DOI: 10.2478/auom-2024-0021
An. Şt. Univ. Ovidius Constanţa **Vol. 32(2),2024, 99–113**

Some bounds on the coupon collector problem with universal coupon

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Abstract

We consider a generalization of the coupon collector problem with unequal probabilities, such that there are two additional coupons in the coupon set: one that speeds up the coupon collection process, and the one that slows it down. We derive some upper and lower bounds on the distribution function of the waiting time until a subcollection or a full collection of coupons is sampled.

1 Introduction

The coupon collector problem (CCP), despite its simple formulation, still attracts considerable interest in the research community, either to modify and generalize the formulation of the problem itself or extend the results for existing versions of the problem.

The basic CCP (collecting coupons of n different types until a subcollection, or a complete collection of coupons is sampled) can be modified or generalized in several ways. Some generalizations are obtained by changing the goal of the collection process (for example, obtaining multiple copies of the original collection, collecting all pairs of the elements, etc.). Another set of generalizations is obtained by introducing additional coupons with special purposes (null coupon in [1],[2], bonus coupon in [7]).

Key Words: coupon collector problem, waiting time, universal coupon, Schur-convexity, bounds.
2010 Mathematics Subject Classification: Primary 60C05.
Received: 25.05.2023
Accepted: 22.10.2023

It is well known that the CCP has several applications in engineering (see, for example, [3]). In particular, the CCP with unequal probabilities has recently been used in biology, to model parasitism (explained in [9]), and in telecommunications, to solve Internet security problems (analyzed in [1],[2],[8]).

In the case of CCP with unequal probabilities (and any of its generalizations), computing quantities such as the expected waiting time until the end of the experiment becomes computationally intensive, and requires some sort of approximation. The determination of lower and upper bounds for quantities related to CCP is motivated by various technical applications, and has been addressed in several papers (see [1],[2],[8],[6],[5]). Although the contexts and formulations of these results vary widely, all authors prove that the complementary cumulative distribution function (ccdf) of the waiting time and the expected waiting time to complete the collection process are Schur-concave (Schur-convex, depending on the definition used) functions of the sampling probability, and conclude that the corresponding lowest bounds (or minimum values) are obtained when all coupons have the same drawing probability.

The goal of this work is to provide different versions of the lower and upper bounds on the waiting time to the end of the experiment when CCP is generalized by adding two types of additional coupons with special purposes, which, to the best of our knowledge, has not been considered yet.

The version of CCP we consider is the following: We assume that the set of coupons, in addition to the elements from $\mathbb{N}_n = \{1, 2, \ldots, n\}$, consists of a null coupon (which is not part of collection) and, in addition, a universal coupon (so called, joker), an element that can substitute for any element from the set $\mathbb{N}_n$ (one at a time). We assume sampling with replacement, that the coupon $k \in \mathbb{N}_n$ is drawn with probability p_k, that the joker is drawn with probability p_J and that the null coupon is drawn with probability p_N ($p_N, p_J < 1$, $\sum_{k=1}^{n} p_k + p_N + p_J = 1$). The quantity of interest is the waiting time $W_{n,c}$ until a subcollection of c, $1 \le c \le n$, different coupons from $\mathbb{N}_n$ is sampled, where some or all elements can be replaced by jokers.

The paper is organized as follows: In Section 2 we compute the ccdf of $W_{n,c}$ and the consequences of this result. In Section 3 we obtain several versions of bounds on the ccdf of $W_{n,c}$. In Section 4 we illustrate the behavior of the proposed bounds with numerical examples. Conclusions are given in Section 5.

2 Distributional properties of $W_{n,c}$

The version of the CCP we consider is the direct generalization of the CCP with null coupon, considered in [1]. We will refer to some of these results, precisely, to Theorem 1 (page 409), which we formulate as a lemma. With $W_{n,c}^{(N)}$ we denote the corresponding waiting time until a subcollection of c, $1 \leq c \leq n$, different coupons from $\mathbb{N}_n$ is sampled. In the rest of the text we will use the notation: $P_K = \sum_{k \in K} p_k$.

Lemma 1 (Anceaume et al. [1]). *For every $n \geq 1$ and $1 \leq c \leq n$, we have for every $t \geq 0$,*

$$P\{W_{n,c}^{(N)} > t\} = \sum_{k=0}^{c-1} (-1)^{c-1-k} \binom{n-k-1}{n-c} \sum_{\substack{K \subset \mathbb{N}_n, \\ |K|=k}} (P_K + p_N)^t . \tag{1}$$

The ccdf, first moment and second moment of $W_{n,c}$ are determined in the following theorem.

Theorem 1. *For the waiting time $W_{n,c}$ the following relations hold:*
1.

$$P\{W_{n,c} > t\} = \sum_{i=0}^{c-1} \binom{t}{i} {p_J}^i \sum_{k=0}^{c-i-1} (-1)^{c-i-k-1} \binom{n-k-1}{n-c+i} \sum_{\substack{K \subset \mathbb{N}_n, \\ |K|=k}} (P_K + p_N)^{t-i} , \tag{2}$$

for $t \geq 0$,
2.

$$E(W_{n,c}) = \sum_{i=0}^{c-1} p_J^i \sum_{k=0}^{c-i-1} (-1)^{c-i-k-1} \binom{n-k-1}{n-c+i} \sum_{\substack{K \subset \mathbb{N}_n, \\ |K|=k}} \frac{1}{(1 - P_K - p_N)^{i+1}}, \tag{3}$$

3.

$$E(W_{n,c}^2) = \sum_{i=0}^{c-1} p_J^i \sum_{k=0}^{c-i-1} (-1)^{c-i-k-1} \binom{n-k-1}{n-c+i} \sum_{\substack{K \subset \mathbb{N}_n, \\ |K|=k}} \frac{2i + 1 + P_K + p_N}{(1 - P_K - p_N)^{i+2}}. \tag{4}$$

Proof. 1. The waiting time $W_{n,c}$ can be written as:

$$W_{n,c} = \min\{t \in \mathbb{N} | Y_t + Z_t = c\}, n \in \mathbb{N}, 1 \leq c \leq n,$$

where Y_t and Z_t are number of standard coupons and number of jokers, respectively, sampled by the time t.

