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A HOMOTOPY THEORY FOR MAPS HAVING STRONGLY CONVEXLY TOTALLY BOUNDED RANGES IN TOPOLOGICAL VECTOR SPACES

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Abstract

This paper presents Leray–Schauder alternatives and a topological transversality (homotopy) theorem for compact upper semicontinuos maps having (strongly) convexly totally bounded ranges.

1. Introduction.

In this paper we introduce essential maps and consider two maps F and G with $F \cong G$ (in a natural way) and we discuss the situation that if either of the maps is essential then the other map will be essential. This seems to be the first very general theory of homotopy in topological vector spaces (without additional structure like local convexity). To achieve this we will consider upper semicontinuous maps having strongly convexly totally bounded ranges or convexly totally bounded ranges. For some initial results in this direction see [1, 6] and the references therein.

Let E be a Hausdorff topological vector space. Strongly convexly totally bounded was considered in the literature (see [4] and the references therein).

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Definition 1.1. A subset K of E is said to be strongly convexly totally bounded (s.c.t.b. for brevity) if for every neighborhood V of 0 there exists a convex subset C of V and a finite subset N of E such that $K \subseteq N + C$.

From [4, Proposition 2.2] we have the following result.

Theorem 1.2. *If K_1 and K_2 are two s.c.t.b. subsets of E then $K_1 \cup K_2$, $K_1 + K_2$, αK_1 (here $\alpha \in \mathbf{R}$), $\overline{K_1}$, $co\,(K_1)$ (and so $\overline{co}\,(K_1)$) are s.c.t.b. subsets of E.*

The analogue of the Schauder fixed point theorem was established in [5] (see also [7] for a different proof).

Theorem 1.3. *Let X be a nonempty convex subset of a Hausdorff topological vector space E and $\Phi : X \to CK(X)$ a upper semicontinuous compact map (here $CK(X)$ denotes the family of nonempty compact convex subsets of X). If $\overline{\Phi(X)}$ is a s.c.t.b. subset of X then Φ has a fixed point.*

Next we discuss a more general notion, namely convexly totally bounded.

Definition 1.4. A subset K of E is said to be convexly totally bounded (c.t.b. for brevity) if for every neighborhood V of 0 there exists a finite set $\{x_i : i \in I\} \subseteq E$ (I finite) and a finite family of convex sets $\{C_i : i \in I\}$ with $C_i \subseteq V$ for each $i \in I$ and $K \subseteq \cup_{i \in I}\,(x_i + C_i)$.

Recall the following results [1, 5].

Theorem 1.5. *If a compact set K is c.t.b. then the set $[0,1]\,K$ is compact and c.t.b.*

Theorem 1.6. *Let X be a convex subset of a Hausdorff topological vector space E and $\Phi : X \to CK(X)$ a upper semicontinuous compact map. If $\overline{\Phi(X)}$ is a c.t.b. subset of X then Φ has a fixed point.*

Every s.c.t.b. set is c.t.b and every c.t.b. set is totally bounded; in locally convex topological vector spaces these three notions are equivalent [8 pp. 277]. Also note every compact set in a locally convex topological vector space is a c.t.b. set.

In [4] it was shown that the convex hull of a c.t.b. set does not need to be c.t.b. However one can modify [4, Proposition 2.2] to obtain the following result.

Theorem 1.7. *If K_1 and K_2 are two c.t.b. subsets of E then $K_1 \cup K_2$, and $K_1 + K_2$ are c.t.b. subsets of E.*

