DOI: 10.2478/auom-2024-0020
An. Şt. Univ. Ovidius Constanţa **Vol. 32(2),2024, 85–97**

On generalized osculating-type curves in Myller configuration

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Abstract

In this study, we examine osculating-type curves with Frenet-type frame in Myller configuration for Euclidean 3-space E_3. We present the necessary characterizations for a curve to be an osculating-type curve. Characterizations originating from the natural structure of Myller configuration are a generalization of osculating curves according to the Frenet frame. Also, we introduce some new results that are not valid for osculating curves. Then, we give an illustrative numerical example supported by a figure.

1 Introduction and Basic Concepts

The theory of curves has quite an importance and applications in several work-frames such as; mathematics, architecture, engineering, etc., and also attracts a lot of researchers. It is quite a fundamental concept of differential geometry in mathematics, as well. Also, moving frames are one of the most important work-frames in differential geometry from the investigation of the Frenet (Serret-Frenet) frame [6, 29]. In the Euclidean 3-space E_3, every unit speed curve $C : I \to E_3$ can be associated with the orthogonal unit vector fields tangent vector field T, principal normal vector field N, and binormal vector field B. The planes spanned by $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are called the osculating plane, rectifying plane, and normal plane at each point of the

Key Words: Myller configuration, osculating-type curve, versor field.
2010 Mathematics Subject Classification: Primary 58A04.
Received: 26.03.2023
Accepted: 29.07.2023

curve C, respectively [14]. Special curve types named osculating, rectifying curves, and normal curves (spherical curves) which satisfy Cesaro's fixed point condition (see [28]) have been examined in several studies [1–5, 8–18, 31, 32].

The curve $C : I \to E_3$ for which the position vector of the curve C always lies in their rectifying plane, is named rectifying curves [2–5, 8, 11–14]. Also, if the position vector of the curve C always lies in its normal plane, the curve is named normal curve [1, 9, 10, 14, 15, 31, 32]. Then, if the position vector of the curve C always lies in its osculating plane, the curve is named osculating curve [14, 16, 18].

In E_3, one describes some versor fields* such as; tangent, principal normal, and binormal, alongside some plane fields such as; rectifying, normal, and osculating planes along the curve C. By and large, a versor field and a plane field are denoted by $(C, \overline{\zeta})$ and (C, π), respectively. The couple $\{(C, \overline{\zeta}), (C, \pi)\}$ where $\overline{\zeta} \in \pi$ is named a Myller configuration and denoted by $\mathfrak{M}(C, \overline{\zeta}, \pi)$ in Euclidean 3-space E_3 [26, 27]. Radu Miron completed some studies with respect to this pair $\left\{(C, \overline{\zeta}), (C, \pi)\right\}$ in 1960 [26]. Provided that the plane π is tangent to the curve C, we have a tangent Myller configuration which is denoted by $\mathfrak{M}_t(C, \overline{\zeta}, \pi)$ [24, 27]. An investigation of the geometry of Myller configurations $\mathfrak{M}(C, \overline{\zeta}, \pi)$ and tangent Myller configuration $\mathfrak{M}_t(C, \overline{\zeta}, \pi)$ is given and examined in the book [27] by Miron. On the other hand, if C is a curve on the surface S, the geometry of the field (C, ζ) on surface S is the geometry of the associated Myller configurations $\mathfrak{M}_t(C, \overline{\zeta}, \pi)$ [27]. It is seen that the theory of geometry for tangent Myller configuration $\mathfrak{M}_t(C, \overline{\zeta}, \pi)$ is a special case of that for general Myller configuration $\mathfrak{M}(C, \overline{\zeta}, \pi)$ [27]. Moreover, the Darboux frame is determined for a Myller configuration [27]. For more detailed information on the concept of Myller configuration, we can refer to the book [27].

In the existing literature, there are several studies based on Myller configuration. Recently, Macsim et al. examined the special curves in a Myller configuration and their properties [25]. Also, Macsim et al. introduced the rectifying-type curves [24] and Bertrand curves [23] in Myller configuration. Rectifying-type curves and Bertrand curves with Frenet-type frame in Myller configuration for Euclidean space E_3 are a generalization of rectifying curves and Bertrand curves with Frenet frame in E_3 [23, 24]. Keskin and Yaylı investigated the rectifying-type curves with rotation minimizing frame in n-dimensional space of $\mathbb{R}$ in [22]. Additionally, versor fields along a curve in a four-dimensional Lorentz space are scrutinized in [7]. Also, Smarandache curves with Frenet-type frame in Myller configuration were examined in [21]. Moreover, osculating-type curves in Myller configuration for Euclidean 4-space E_4 [19] and rectifying-type curves in Myller configuration for Euclidean 4-space E_4 [20] were studied.