The statement (2) follows from the following relations:

$$P\{W_{n,c} > t\} = P\{Y_t + Z_t \le c-1\} \tag{5}$$
$$= \sum_{i=0}^{c-1} P\{Z_t = i\} P\{Y_t + Z_t \le c-1 | Z_t = i\}$$
$$= \sum_{i=0}^{c-1} P\{Z_t = i\} P\{Y_{t-i} \le c-1-i | Z_{t-i} = 0\}$$
$$= \sum_{i=0}^{c-1} P\{Z_t = i\} \frac{P\{Y_{t-i} \le c-1-i, Z_{t-i} = 0\}}{P\{Z_{t-i} = 0\}}$$
$$= \sum_{i=0}^{c-1} P\{Z_t = i\} \frac{P\left\{W_{n,c-i}^{(N)} > t-i\right\}}{P\{Z_{t-i} = 0\}}$$
$$= \sum_{i=0}^{c-1} \binom{t}{i} {p_J}^i (1-p_J)^{t-i} \sum_{k=0}^{c-i-1} (-1)^{c-i-1-k} \binom{n-k-1}{n-c+i} \sum_{\substack{K \subset \mathbb{N}_n, \\ |K|=k}} \left(\frac{P_K + p_N}{1-p_J}\right)^{t-i},$$

where the last line follows from Lemma 1.
2. The statement (3) follows from the following relation:

$$E(W_{n,c}) = \sum_{t=0}^{+\infty} P\{W_{n,c} > t\} \tag{6}$$
$$= \sum_{i=0}^{c-1} {p_J}^i \sum_{k=0}^{c-i-1} (-1)^{c-i-k-1} \binom{n-k-1}{n-c+i} \sum_{\substack{K \subset \mathbb{N}_n, \\ |K|=k}} \sum_{t=i}^{+\infty} \binom{t}{i} (P_K + p_N)^{t-i}.$$

and from the equality:

$$\sum_{t=i}^{+\infty} \binom{t}{i} a^{t-i} = \frac{1}{(1-a)^{i+1}}, \quad |a| < 1. \tag{7}$$

3. We have

$$E({W_{n,c}}^2) = \sum_{t=0}^{+\infty} P\{W_{n,c} > t\} + 2 \sum_{t=0}^{+\infty} t P\{W_{n,c} > t\}. \tag{8}$$

The statement (4) is a consequence of (8) and the relation:

$$\sum_{t=i}^{+\infty} t \binom{t}{i} a^{t-i} = \frac{i}{(1-a)^{i+1}} + \frac{a(i+1)}{(1-a)^{i+2}} = \frac{i+a}{(1-a)^{i+2}}, \quad |a| < 1. \tag{9}$$

□

Remark 1. *As we have $P\{W_{n,c} \geq c\} = 1$, Theorem 1 leads to the following combinatorial identities:*

$$P\{W_{n,c} > t\} = \sum_{i=0}^{c-1} \binom{t}{i} {p_J}^i \sum_{k=0}^{c-i-1} (-1)^{c-i-k-1} \binom{n-k-1}{n-c+i} \sum_{\substack{K \subset \mathbb{N}_n, \\ |K|=k}} (P_K + p_N)^{t-i} = 1, \tag{10}$$

that hold for any $n \in \mathbb{N}$, $1 \leq c \leq n$, $0 \leq t \leq c-1$, $0 \leq p_N, p_J < 1$, $0 \leq p_1, \ldots, p_n \leq 1$ and $\sum_{k=1}^{n} p_k + p_N + p_J = 1$.

3 Bounds on the ccdf of $W_{n,c}$

In [8], the author considers the CCP with unequal probabilities, and proposes two sets of upper and lower bounds on the ccdf of the waiting time until the full set of coupons is collected. The first set of bounds is obtained by direct combinatorial reasoning, and the other is derived by majorization theory. Numerical experiments have shown that the first set of bounds is tighter in most cases and is also useful for obtaining asymptotic results.

We have focused on the refinement of the bounds obtained using the majorization theory. Since the paper [8] was the basis for this work, we adopt most of the definitions and notations from there. For completeness, we include them in this paper.

Definition 1. *Let $(p_{(1)}, p_{(2)}, \ldots, p_{(n)})$ denote the coordinates of the vector $\mathbf{p} = (p_1, \ldots, p_n)$ ordered in ascending order. Vector $\mathbf{p} = (p_1, \ldots, p_n)$ is said to majorize vector $\mathbf{q} = (q_1, \ldots, q_n)$ (in notation, $\mathbf{q} \prec \mathbf{p}$) if*

$$\sum_{i=1}^{k} q_{(i)} \leq \sum_{i=1}^{k} p_{(i)}, k = 1, 2, \ldots, n-1, \text{ and } \sum_{i=1}^{n} q_{(i)} = \sum_{i=1}^{n} p_{(i)}. \tag{11}$$

Definition 2. *A real valued function f defined on $\mathbb{R}^n$ is said to be Schur-convex (concave) if*

$$\mathbf{q} \prec \mathbf{p} \rightarrow f(\mathbf{q}) \leq (\geq) f(\mathbf{p}).$$

We will also need the next lemma.

Lemma 2. *[Marshall and Olkin [4]]. A function f defined on $\mathbb{R}^n$ is Schur-convex (concave) iff f is symmetric and $f(\lambda q, (1-\lambda)q, p_3, \ldots, p_n)$ is a non-decreasing (non-increasing) function of λ for $\lambda \in (0, 1/2]$.*

In the rest of the text, we will denote $W_{n,c}(\mathbf{p})$ to indicate that the waiting time corresponds to the particular sampling probability vector $\mathbf{p}$. We will use analogous notation for the first and second moments of $W_{n,c}$, where required.

Lemma 3. *The ccdf (2) is Schur-concave function of the sampling probability* $(p_1, \ldots, p_n)$.

Proof. The function (2) is symmetric, and we will apply Lemma 2. Let

$$h(p,t) = \sum_{\substack{K \subset \mathbb{N}_n \setminus \{1,2\}, \\ |K|=k}} (p + P_K + p_N)^{t-i}. \tag{12}$$

We have

$$\begin{aligned} P\{W_{n,c} > t\} &= \sum_{i=0}^{c-2} \binom{t}{i} {p_J}^i \sum_{k=0}^{c-i-2} (-1)^{c-i-k} \binom{n-k-2}{n-c+i} h(p_1,t) \\ &+ \sum_{i=0}^{c-2} \binom{t}{i} {p_J}^i \sum_{k=0}^{c-i-2} (-1)^{c-i-k} \binom{n-k-2}{n-c+i} h(p_2,t) \\ &+ \sum_{i=0}^{c-3} \binom{t}{i} {p_J}^i \sum_{k=0}^{c-i-3} (-1)^{c-i-k-1} \binom{n-k-3}{n-c+i} h(p_1+p_2,t) \\ &+ D, \end{aligned}$$

where the term D does not depend on p_1, or p_2.
Let $f(\lambda) = P\{W_{n,c}(\mathbf{p}^*) > t\}$, $\mathbf{p}^* = (\lambda q, (1-\lambda)q, p_3, \ldots, p_n)$, and $\lambda \in (0, 1/2]$.