Proof: Let V be a circled closed neighborhood of 0 in E. Now there exists a finite set $\{x_i : i \in I_1\} \subseteq E$, I_1 finite, (respectively, a finite set $\{y_i : i \in I_2\} \subseteq E$, I_2 finite) and a finite family of convex sets $\{C_i : i \in I_1\}$ (respectively, a finite family of convex sets $\{D_i : i \in I_2\}$) with $C_i \subseteq V$ for each $i \in I_1$ (respectively, $D_i \subseteq V$ for each $i \in I_2$) and $K_1 \subseteq \cup_{i\in I_1} (x_i + C_i)$ (respectively, $K_2 \subseteq \cup_{i\in I_2} (y_i + D_i)$). Also since $co(C_i \cup \{0\}) \subseteq V$ and $co(D_i \cup \{0\}) \subseteq V$ we may assume $0 \in C_i$ ($i \in I_1$) and $0 \in D_i$ ($i \in I_2$). Also by adding 0 if necessaruy we may assume that $I_1 = I_2 = I$ so

$$K_1 \subseteq \cup_{i\in I} (x_i + C_i) \quad \text{and} \quad K_2 \subseteq \cup_{i\in I} (y_i + D_i).$$

Note for each $i \in I$ and $j \in I$ that $M_{i,j} = C_i + D_j$ is a convex set, it contains C_i and D_j (since $0 \in C_i$ and $0 \in D_j$), $M_{i,j} \subseteq V + V$ with

$$K_1 + K_2 \subseteq \cup_{i\in I,\, j\in I} \{(x_i + y_j) + M_{i,j}\}$$

and (since $C_i \subseteq M_{i,j}$ and $D_j \subseteq M_{i,j}$)

$$K_1 \cup K_2 \subseteq \cup_{i\in I,\, j\in I} \{(x_i \cup y_j) + M_{i,j}\}. \qquad \square$$

2. Homotopy Results.

Let E be a Hausdorff topological vector space and U an open subset of E. We begin by defining the class of maps.

Definition 2.1. We say $F \in IPTW(\overline{U}, E)$ if $F : \overline{U} \to CK(E)$ is a upper semicontinuous compact map with $\overline{F(\overline{U})}$ a s.c.t.b. subset of E; here $\overline{U}$ denotes the closure of U in E.

Definition 2.2. We say $F \in IPTW_{\partial U}(\overline{U}, E)$ if $F \in IPTW(\overline{U}, E)$ and $x \notin F(x)$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

Now we introduce the notion of an essential map.

Definition 2.3. We say $F \in IPTW_{\partial U}(\overline{U}, E)$ is essential in $IPTW_{\partial U}(\overline{U}, E)$ if for every $G \in IPTW_{\partial U}(\overline{U}, E)$ with $G|_{\partial U} = F|_{\partial U}$ there exists a $x \in U$ with $x \in G(x)$.

Next we present the notion of homotopy.

Definition 2.4. Let $F, G \in IPTW_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $IPTW_{\partial U}(\overline{U}, E)$ if there exists an upper semicontinuous, compact map $H : \overline{U} \times [0, 1] \to CK(E)$ with $\overline{H(\overline{U} \times [0, 1])}$ a s.c.t.b. subset of E, $x \notin H_t(x)$ for any $x \in \partial U$ and $t \in (0, 1)$ (here $H_t(x) = H(x, t)$), $H_0 = F$ and $H_1 = G$.

Remark 2.5. We note that $\cong$ in $IPTW_{\partial U}(\overline{U}, E)$ is an equivalence relation. To see this we need only show transitivity. Let F, G, Ψ be maps in $IPTW_{\partial U}(\overline{U}, E)$ with $F \cong G$ in $IPTW_{\partial U}(\overline{U}, E)$ and $G \cong \Psi$ in $IPTW_{\partial U}(\overline{U}, E)$ i.e.suppose there exists an upper semicontinuous, compact map $H : \overline{U} \times [0, 1] \to CK(E)$ (respectively, $N : \overline{U} \times [0, 1] \to CK(E)$) with $\overline{H(\overline{U} \times [0, 1])}$

