulation,* 77, (2019), 305-311.

[24] D.S. Oliveira, E. Capelas de Oliveira, On a Caputo-type fractional derivative, *Advances in Pure and Applied Mathematics,* 10(2), (2019), 81–91

[25] J.V.C. Sousa, E.C. De Oliveira, On the ψ-Hilfer fractional derivative, *Communications in Nonlinear Science and Numerical Simulation,* 60, (2018), 72-91.

[26] I. Podlubny, Fractional differential equations, Academic Press, (1999).

[27] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, (2006).

[28] E. Mittal, S. Joshi, Note on k-generalized fractional derivative, *Discret. Contin. Dyn. Syst.,* 13, (2020), 797–804.

[29] S.K. Magar, P.V. Dole, K.P. Ghadle, Pranhakar and Hilfer-Prabhakar fractional derivatives in the setting of y-fractional calculus and its applications, *Krak. J. Math.,* 48 ,2024, 515–533.

[30] P. Agarwal, J. Tariboon, S.K. Ntouyas, Some generalized Riemann-Liouville k-fractional integral inequalities, *J. Ineq. Appl.,* 2016, (2016), 122.

[31] G. Farid, A. Javed, A.U. Rehman, On Hadamard inequalities for n-times differentiable functions which are relative convex via Caputo k-fractional derivatives, *Nonlinear Anal. Forum,* 22, (2017), 17–28.

[32] M.K. Azam, G. Farid, M.A. Rehman, Study of generalized type k-fractional derivatives, *Adv. Differ. Equ.,* 2017, (2017), 249.

[33] L.G. Romero, L.L. Luque, G.A. Dorrego, R.A. Cerutti, On the k-Riemann-Liouville fractional derivative, *Int. J. Contemp. Math. Sci.,* 8, (2013), 41–51.

Marija PAUNOVIC,
Department of Natural Science,
University of Kragujevac, Faculty of Hotel Management and Tourism,
Vojvodjanska bb, 36210 Vrnjacka Banja, Serbia.
Email: majap@rcub.bg.ac.rs
Department of Computer Science,
University MB, Faculty of Business and Law,
11000 Belgrade, Serbia.

Babak MOHAMMADI,
Department of Mathematics,
Marand Branch, Islamic Azad University,
Marand, Iran.
Corresponding author
Email: bmohammadi@marandiau.ac.ir

Vahid PARVANEH,
Department of Mathematics,
Gilan-E-Gharb Branch, Islamic Azad University,
Gilan-E-Gharb, Iran.
Email: zam.dalahoo@gmail.com