$$\begin{aligned} \frac{\partial f(\lambda)}{\partial \lambda} &= q \sum_{i=0}^{c-2} (t-i) \binom{t}{i} {p_J}^i \sum_{k=0}^{c-i-2} (-1)^{c-i-k} \binom{n-k-2}{n-c+i} h(\lambda q, t-1) \\ &- q \sum_{i=0}^{c-2} (t-i) \binom{t}{i} {p_J}^i \sum_{k=0}^{c-i-2} (-1)^{c-i-k} \binom{n-k-2}{n-c+i} h((1-\lambda)q, t-1) \\ &= qt \sum_{i=0}^{c-2} \binom{t-1}{i} {p_J}^i \sum_{k=0}^{c-i-2} (-1)^{c-i-k} \binom{n-k-2}{n-c+i} \left(h(\lambda q, t-1) - h((1-\lambda)q, t-1)\right). \end{aligned} \tag{13}$$

For $t \geq 1$, we consider the function:

$$l(a) = \sum_{i=0}^{c-2} \binom{t-1}{i} {p_J}^i \sum_{k=0}^{c-i-2} (-1)^{c-i-k} \binom{n-k-2}{n-c+i} h(a, t-1). \tag{14}$$

We define $W_{n-2,c}^{(a+p_N)}$ as the waiting time until a subcollection of size c is sampled, in case when the null coupon is sampled with probability $a + p_N$ (instead

of p_N), and the set of standard coupons is $\mathbb{N}_n \setminus \{1,2\} = \{3,\dots,n\}$ (instead of $\mathbb{N}_n$).
For $t \geq 2$, we have:

$$\begin{aligned}
\frac{\partial l(a)}{\partial a} &= \sum_{i=0}^{c-2}(t-i-1)\binom{t-1}{i}{p_J}^i\sum_{k=0}^{c-i-2}(-1)^{c-i-k}\binom{n-k-2}{n-c+i}h(a,t-2)\\
&= (t-1)\sum_{i=0}^{c-2}\binom{t-2}{i}{p_J}^i\sum_{k=0}^{c-i-2}(-1)^{c-i-k}\binom{n-k-2}{n-c+i}h(a,t-2)\\
&= (t-1)\sum_{i=0}^{c-2}\binom{t-2}{i}{p_J}^i\sum_{k=0}^{c-i-2}(-1)^{c-i-k}\binom{n-k-3}{n-c+i-1}h(a,t-2)\\
&\quad +(t-1)\sum_{i=0}^{c-3}\binom{t-2}{i}{p_J}^i\sum_{k=0}^{c-i-3}(-1)^{c-i-k}\binom{n-k-3}{n-c+i}h(a,t-2)\\
&= (t-1)\left(P\{W_{n-2,c-1}^{(a+p_N)} > t-2\} - P\{W_{n-2,c-2}^{(a+p_N)} > t-2\}\right) \geq 0. \quad (15)
\end{aligned}$$

Therefore, $l(a)$ is an increasing function of a, and we obtain

$$\frac{\partial f(\lambda)}{\partial \lambda} = qt(l(\lambda q) - l((1-\lambda)q)) \leq 0, \tag{16}$$

which completes the proof of the Lemma. □

Next, we prove another lemma that we will need later.

Lemma 4. *The ccdf (2) is an increasing function of (any) sampling probability.*

Proof. The ccdf (2) is symmetric on $p_1, p_2, \dots, p_n$, therefore it is enough to prove that it is increasing function of p_1.
Let

$$q(p,t) = \sum_{\substack{K\subset\mathbb{N}_n\setminus\{1\},\\ |K|=k}} (p + P_K + p_N)^{t-i}. \tag{17}$$

We have

$$P\{W_{n,c} > t\} = r(p_1) = \sum_{i=0}^{c-2}\binom{t}{i}{p_J}^i\sum_{k=0}^{c-i-2}(-1)^{c-i-k}\binom{n-k-2}{n-c+i}q(p_1,t) + C, \tag{18}$$

where the term C does not depend on p_1.
For $t \geq 1$ we obtain:

$$\begin{aligned}\frac{\partial r(p_1)}{\partial p_1} &= \sum_{i=0}^{c-2}(t-i)\binom{t}{i}{p_J}^i \sum_{k=0}^{c-i-2}(-1)^{c-i-k}\binom{n-k-2}{n-c+i}q(p_1,t-1)\\ &= t\sum_{i=0}^{c-2}\binom{t-1}{i}{p_J}^i \sum_{k=0}^{c-i-2}(-1)^{c-i-k}\binom{n-k-2}{n-c+i}q(p_1,t-1). \quad (19)\end{aligned}$$

We define $W_{n-1,c}^{(a+p_N)}$ as the waiting time until a subcollection of size c is sampled, in case when the null coupon is sampled with probability $a+p_N$ (instead of p_N), and the set of standard coupons is $\mathbb{N}_n \setminus \{1\} = \{2,\ldots,n\}$ (instead of $\mathbb{N}_n$). Now we have:

$$\frac{\partial r(p_1)}{\partial p_1} = tP\{W_{n-1,c-1}^{(p_1+p_N)} > t-1\} \geq 0,$$

which completes the proof of the Lemma. □

From now on, we assume that the probability vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is already in ascending order.

Next, we define the following vectors:

$$\begin{aligned}\mathbf{p}^{(s,1)} &= \left(p_1,\ldots,p_s,\underbrace{\sum_{k=s+1}^{n}\frac{p_k}{n-s},\ldots,\sum_{k=s+1}^{n}\frac{p_k}{n-s}}_{n-s}\right), s\in\{1,\ldots,n-1\},\\ \mathbf{p}^{(0,1)} &= (\underbrace{p_{ave},\ldots,p_{ave}}_{n}), p_{ave} = \frac{1}{n}\sum_{k=1}^{n}p_k,\\ \mathbf{p}^{(s,2)} &= \left(\underbrace{0,\ldots,0}_{s-1},\underbrace{p_s,\ldots,p_s}_{n-s},\sum_{k=1}^{n}p_k-(n-s)p_s\right), s\in\{1,\ldots,n\},\\ \mathbf{p}^{(s,3)} &= \left(\underbrace{0,\ldots,0}_{s-1},p_s,\ldots,p_{n-1},\sum_{k=1}^{n}p_k-\sum_{k=s}^{n-1}p_k\right), s\in\{1,\ldots,n-1\},\\ \mathbf{p}^{(n,3)} &= \left(\underbrace{0,\ldots,0}_{n-1},\sum_{k=1}^{n}p_k\right). \quad (20)\end{aligned}$$

It is easy to see that

$$\mathbf{p}^{(n,2)} = \mathbf{p}^{(n,3)}, \mathbf{p}^{(n-1,2)} = \mathbf{p}^{(n-1,3)} \text{ and } \mathbf{p}^{(n-1,1)} = \mathbf{p}^{(1,3)} = \mathbf{p}. \tag{21}$$

Further relations between vectors are described in the next lemma.