(respectively, $\overline{N(\overline{U} \times [0,1])}$) a s.c.t.b. subset of E, $x \notin H_t(x)$ (respectively $x \notin N_t(x)$) for $x \in \partial U$ and $t \in (0,1)$, $H_0 = F$ (respectively, $N_0 = G$) and $H_1 = G$ (respectively, $N_1 = \Psi$). Let

$$\Phi(x,t) = \begin{cases} H(x, 2\,t), & t \in \left[0, \frac{1}{2}\right],\ x \in \overline{U} \\ N(x, 2\,t - 1), & t \in \left[\frac{1}{2}, 1\right],\ x \in \overline{U}. \end{cases}$$

Note $\Phi : \overline{U} \times [0,1] \to CK(E)$ is a upper semicontinuous compact map with $x \notin \Phi_t(x)$ for $x \in \partial U$ and $t \in (0,1)$ and $\Phi_0 = F$ and $\Phi_1 = \Psi$. Also $\Phi(\overline{U} \times [0,1]) = K_1 \cup K_2$ where $K_1 = H(\overline{U} \times [0,1])$ and $K_2 = N(\overline{U} \times [0,1])$ and note $\overline{K_1}$, $\overline{K_2}$ are s.c.t.b. subsets of E. Now Theorem 1.2 guarantees that $\overline{\Phi(\overline{U} \times [0,1])} = \overline{K_1 \cup K_2} = \overline{K_1} \cup \overline{K_2}$ is a s.c.t.b. subset of E.

Next we present a result which will then generate a Leray–Schauder alternative and a topological transversality theorem.

Theorem 2.6. *Let E be a Hausdorff topological vector space, U an open subset of E and $F \in IPTW_{\partial U}(\overline{U}, E)$. Assume $G \in IPTW_{\partial U}(\overline{U}, E)$ is essential in $IPTW_{\partial U}(\overline{U}, E)$ and suppose the following holds:*

$$(2.1) \qquad \begin{cases} \textit{for any } \ \theta \in IPTW_{\partial U}(\overline{U}, E) \ \ \textit{with} \ \ \theta|_{\partial U} = F|_{\partial U} \\ \textit{we have } \ G \cong \theta \ \ \textit{in} \ \ IPTW_{\partial U}(\overline{U}, E). \end{cases}$$

Then F is essential in $IPTW_{\partial U}(\overline{U}, E)$.

Proof: Consider any map $\theta \in IPTW_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = F|_{\partial U}$. We must show there exists a $x \in U$ with $x \in \theta(x)$. Now (2.1) guarantees that there exists a upper semicontinuous, compact map $H : \overline{U} \times [0,1] \to CK(E)$, $\overline{H(\overline{U} \times [0,1])}$ a s.c.t.b. subset of E, $x \notin H_t(x)$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H_t(x) = H(x,t)$), $H_0 = G$ and $H_1 = \theta$. Let

$$\Omega = \left\{ x \in \overline{U} : \ x \in H(x,t) \ \text{ for some } \ t \in [0,1] \right\}.$$

Now $\Omega \neq \emptyset$ (note G is essential in $IPTW_{\partial U}(\overline{U}, E)$, Ω is closed (since H is upper semicontinuous) and in in fact Ω is compact (since H is compact). Also note $\Omega \cap \partial U = \emptyset$ since $x \notin H_t(x)$ for any $x \in \partial U$ and $t \in [0,1]$. Now since Hausdorff topological vector spaces are completely regular there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. Define a map $R : \overline{U} \to CK(E)$ by $R(x) = H(x, \mu(x)) = H_{\mu(x)}(x) = H \circ g(x)$ where $g : \overline{U} \to \overline{U} \times [0,1]$ is given by $g(x) = (x, \mu(x))$. Note R is a upper semicontinuous compact map with $R|_{\partial U} = G|_{\partial U}$ since if $x \in \partial U$ then $R(x) = H(x,0) = G(x)$. Also since $\overline{R(\overline{U})} \subseteq \overline{H(\overline{U} \times [0,1])}$ then $\overline{R(\overline{U})}$ is a s.c.t.b. subset of E. Thus $R \in IPTW_{\partial U}(\overline{U}, E)$ with $R|_{\partial U} = G|_{\partial U}$. Now since