Lemma 5. *The following relations hold:*

$$\mathbf{p}^{(s,1)} \prec \mathbf{p}^{(s-1,1)}, \quad 1 \le s \le n-1. \tag{22}$$

$$\mathbf{p}^{(s,3)} \prec \mathbf{p}^{(s-1,3)}, \quad 2 \le s \le n. \tag{23}$$

$$\mathbf{p}^{(s,2)} \prec \mathbf{p}^{(s,3)}, \quad 1 \le s \le n. \tag{24}$$

Proof. For proving (22) it is enough to check that the relation

$$p_s + k \sum_{i=s+1}^{n} \frac{p_i}{n-s} \le (k+1) \sum_{i=s}^{n} \frac{p_i}{n-s+1} \tag{25}$$

holds for any $0 \le k \le n-s$. However, (25) can be reduced to the obvious inequality:

$$(n-s-k)\left(\sum_{i=s+1}^{n} p_i - (n-s)p_s\right) \ge 0. \tag{26}$$

Relations (23) and (24) trivially hold. □

The set of possible lower and upper bounds is obtained in the next theorem.

Theorem 2. *Lower bounds for the ccdf (2) are given by*

$$L^{(s,1)}(t) = P\{W_{n,c}(\mathbf{p}^{(s,1)}) > t\}, \text{ for } s \in \{0, \ldots, n-2\}. \tag{27}$$

For $s \in \{1, \ldots, n\}$, the upper bounds for the ccdf (2) are given by

$$U^{(s,2)}(t) = P\{W_{n,c}(\mathbf{p}^{(s,2)}) > t\} \text{ and } U^{(s,3)}(t) = P\{W_{n,c}(\mathbf{p}^{(s,3)}) > t\}. \tag{28}$$

Proof. All the bounds follow from Lemma 2, Lemma 3, and Lemma 5. □

The exact expressions for all the bounds obtained in Theorem 2 are displayed in Table 1.

Remark 2. *From Lemma 3 and Lemma 5 we have the following relationship between the bounds proposed in Theorem 2:*

$$L^{(0,1)}(t) \le \cdots \le L^{(n-1,1)}(t) \le P\{W_{n,c}(\mathbf{p}) > t\} \le U^{(2,3)}(t) \le \cdots \le U^{(n,3)}(t),$$

and $P\{W_{n,c}(\mathbf{p}) > t\} \le U^{(s,3)}(t) \le U^{(s,2)}(t), s \in \{2, \ldots, n\}$. Obviously, the bounds of the type $U^{(s,2)}$ are computationally simpler than the bounds $U^{(s,3)}$.

bound	exact formula
$L^{(s,1)}(t)$	$\sum_{i=0}^{c-1}\binom{t}{i}{p_J}^i\sum_{k=0}^{c-i-1}(-1)^{c-i-k-1}\binom{n-k-1}{n-c+i}\sum_{j=0}^{\min\{k,s\}}\sum_{\substack{K\subset\{1,\dots,s\},\\ \lvert K\rvert=j}}\binom{n-s}{k-j}\left(P_K+p_N+\frac{k-j}{n-s}\sum_{l=s+1}^{n}p_l\right)^{t-i}$
$U^{(s,2)}(t)$	$\sum_{i=0}^{c-2}\binom{t}{i}{p_J}^i\sum_{k=0}^{c-i-2}(-1)^{c-i-k}\binom{n-k-2}{n-c+i}\sum_{j=0}^{k}\binom{n-s}{j}\binom{s-1}{k-j}\left(1-p_J-(n-s-j)p_s\right)^{t-i}$ $+\sum_{i=0}^{c-1}\binom{t}{i}{p_J}^i\sum_{k=0}^{c-i-1}(-1)^{c-i-k-1}\binom{n-k-1}{n-c+i}\sum_{j=0}^{k}\binom{n-s}{j}\binom{s-1}{k-j}\left(jp_s+p_N\right)^{t-i}$
$U^{(s,3)}(t)$	$\sum_{i=0}^{c-2}\binom{t}{i}{p_J}^i\sum_{k=0}^{c-i-2}(-1)^{c-i-k}\binom{n-k-2}{n-c+i}\sum_{j=0}^{\min\{k,n-s\}}\sum_{\substack{K\subset\{s,\dots,n-1\},\\ \lvert K\rvert=j}}\binom{s-1}{k-j}\left(P_K+1-p_J-\sum_{l=s}^{n-1}p_l\right)^{t-i}$ $+\sum_{i=0}^{c-1}\binom{t}{i}{p_J}^i\sum_{k=0}^{c-i-1}(-1)^{c-i-k-1}\binom{n-k-1}{n-c+i}\sum_{j=0}^{\min\{k,n-s\}}\sum_{\substack{K\subset\{s,\dots,n-1\},\\ \lvert K\rvert=j}}\binom{s-1}{k-j}\left(P_K+p_N\right)^{t-i}$

Table 1: Exact expressions for the bounds $L^{(s,1)}(t)$, $U^{(s,2)}(t)$ and $U^{(s,3)}(t)$

bound	exact formula
$L^{(0,1)}(t)$	$\sum_{i=0}^{c-1} \binom{t}{i} {p_J}^i \sum_{k=0}^{c-i-1} (-1)^{c-i-k-1} \binom{n-k-1}{n-c+i} \binom{n}{k} (kp_{ave} + p_N)^{t-i}$
$U^{(n-1,3)}(t)$	$\sum_{i=0}^{c-3} \binom{t}{i} {p_J}^i (1-p_J)^{t-i} + \binom{t}{c-1} p_J^{c-1} p_N^{t-c+1} + \binom{t}{c-2} p_J^{c-2} \left((p_{n-1}+p_N)^{t-c+2} + (1-p_J-p_{n-1})^{t-c+2} - p_N^{t-c+2} \right)$
$U^{(n,3)}(t)$	$\sum_{i=0}^{c-2} \binom{t}{i} {p_J}^i (1-p_J)^{t-i} + \binom{t}{c-1} p_J^{c-1} p_N^{t-c+1}$
$U^{(1,2)}(t)$	$\sum_{i=0}^{c-2} \binom{t}{i} {p_J}^i \sum_{k=0}^{c-i-2} (-1)^{c-i-k} \binom{n-k-2}{n-c+i} \binom{n-1}{k} (1-p_J-(n-1-k)p_1)^{t-i} + \sum_{i=0}^{c-1} \binom{t}{i} {p_J}^i \sum_{k=0}^{c-i-1} (-1)^{c-i-k-1} \binom{n-k-1}{n-c+i} \binom{n-1}{k} (kp_1+p_N)^{t-i}$

Table 2: Exact expressions for the bounds $L^{(0,1)}(t)$, $U^{(n-1,3)}(t)$, $U^{(n,3)}(t)$ and $U^{(1,2)}(t)$

Simpler formulas for particular cases of lower and upper bounds, obtained with some combinatorics, are displayed in Table 2.

Proposition 1. *The first and second moments of $W_{n,c}$, given in Theorem 1, are Schur-concave functions of the sampling probability.*

Proof. The statements follow from Lemma 3 and similar considerations as in the proof of Corollary 2 in [6]. □

Remark 3. *From Proposition 1 follows that it is possible to construct upper and lower bounds for $E(W_{n,c}(\mathbf{p}))$ and $Var(W_{n,c}(\mathbf{p}))$ by combining upper and lower bounds for the ccdf of $W_{n,c}$, proposed in Theorem 2. We can also conclude (using the same argument as in [6], Corollary 1), that the ccdf of $W_{n,c}(\mathbf{p})$ is minimized for $\mathbf{p} = \mathbf{p}^{(0,1)}$, and the same holds for $E(W_{n,c}(\mathbf{p}))$ and $E(W^2_{n,c}(\mathbf{p}))$.*

Using the representation (5) and the relation:

$$P\{Y_t = 0, Z_t \le c-1\} \le P\{Y_t + Z_t \le c-1\} \le P\{Z_t \le c-1\}, \tag{29}$$

we obtain another pair of simple, trivial lower (L^*) and upper (U^*) bounds for the ccdf (2):

$$L^*(t) = \sum_{i=0}^{c-1} \binom{t}{i} p_J^i p_N^{t-i}, \text{ and } U^*(t) = \sum_{i=0}^{c-1} \binom{t}{i} p_J^i (1-p_J)^{t-i}. \tag{30}$$

The bounds L^* and U^* are less sharp than any other bounds proposed in this work, which is clear from the following considerations.
For the bound U^* we have:

$$U^*(t) \geq U^{(n,3)}(t) = U^{(n,2)}(t). \tag{31}$$

For the bound L^*, using the well known inequality

$$\sum_{k=0}^{c-1} (-1)^{c-1-k} \binom{n-k-1}{n-c} \binom{n}{k} = 1, \tag{32}$$

we obtain:

$$L^*(t) = \sum_{i=0}^{c-1} \binom{t}{i} p_J^i \sum_{k=0}^{c-i-1} (-1)^{c-i-1-k} \binom{n-k-1}{n-c+i} \binom{n}{k} p_N^{t-i}. \tag{33}$$

Using Lemma 4 we obtain

$$L^*(t) \leq L^{(0,1)}(t). \tag{34}$$

Conclusion follows from Remark 2.

4 Numerical results

In this section we provide numerical comparison of the efficiency of the sets of bounds proposed in Section 3. We consider the following combinations of additional parameters: $(p_N, p_J) \in \{(1/22, 1/22), (1/3, 1/4), (1/3, 0)\}$, $(n, c) \in \{(20, 10), (20, 20)\}$, and check the behavior of the proposed bounds on the following distributions:

$$\mathbf{q}(1) = (q, q(1-q), q(1-q)^2, \ldots q(1-q)^{19}), q = 1 - (p_J + p_N)^{\frac{1}{20}},$$

and

$$\mathbf{q}(4) = (\underbrace{q, \ldots, q}_{4}, \underbrace{q(1-q), \ldots, q(1-q)}_{4}, \ldots, \underbrace{q(1-q)^4, \ldots, q(1-q)^4}_{4}),$$

where

$$q = 1 - \left(1 - \frac{1 - p_J - p_N}{4}\right)^{\frac{1}{5}}.$$

We include results for the bounds $L^{(0,1)}(t)$, $L^{(1,1)}(t)$, $L^{(2,1)}(t)$, $U^{(19,3)}(t)$, $U^{(20,3)}(t)$, $U^{(1,2)}(t)$, $U^{(10,2)}(t)$, which are chosen for their relative computational simplicity. For comparison, we also include the bounds $L^{(10,1)}(t)$ and

$U^{(10,3)}(t)$ that are not so computationally simple, but are tight. Results are presented in Tables 4 - 9.

Conclusions are the following:

1. In most cases, the bounds for the ccdf of $W_{20,20}$ are more tight than the same type of bounds for the ccdf of $W_{20,10}$, and the bounds seems to be the most tight for $(p_N, p_J) = (1/3, 1/4)$. This is not surprising, because of the following relation:

$$U^*(t) \leq \frac{(1-p_J)^{t-c+1}}{{p_N}^t} L^*(t). \tag{35}$$

Therefore, the accuracy of the proposed bounds strongly depends both on the probabilities p_J and p_N, and the size of the portion of the collection sampled.

2. In most cases considered, the combination of bounds $L^{(2,1)}(t)$, and $U^{(1,2)}(t)$ seems to be the most appropriate, taking into consideration both tightness and computational effort. We indicated these cases by bold font.

3. The only case where none of the proposed bounds seems to work so well is the case $(n, c) = (20, 10)$, $(p_N, p_J) = (1/22, 1/22)$ in Table 4. This can be explained by (35), as well.

4. In most cases considered, the bound $L^{(10,1)}(t)$ is relatively close to the true ccdf of $W_{n,c}$, which confirms the trade - off between the accuracy of the bounds and computational effort.

Finally, for their overall performance, we can recommend the pair of bounds $L^{(2,1)}(t)$, and $U^{(1,2)}(t)$.