G is essential in $IPTW_{\partial U}(\overline{U}, E)$ then there exists a $x \in U$ with $x \in R(x)$ i.e. $x \in H_{\mu(x)}(x) = H(x, \mu(x))$. Thus $x \in \Omega$ so $\mu(x) = 1$ and as a result $x \in H_1(x) = \theta(x)$. □

To establish the topological transversality (homotopy) theorem we must first prove the following:

$$\text{(2.2)} \qquad \begin{cases} \text{if } F\ G \in IPTW_{\partial U}(\overline{U}, E) \text{ with } F|_{\partial U} = G|_{\partial U} \\ \text{then } F \cong G \text{ in } IPTW_{\partial U}(\overline{U}, E). \end{cases}$$

To see this let $H(x,t) = (1-t)\,F(x) + t\,G(x)$ and note $H : \overline{U} \times [0,1] \to CK(E)$ is a upper semicontinuous compact map (to see the compactness note $D_1 = [0,1]\,\overline{F(\overline{U})}$ (note $D_1 = f([0,1] \times \overline{F(\overline{U})})$ where $f(\alpha, z) = \alpha\, z$ for $\alpha \in [0,1]$, $z \in \overline{F((\overline{U})})$ and $D_2 = [0,1]\,\overline{G(\overline{U})}$ are compact so $D_1 + D_2$ is compact (see [3, pp. 121]) and as a result $\overline{H(\overline{U} \times [0,1])}$ is compact). Also note if $x \in \partial U$ and $t \in (0,1)$ then $H_t(x) = (1-t)\,F(x) + t\,G(x) = F(x)$ since $F|_{\partial U} = G|_{\partial U}$ and as a result $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0,1)$. Finally note since $J_1 = \overline{F(\overline{U})}$ and $J_2 = \overline{G(\overline{U})}$ are s.c.t.b. subsets of E then from Theorem 1.2 we have that $\overline{co}(J_1 \cup J_2)$ is a s.c.t.b. subset of E and as a result since $\overline{H(\overline{U} \times [0,1])} \subseteq \overline{co}(J_1 \cup J_2)$ we have that $\overline{H(\overline{U} \times [0,1])}$ is a s.c.t.b. subset of E. Thus since $H_0 = F$ and $H_1 = G$ we have $F \cong G$ in $IPTW_{\partial U}(\overline{U}, E)$.

Remark 2.7. From (2.2) note in (2.1) since $\theta \in IPWT_{\partial U}(\overline{U}, E)$ and $\theta|_{\partial U} = F|_{\partial U}$ then $\theta \cong F$ in $IPWT_{\partial U}(\overline{U}, E)$.

Theorem 2.8. *Let E be a Hausdorff topological vector space and U an open subset of E. Suppose F and G are two maps in $IPTW_{\partial U}(\overline{U}, E)$ with $F \cong G$ in $IPTW_{\partial U}(\overline{U}, E)$. Now F is essential in $IPTW_{\partial U}(\overline{U}, E)$ if and only if G is essential in $IPTW_{\partial U}(\overline{U}, E)$.*

Proof: Assume G is essential in $IPTW_{\partial U}(\overline{U}, E)$. We will use Theorem 2.6 to show F is essential in $IPTW_{\partial U}(\overline{U}, E)$. Consider any map $\theta \in IPTW_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = F|_{\partial U}$. Now (2.2) guarantees that $\theta \cong F$ in $IPTW_{\partial U}(\overline{U}, E)$ and this together with $F \cong G$ in $IPTW_{\partial U}(\overline{U}, E)$ guarantees (see Remark 2.5) that $\theta \cong G$ in $IPTW_{\partial U}(\overline{U}, E)$. Thus (2.1) holds so Theorem 2.6 guarantees that F is essential in $IPTW_{\partial U}(\overline{U}, E)$. A similar argument shows if F is essential in $IPTW_{\partial U}(\overline{U}, E)$ then G is essential in $IPTW_{\partial U}(\overline{U}, E)$. □

Next we present an example of an essential maps and then we will give two general Leray-Schauder alternatives in topological vector spaces.