Remark 4. *When we reduce the problem to the CCP with unequal probabilities (by setting probabilities of additional coupons to zero), we can compare the bounds we propose to those considered in [8]. The lower bound obtained in [8] using majorization (which we denote as $L_1^{(S)}$) coincides with the bound $L^{(0,1)}$ considered in this paper, and the corresponding upper bound in [8] (which we denote as $U_1^{(S)}$), is, in our experience, less tight than any upper bound we considered.*

For illustration, we compare in Table 3 the pair of bounds $L^{(2,1)}$, and $U^{(1,2)}$, recommended in this paper, with the bounds $L_1^{(S)}$ and $U_1^{(S)}$. We also include the upper and lower bounds obtained by direct probabilistic arguments in [8] (which we denote as $U_2^{(S)}$ and $L_2^{(S)}$, respectively), which are specific to the variant of the problem considered. The results for the bounds $L_1^{(S)}$, $U_1^{(S)}$, $L_2^{(S)}$ and $U_2^{(S)}$ are taken from Table 1, p. 161 in [8]. We can observe that the pair of bounds $(L^{(2,1)}(t), U^{(1,2)}(t))$ is tighter than the pair of bounds $(L_1^{(S)}(t), U_1^{(S)}(t))$ in most of the cases considered, but less tight than the pair $(L_2^{(S)}(t), U_2^{(S)}(t))$.

t	$P\{W_{20,20}>t\}$	$L_1^{(S)}(t)$	$L^{(2,1)}(t)$	$L_2^{(S)}(t)$	$U_1^{(S)}(t)$	$U^{(1,2)}(t)$	$U_2^{(S)}(t)$
50	9.9915e-01	9.9859e-01	9.9877e-01	-6.8488	9.9999e-01	9.9998e-01	5.1241
100	7.9769e-01	7.2222e-01	7.4740e-01	5.3002e-01	9.7367e-01	9.6798e-01	1.4451
200	1.3314e-01	6.7771e-02	9.2527e-02	1.3290e-01	3.9348e-01	3.7786e-01	1.4095e-01
500	2.6724e-04	1.4339e-05	1.6181e-04	2.6724e-04	1.8377e-03	1.7459e-03	2.6726e-04
1000	1.5259e-08	1.0281e-11	1.2326e-08	1.5259e-08	1.6911e-07	1.6065e-07	1.5259e-08

Table 3: Comparison of the bounds $L^{(2,1)}$ and $U^{(1,2)}$ with those obtained in [8], Table 1, p. 161 (the same sampling distribution)

5 Conclusions

We have presented introductory results related to the extension of the CCP with unequal probabilities, dealing with the situation when there are additional coupons in the coupon set: null coupon that slows down the collection process, and universal coupon that speeds it up. We derived a class of bounds for the ccdf of the waiting time until the end of the experiment, by refining the bounds proposed in [8]. The quality of the proposed bounds is tested in numerical experiments, and we indicate the specific bounds from the class with the most desirable properties. The bounds we derive may be as well applied to functions of sampling probability that require approximation in other contexts.

Acknowledgments

This work was supported by Ministry of Science, Technological Development and Innovation of the Republic of Serbia: Grant No. 451 - 03 - 47/2023 - 01/200104.

The authors would also like to thank the anonymous referee for his/her useful comments and suggestions, which significantly improved the manuscript.

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t	$P\{W_{20,10}>t\}$	$L^{(0,1)}(t)$	$L^{(1,1)}(t)$	$L^{(2,1)}(t)$	$L^{(10,1)}(t)$	$U^{(19,3)}(t)$	$U^{(20,3)}(t)$	$U^{(1,2)}(t)$	$U^{(10,2)}(t)$	$U^{(10,3)}(t)$
50	4.6973e-07	1.2441e-12	3.9312e-12	1.2527e-11	6.4489e-08	9.9962e-01	9.9825e-01	1.7929e-01	1.2223e-01	1.3384e-02
100	2.4998e-15	9.6148e-30	1.6986e-28	3.1685e-27	7.7998e-17	9.6141e-01	9.1446e-01	1.6048e-04	3.0781e-04	1.1138e-05
200	7.3601e-31	5.5039e-64	3.1215e-61	2.0619e-58	5.5487e-34	4.4042e-01	3.0777e-01	2.4914e-12	1.5286e-09	2.058e-11
500	1.8795e-76	1.0324e-166	1.9389e-159	5.6947e-152	2.2927e-85	2.8248e-04	9.0905e-05	4.7788e-37	2.9585e-25	6.0488e-28
1000	3.314e-152	0.0000e+00	0.0000e+00	6.67e-308	5.2563e-171	4.7809e-12	7.8929e-13	2.9632e-78	1.9451e-51	3.2073e-55

t	$P\{W_{20,20}>t\}$	$L^{(0,1)}(t)$	$L^{(1,1)}(t)$	$L^{(2,1)}(t)$	$L^{(10,1)}(t)$	$U^{(19,3)}(t)$	$U^{(20,3)}(t)$	$U^{(1,2)}(t)$	$U^{(10,2)}(t)$	$U^{(10,3)}(t)$
50	7.6509e-01	3.2661e-01	3.9424e-01	**4.5853e-01**	7.3277e-01	1.0000e+00	1.0000e+00	**9.9929e-01**	9.9996e-01	9.9984e-01
100	5.2792e-02	1.8106e-03	5.4955e-03	**1.1016e-02**	5.0344e-02	1.0000e-01	1.0000e-01	**6.3534e-01**	9.6915e-01	9.6347e-01
200	5.0073e-05	1.0565e-07	8.0432e-06	**1.8105e-05**	4.9933e-05	9.979e-01	9.9521e-01	**3.4966e-03**	4.4172e-01	4.4063e-01
500	3.1668e-13	4.0243e-20	1.7698e-13	**2.6179e-13**	3.1668e-13	1.8333e-01	1.285e-01	**5.0153e-12**	2.8248e-04	2.8248e-04
1000	3.9157e-26	8.0974e-41	3.1323e-26	**3.787e-26**	3.916e-26	2.117e-06	7.8807e-07	**5.9623e-25**	4.7809e-12	4.7809e-12

Table 4: Results for $c = 10$ and $c = 20$, $p_N = \frac{1}{22}$, $p_J = \frac{1}{22}$, $\mathbf{p} = \mathbf{q}(1)$

t	$P\{W_{20,10}>t\}$	$L^{(0,1)}(t)$	$L^{(1,1)}(t)$	$L^{(2,1)}(t)$	$L^{(10,1)}(t)$	$U^{(19,3)}(t)$	$U^{(20,3)}(t)$	$U^{(1,2)}(t)$	$U^{(10,2)}(t)$	$U^{(10,3)}(t)$
50	4.4893e-09	3.5551e-09	3.6543e-09	**3.7498e-09**	4.3199e-09	9.1597e-02	5.5857e-02	**2.8412e-07**	1.3948e-04	5.9733e-05
100	3.2566e-23	1.2471e-23	1.3942e-23	**1.5489e-23**	2.7879e-23	1.2085e-05	3.3484e-06	**2.1398e-19**	1.0828e-12	2.5742e-13
200	1.2508e-50	3.7427e-52	5.5723e-52	**8.1643e-52**	7.5319e-51	9.8403e-16	1.2123e-16	**7.0747e-44**	1.1212e-28	2.3673e-29
500	3.0634e-130	3.7548e-137	1.9177e-136	**9.6537e-136**	6.1192e-131	4.9773e-50	2.4075e-51	**4.0447e-116**	1.241e-73	9.3361e-75
1000	4.1565e-261	8.3941e-279	3.9355e-277	**1.8689e-275**	1.4372e-262	4.3368e-110	1.0449e-111	**2.164e-236**	3.4208e-148	7.7398e-150

t	$P\{W_{20,20}>t\}$	$L^{(0,1)}(t)$	$L^{(1,1)}(t)$	$L^{(2,1)}(t)$	$L^{(10,1)}(t)$	$U^{(19,3)}(t)$	$U^{(20,3)}(t)$	$U^{(1,2)}(t)$	$U^{(10,2)}(t)$	$U^{(10,3)}(t)$
50	2.3151e-02	2.1275e-02	2.1509e-02	**2.1727e-02**	2.2871e-02	9.7127e-01	9.5345e-01	**7.3e-02**	4.5312e-01	3.956e-01
100	4.3526e-10	2.8867e-10	3.0563e-10	**3.2192e-10**	4.1349e-10	6.3011e-02	3.9051e-02	**1.0079e-08**	6.5431e-05	4.4039e-05
200	7.0316e-26	3.217e-26	3.7087e-26	**4.1872e-26**	6.6384e-26	6.9418e-09	2.0081e-09	**1.3897e-24**	1.3489e-15	1.1734e-15
500	2.7148e-67	5.1856e-68	1.1508e-67	**1.6144e-67**	2.6781e-67	4.3072e-39	4.7812e-40	**1.4378e-66**	4.9785e-50	4.9776e-50
1000	1.186e-134	1.3423e-136	5.5198e-135	**8.4902e-135**	1.185e-134	4.2107e-96	2.3056e-97	**1.0317e-133**	4.3368e-110	4.3368e-110

Table 5: Results for $c = 10$ and $c = 20$, $p_N = \frac{1}{3}$, $p_J = \frac{1}{4}$, $\mathbf{p} = \mathbf{q}(1)$

t	$P\{W_{20,10}>t\}$	$L^{(0,1)}(t)$	$L^{(1,1)}(t)$	$L^{(2,1)}(t)$	$L^{(10,1)}(t)$	$U^{(19,3)}(t)$	$U^{(20,3)}(t)$	$U^{(1,2)}(t)$	$U^{(10,2)}(t)$	$U^{(10,3)}(t)$
50	4.9493e-05	1.1207e-05	1.3107e-05	**1.5242e-05**	3.7927e-05	1	1	**4.7996e-02**	6.6713e-01	4.1167e-01
100	5.7068e-13	2.3565e-15	4.0932e-15	**7.0456e-15**	2.3849e-13	1	1	**3.1413e-06**	6.3662e-02	1.4619e-02
200	5.1495e-28	3.5613e-35	1.7077e-34	**8.2361e-34**	7.9031e-29	1	1	**5.0312e-16**	1.3092e-04	9.9319e-06
500	7.5013e-71	1.1e-94	1.3358e-92	**1.7535e-90**	5.8111e-73	1	1	**3.1782e-46**	6.6062e-13	1.0572e-14
1000	2.5455e-141	7.2036e-194	1.9317e-189	**6.3243e-185**	3.3628e-146	1	1	**1.3365e-96**	9.6982e-27	3.9781e-29
t	$P\{W_{20,20}>t\}$	$L^{(0,1)}(t)$	$L^{(1,1)}(t)$	$L^{(2,1)}(t)$	$L^{(10,1)}(t)$	$U^{(19,3)}(t)$	$U^{(20,3)}(t)$	$U^{(1,2)}(t)$	$U^{(10,2)}(t)$	$U^{(10,3)}(t)$
50	9.9636e-01	9.914e-01	9.9243e-01	**9.9324e-01**	9.9599e-01	1	1	**9.9998e-01**	1	1
100	6.9075e-01	5.0996e-01	5.4733e-01	**5.7748e-01**	6.7938e-01	1	1	**9.6267e-01**	1	1
200	9.4587e-02	2.2526e-02	4.0307e-02	**5.4226e-02**	9.2704e-02	1	1	**3.5352e-01**	1	1
500	1.6997e-04	8.6976e-07	7.534e-05	**1.1863e-04**	1.6987e-04	1	1	**1.4201e-03**	1	1
1000	8.3309e-09	3.7825e-14	5.5925e-09	**7.4845e-09**	8.3309e-09	1	1	**1.0626e-07**	1	1

Table 6: Results for $c = 10$ and $c = 20$, $p_N = \frac{1}{3}$, $p_J = 0$, $\mathbf{p} = \mathbf{q}(1)$

t	$P\{W_{20,10}>t\}$	$L^{(0,1)}(t)$	$L^{(1,1)}(t)$	$L^{(2,1)}(t)$	$L^{(10,1)}(t)$	$U^{(19,3)}(t)$	$U^{(20,3)}(t)$	$U^{(1,2)}(t)$	$U^{(10,2)}(t)$	$U^{(10,3)}(t)$
50	1.7309e-12	1.2441e-12	1.2856e-12	**1.3332e-12**	1.6437e-12	9.9962e-01	9.9837e-01	**1.0127e-10**	3.3243e-02	2.5113e-02
100	3.4982e-29	9.6148e-30	1.0914e-29	**1.2568e-29**	2.8683e-29	9.6141e-01	9.1478e-01	**1.531e-25**	2.0743e-05	1.2166e-05
200	5.8213e-62	5.5039e-64	8.5896e-64	**1.4128e-63**	3.0007e-62	4.4042e-01	3.0777e-01	**3.1022e-55**	8.7877e-12	3.3596e-12
500	5.4671e-158	1.0324e-166	6.1113e-166	**4.441e-165**	5.4135e-159	2.8248e-04	9.0905e-05	**2.5796e-144**	7.574e-31	1.1872e-31
1000	2.2448e-316	0.0000e+00	0.0000e+00	**0.0000e+00**	1.2367e-318	4.7809e-12	7.8929e-13	**8.8039e-293**	1.2748e-62	1.0761e-63
t	$P\{W_{20,20}>t\}$	$L^{(0,1)}(t)$	$L^{(1,1)}(t)$	$L^{(2,1)}(t)$	$L^{(10,1)}(t)$	$U^{(19,3)}(t)$	$U^{(20,3)}(t)$	$U^{(1,2)}(t)$	$U^{(10,2)}(t)$	$U^{(10,3)}(t)$
50	3.3318e-01	3.2661e-01	3.2732e-01	**3.281e-01**	3.3228e-01	1.0000e+00	1.0000e+00	**4.1779e-01**	9.9989e-01	9.9988e-01
100	1.9581e-03	1.8106e-03	1.8275e-03	**1.846e-03**	1.9405e-03	1.0000+00	1.0000+00	**3.2062e-03**	9.6407e-01	9.6354e-01
200	1.3525e-07	1.0565e-07	1.0959e-07	**1.1383e-07**	1.3272e-07	9.979e-01	9.9521e-01	**2.7477e-07**	4.4054e-01	4.405e-01
500	1.3991e-19	4.0243e-20	5.8095e-20	**7.6594e-20**	1.3703e-19	1.8333e-01	1.285e-01	**4.6731e-19**	2.8248e-04	2.8248e-04
1000	2.6655e-39	8.0974e-41	6.64e-40	**1.2516e-39**	2.6567e-39	2.117e-06	7.8807e-07	**1.1494e-38**	4.7809e-12	4.7809e-12