Theorem 2.9. *Let E be a Hausdorff topological vector space, U an open subset of E and $0 \in U$. Then the zero map is essential in $IPTW_{\partial U}(\overline{U}, E)$.*

Proof: Let $F(x) = \{0\}$ for $x \in \overline{U}$ (i.e. F is the zero map). Consider any map

$\theta \in IPTW_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = F|_{\partial U} = \{0\}$. We must show there exists a $x \in U$ with $x \in \theta(x)$. Let

$$J(x) = \begin{cases} \theta(x), & x \in \overline{U} \\ \{0\}, & x \in E \backslash \overline{U}. \end{cases}$$

Note $J : E \to CK(E)$ is a upper semicontinuous, compact map. Also note since $\overline{\theta(\overline{U})}$ is a s.c.t.b. subset of E then $\overline{J(\overline{U})} \subseteq \overline{co}\,(\overline{\theta(\overline{U})} \cup \{0\})$ and Theorem 1.2 guarantee that $\overline{J(\overline{U})}$ is a s.c.t.b. subset of E. Now Theorem 1.3 guarantees that there exists a $x \in E$ with $x \in J(x)$. If $x \in E \backslash U$ then $J(x) = \{0\}$, a contradiction since $0 \in U$. Thus $x \in U$ and so $x \in \theta(x)$. $\square$

Next we present a very general Leray–Schauder alternative from Theorem 2.6.

Theorem 2.10. *Let E be a Hausdorff topological vector space, U an open subset of E and $F \in IPTW_{\partial U}(\overline{U}, E)$. Assume $G \in IPTW_{\partial U}(\overline{U}, E)$ is essential in $IPTW_{\partial U}(\overline{U}, E)$ and $x \notin t\,F(x) + (1-t)\,G(x)$ for $x \in \partial U$ and $t \in (0,1)$. Then F is essential in $IPTW_{\partial U}(\overline{U}, E)$ (in particular F has a fixed point in U).*

Proof: Let $\theta \in IPTW_{\partial U}(\overline{U}, E)$ with $\theta|_{\partial U} = F|_{\partial U}$. Let $H(x,t) = t\,\theta(x) + (1-t)\,G(x)$ and the argument after (2.2) guarantees that $H : \overline{U} \times [0,1] \to CK(E)$ is a upper semicontinuous compact map and $\overline{H(\overline{U} \times [0,1])}$ is a s.c.t.b. subset of E. Also note if $x \in \partial U$ and $t \in (0,1)$ then since $\theta|_{\partial U} = F|_{\partial U}$ we have $H(x,t) = t\,\theta(x) + (1-t)\,G(x) = t\,F(x) + (1-t)\,G(x)$ so $x \notin H_t(x)$ for $x \in \partial\,U$ and $t \in (0,1)$. Finally note $H_0 = G$ and $H_1 = \theta$ so $G \cong \theta$ in $IPTW_{\partial U}(\overline{U}, E)$ i.e. (2.1) holds. Thus F is essential in $IPTW_{\partial U}(\overline{U}, E)$ from Theorem 2.6. $\square$

Now we combine Theorem 2.9 and Theorem 2.10 to obtain the following result.

Theorem 2.11. *Let E be a Hausdorff topological vector space, U an open subset of E, $0 \in U$ and $F \in IPTW_{\partial U}(\overline{U}, E)$ with $x \notin t\,F(x)$ for $x \in \partial U$ and $t \in (0,1)$. Then F is essential in $IPTW_{\partial U}(\overline{U}, E)$ (in particular F has a fixed point in U).*

Proof: Let G be the zero map which we know from Theorem 2.9 is essential in $IPTW_{\partial U}(\overline{U}, E)$. Now apply Theorem 2.10. $\square$

Next we discuss a more general situation. Let E be a Hausdorff topological vector space and U an open subset of E.