Table 7: Results for $c = 10$ and $c = 20$, $p_N = \frac{1}{22}$, $p_J = \frac{1}{22}$, $\mathbf{p} = \mathbf{q}(4)$

t	$P\{W_{20,10}>t\}$	$L^{(0,1)}(t)$	$L^{(1,1)}(t)$	$L^{(2,1)}(t)$	$L^{(10,1)}(t)$	$U^{(19,3)}(t)$	$U^{(20,3)}(t)$	$U^{(1,2)}(t)$	$U^{(10,2)}(t)$	$U^{(10,3)}(t)$
50	3.5891e-09	3.5551e-09	3.5586e-09	**3.5624e-09**	3.584e-09	9.1597e-02	5.8712e-02	**5.0364e-09**	1.1794e-04	1.0145e-04
100	1.2983e-23	1.2471e-23	1.2522e-23	**1.2579e-23**	1.2905e-23	1.2085e-05	3.595e-06	**4.0878e-23**	7.9932e-13	6.1249e-13
200	4.3909e-52	3.7427e-52	3.804e-52	**3.8734e-52**	4.2879e-52	9.8403e-16	1.2324e-16	**7.5454e-51**	7.8368e-29	5.765e-29
500	9.9616e-137	3.7548e-137	4.1388e-137	**4.6144e-137**	8.623e-137	4.9773e-50	2.4075e-51	**2.2162e-133**	6.0021e-74	3.2912e-74
1000	3.1334e-277	8.3941e-279	1.1903e-278	**1.7622e-278**	1.8835e-277	4.3368e-110	1.0449e-111	**6.4983e-271**	8.003e-149	2.6905e-149

t	$P\{W_{20,20}>t\}$	$L^{(0,1)}(t)$	$L^{(1,1)}(t)$	$L^{(2,1)}(t)$	$L^{(10,1)}(t)$	$U^{(19,3)}(t)$	$U^{(20,3)}(t)$	$U^{(1,2)}(t)$	$U^{(10,2)}(t)$	$U^{(10,3)}(t)$
50	2.1349e-02	2.1275e-02	2.1283e-02	**2.1291e-02**	2.1338e-02	9.7127e-01	9.5516e-01	**2.3584e-02**	4.4111e-01	4.3059e-01
100	2.9363e-10	2.8867e-10	2.8919e-10	**2.8977e-10**	2.9293e-10	6.3011e-02	3.9897e-02	**4.2689e-10**	5.988e-05	5.5515e-05
200	3.3263e-26	3.217e-26	3.2289e-26	**3.242e-26**	3.3119e-26	6.9418e-09	2.0231e-09	**5.0834e-26**	1.298e-15	1.2599e-15
1000	1.9281e-136	1.3423e-136	1.4268e-136	**1.5171e-136**	1.8882e-136	4.2107e-96	2.3056e-97	**4.4168e-136**	4.3368e-110	4.3368e-110

Table 8: Results for $c = 10$ and $c = 20$, $p_N = \frac{1}{3}$, $p_J = \frac{1}{4}$, $\mathbf{p} = \mathbf{q}(4)$

t	$P\{W_{20,10}>t\}$	$L^{(0,1)}(t)$	$L^{(1,1)}(t)$	$L^{(2,1)}(t)$	$L^{(10,1)}(t)$	$U^{(19,3)}(t)$	$U^{(20,3)}(t)$	$U^{(1,2)}(t)$	$U^{(10,2)}(t)$	$U^{(10,3)}(t)$
50	1.1715e-05	1.1207e-05	1.1257e-05	**1.1314e-05**	1.1637e-05	1	1	**3.4811e-05**	5.8862e-01	5.4915e-01
100	2.843e-15	2.3565e-15	2.4013e-15	**2.4522e-15**	2.7627e-15	1	1	**6.0092e-14**	3.9726e-02	3.1275e-02
200	7.4844e-35	3.5613e-35	3.8333e-35	**4.1613e-35**	6.6838e-35	1	1	**5.383e-32**	4.6471e-05	2.8275e-05
500	7.3392e-93	1.1e-94	1.6413e-94	**2.572e-94**	4.0418e-93	1	1	**3.2394e-86**	4.9227e-14	1.631e-14
1000	4.461e-188	7.2036e-194	2.392e-193	**9.201e-193**	8.6959e-189	1	1	**1.3884e-176**	5.3851e-29	8.9056e-30

t	$P\{W_{20,20}>t\}$	$L^{(0,1)}(t)$	$L^{(1,1)}(t)$	$L^{(2,1)}(t)$	$L^{(10,1)}(t)$	$U^{(19,3)}(t)$	$U^{(20,3)}(t)$	$U^{(1,2)}(t)$	$U^{(10,2)}(t)$	$U^{(10,3)}(t)$
50	9.9159e-01	9.914e-01	9.9142e-01	**9.9145e-01**	9.9157e-01	1	1	**9.944e-01**	1	1
100	5.1567e-01	5.0996e-01	5.1059e-01	**5.1128e-01**	5.1493e-01	1	1	**5.8266e-01**	1	1
200	2.3945e-02	2.2526e-02	2.2693e-02	**2.2876e-02**	2.3783e-02	1	1	**3.4855e-02**	1	1
500	1.2463e-06	8.6976e-07	9.2346e-07	**9.8077e-07**	1.2196e-06	1	1	**2.8337e-06**	1	1
1000	1.2683e-13	3.7825e-14	5.3799e-14	**7.0363e-14**	1.2427e-13	1	1	**4.2262e-13**	1	1

Table 9: Results for $c = 10$ and $c = 20$, $p_N = \frac{1}{3}$, $p_J = 0$, $\mathbf{p} = \mathbf{q}(4)$