Definition 2.12. We say $F \in I(\overline{U}, E)$ if $F : \overline{U} \to CK(E)$ is a upper semicontinuous compact map with $\overline{F(\overline{U})}$ a c.t.b. subset of E.

Definition 2.13. We say $F \in I_{\partial U}(\overline{U}, E)$ if $F \in I(\overline{U}, E)$ and $x \notin F(x)$ for $x \in \partial U$.

Definition 2.14. We say $F \in I_{\partial U}(\overline{U}, E)$ is essential in $I_{\partial U}(\overline{U}, E)$ if for every $G \in I_{\partial U}(\overline{U}, E)$ with $G|_{\partial U} = F|_{\partial U}$ there exists a $x \in U$ with $x \in G(x)$.

Definition 2.15. Let F, $G \in I_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $I_{\partial U}(\overline{U}, E)$ if there exists an upper semicontinuous, compact map $H : \overline{U} \times [0,1] \to CK(E)$ with $\overline{H(\overline{U} \times [0,1])}$ a c.t.b. subset of E, $x \notin H_t(x)$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H_t(x) = H(x,t)$), $H_0 = F$ and $H_1 = G$.

We note that $\cong$ in $I_{\partial U}(\overline{U}, E)$ is an equivalence relation; the proof is as in Remark 2.5 where Theorem 1.2 is replaced by Theorem 1.7. The same reasoning as in Theorem 2.6 (with only the words s.c.t.b. are replaced by c.t.b.) immediately gives our next result.

Theorem 2.16. *Let E be a Hausdorff topological vector space, U an open subset of E and $F \in I_{\partial U}(\overline{U}, E)$. Assume $G \in I_{\partial U}(\overline{U}, E)$ is essential in $I_{\partial U}(\overline{U}, E)$ and suppose the following holds:*

$$(2.3) \qquad \begin{cases} \text{for any } \theta \in I_{\partial U}(\overline{U}, E) \text{ with } \theta|_{\partial U} = F|_{\partial U} \\ \text{we have } G \cong \theta \text{ in } I_{\partial U}(\overline{U}, E). \end{cases}$$

Then F is essential in $I_{\partial U}(\overline{U}, E)$.

To establish the topological transversality (homotopy) theorem we must first prove the following:

$$(2.4) \qquad \begin{cases} \text{if } F\ G \in I_{\partial U}(\overline{U}, E) \text{ with } F|_{\partial U} = G|_{\partial U} \\ \text{then } F \cong G \text{ in } I_{\partial U}(\overline{U}, E). \end{cases}$$

To see this let $H(x,t) = (1-t)\,F(x) + t\,G(x)$ and (see the argument after (2.2)) $H : \overline{U} \times [0,1] \to CK(E)$ is a upper semicontinuous compact map and $x \notin H_t(x)$ for $x \in \partial U$ and $t \in (0,1)$. Also note $\overline{F(\overline{U})}$ and $\overline{G(\overline{U})}$ are c.t.b. subsets of E so from Theorem 1.5 we have that $D_1 = [0,1]\,\overline{F(\overline{U})}$ and $D_2 = [0,1]\,\overline{G(\overline{U})}$ are c.t.b. subsets of E and from Theorem 1.7 we have that $D_1 + D_2$ is a c.t.b. subset of E and so $\overline{H(\overline{U} \times [0,1])}$ is a c.t.b. subset of E. Also $H_0 = F$ and $H_1 = G$ so $F \cong G$ in $I_{\partial U}(\overline{U}, E)$.

The same reasoning as in Theorem 2.8 immediately gives our next result.

Theorem 2.17. *Let E be a Hausdorff topological vector space and U an open subset of E. Suppose F and G are two maps in $I_{\partial U}(\overline{U}, E)$ with $F \cong G$ in $I_{\partial U}(\overline{U}, E)$. Now F is essential in $I_{\partial U}(\overline{U}, E)$ if and only if G is essential in $I_{\partial U}(\overline{U}, E)$.*

Theorem 2.18. *Let E be a Hausdorff topological vector space, U an open subset of E and $0 \in U$. Then the zero map is essential in $I_{\partial U}(\overline{U}, E)$.*

Proof: Let F, θ and J be as in Theorem 2.9. and note $J : E \to CK(E)$ is a upper semicontinuous, compact map. Also note since $\overline{\theta(\overline{U})}$ is a c.t.b. subset of E then $[0,1]\,\overline{\theta(\overline{U})}$ is a c.t.b. subset of E so $\overline{J(\overline{U})}$ is a c.t.b. subset of E. Now Theorem 1.6 guarantees that there exists a $x \in E$ with $x \in J(x)$. If $x \in E\backslash U$ then $J(x) = \{0\}$, a contradiction since $0 \in U$. Thus $x \in U$ and so $x \in \theta(x)$. □

The same reasoning as in Theorem 2.10 (with only the words s.c.t.b. are replaced by c.t.b.) and Theorem 2.11 immediately gives our next results.

Theorem 2.19. *Let E be a Hausdorff topological vector space, U an open subset of E and $F \in I_{\partial U}(\overline{U}, E)$. Assume $G \in I_{\partial U}(\overline{U}, E)$ is essential in $I_{\partial U}(\overline{U}, E)$ and $x \notin t\,F(x) + (1-t)\,G(x)$ for $x \in \partial U$ and $t \in (0,1)$. Then F is essential in $I_{\partial U}(\overline{U}, E)$ (in particular F has a fixed point in U).*

Theorem 2.20. *Let E be a Hausdorff topological vector space, U an open subset of E, $0 \in U$ and $F \in I_{\partial U}(\overline{U}, E)$ with $x \notin t\,F(x)$ for $x \in \partial U$ and $t \in (0,1)$. Then F is essential in $I_{\partial U}(\overline{U}, E)$ (in particular F has a fixed point in U).*

Many problems which arise naturally in differential and integral inclusion can be formulated in the form $x \in F\,x$. A simple example is $y''(t) = -\,e^{y(t)}$, $t \in [0,1]$ with $y(0) = y(1) = 0$ which models the steady state temperature in a rod with temperature dependent internal heating and note it can be rewritten in the form

$$y(t) = \int_0^1 G(t,s)\,e^{y(s)}\,ds \equiv F\,y(t)$$

where

$$G(t,s) = \begin{cases} t\,(1-s), & 0 \le t \le s \le 1 \\ s\,(1-t), & 0 \le s \le t \le 1. \end{cases}$$

Homotopy theory is a useful tool to establish whether the fixed point problem has a solution. The idea is to relate the problem above with the simpler problem $y''(t) = 0$, $t \in [0,1]$ with $y(0) = y(1) = 0$ via the family $y''(t) = -\,\lambda\,e^{y(t)}$, $t \in [0,1]$, $0 \le \lambda \le 1$, with $y(0) = y(1) = 0$ and to apply a homotopy result. As a result in general to establish existence for differential and integral inclusions one rewrites the problem as a fixed point problem and set up a family of problems relating the fixed point problem one is considering with a simpler problem (whose solution is known). The idea then is to use a homotopy result (usually a Leray–Schauder alternative or a topological transversality theorem) to guarantee a solution to the fixed point problem considered. In the non-normable setting, say in a locally convex topological vector space setting, the situation can be more complicated and usually here one uses a Leray–Schauder alternative or a topological transversality theorem to establish a Furi–Pera type result (see [2, Chapter 8]) which will then guarantee a solution to the fixed point problem considered. Our paper is the first theory of homotopy in topological vector spaces (without additional structure like local convexity) and our hope is to use this theory to establish an applicable theory of Furi-Pera type in a general setting.

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