*unit vector fields

Let $(C, \overline{\zeta})$ be a versor field and $\overline{r}(s)$ be a position vector of the curve C where s is the arc-length on the curve C. For Frenet-type frame $\mathcal{R}_F = \{P, \overline{\zeta}_1, \overline{\zeta}_2, \overline{\zeta}_3\}$ of versor field, we can write:

$$\overline{r}'(s) = \varrho_1(s)\overline{\zeta}_1(s) + \varrho_2(s)\overline{\zeta}_2(s) + \varrho_3(s)\overline{\zeta}_3(s), \tag{1.1}$$

where $\varrho_1^2(s) + \varrho_2^2(s) + \varrho_3^2(s) = 1$. Also, the following derivative formulas hold:

$$\begin{cases} \overline{\zeta}_1'(s) = \mathcal{K}_1(s)\overline{\zeta}_2(s), \\ \overline{\zeta}_2'(s) = -\mathcal{K}_1(s)\overline{\zeta}_1(s) + \mathcal{K}_2(s)\overline{\zeta}_3(s), \\ \overline{\zeta}_3'(s) = -\mathcal{K}_2(s)\overline{\zeta}_2(s), \end{cases} \tag{1.2}$$

where $\mathcal{K}_1 > 0$. $\mathcal{K}_1$-curvature and $\mathcal{K}_2$-torsion have the same geometrical interpretation as the curvature and torsion of a curve in E_3. It is clear that, if $\varrho_1(s) = 1$, $\varrho_2(s) = 0$ and $\varrho_3(s) = 0$, we have Frenet equations of a curve in Euclidean 3-space E_3 [27]. Additionally, the fundamental theorem of invariants for versor field $(C, \overline{\zeta})$ is given as follows:

Theorem 1.1 ([27]). *If the invariants $\mathcal{K}_1(s), \mathcal{K}_2(s), \varrho_1(s), \varrho_2(s), \varrho_3(s)$, with $\varrho_1^2(s) + \varrho_2^2(s) + \varrho_3^2(s) = 1$ are smooth functions for $s \in [a, b]$, then there exist a curve $C : [a, b] \to E_3$ parameterized by arc-length s and a versor field $\overline{\zeta}(s)$, $s \in [a, b]$, whose the curvature, torsion and the functions $\varrho_i(s)$ are $\mathcal{K}_1(s), \mathcal{K}_2(s)$ and $\varrho_i(s), i = 1, 2, 3$. Any two such versor fields $\left(C, \overline{\zeta}\right)$ differ by a proper Euclidean motion.*

In this work, we introduce osculating-type curves in a Myller configuration. We express the necessary conditions for a curve to be an osculating-type curve with Frenet-type frame in Myller configuration for E_3. We have shown that well-known situations in Euclidean space do not always hold. Considering the versor field and plane areas due to the structure of the osculating curves, we believe that it is a work that sheds light. The osculating curves with Frenet frame in E_3 are one of the special cases for the generalized osculating-type curves since the geometry of versor fields along a curve with Myller configuration in E_3 is a generalization of the usual theory of curves in E_3 (see for generalization [7, 27]).

2 Generalized Osculating-Type Curves

In this section, we introduce the osculating-type curves with Frenet-type frame in Myller configuration for Euclidean space E_3. We acquire the conditions for being osculating-type curves in Myller configuration and some characterizations with respect to them. Also, we get some corollaries for the cases of invariants $\varrho_1(s), \varrho_2(s)$ and $\varrho_3(s)$ with the condition $\varrho_1^2(s)+\varrho_2^2(s)+\varrho_3^2(s)=1$. Then, we see the osculating-type curves in Myller configuration for Euclidean space E_3 is a generalization of osculating curves in Euclidean space E_3.

Definition 2.1. *$\overline{r}(s): I \to E_3$ is called osculating-type curve with Frenet-type frame in Myller configuration for Euclidean space E_3 if*

$$\overline{r}(s)=\eta(s)\overline{\zeta}_1(s)+\omega(s)\overline{\zeta}_2(s), \quad s\in I, \tag{2.1}$$

where $\eta(s)$ and $\omega(s)$ are smooth functions.

Before starting the relevant theorems, let us give some preliminary preparations:

By differentiating the equation (2.1) and using the equation (1.1), we get as follows:

$$\begin{aligned}\varrho_1(s)\overline{\zeta}_1(s)+\varrho_2(s)\overline{\zeta}_2(s)+\varrho_3(s)\overline{\zeta}_3(s) =& \eta'(s)\overline{\zeta}_1(s)+\eta(s)\mathcal{K}_1(s)\overline{\zeta}_2(s)\\ &+\omega'(s)\overline{\zeta}_2(s)\\ &+\omega(s)\begin{pmatrix}-\mathcal{K}_1(s)\overline{\zeta}_1(s)\\ +\mathcal{K}_2(s)\overline{\zeta}_3(s)\end{pmatrix}.\end{aligned} \tag{2.2}$$

Then, we have:

$$\begin{cases}\eta'(s)-\omega(s)\mathcal{K}_1(s)=\varrho_1(s),\\ \omega'(s)+\eta(s)\mathcal{K}_1(s)=\varrho_2(s),\\ \omega(s)\mathcal{K}_2(s)=\varrho_3(s).\end{cases} \tag{2.3}$$

We should examine the solutions of equation (2.3) according to the cases $\varrho_3(s)=0$ or $\varrho_3(s)\neq 0$. First, let us examine the situation $\varrho_3(s)\neq 0$. Since $\varrho_1(s)=1$, $\varrho_2(s)=\varrho_3(s)=0$ is accepted for the Frenet frame in Euclidean space E_3, the case $\varrho_3\neq 0$ will be a new classification that does not correspond to Frenet frame in E_3.

Theorem 2.1. *Let $\overline{r}(s): I\to E_3$ be a curve with Frenet-type frame in Myller configuration for Euclidean space E_3. Then, $\overline{r}(s)$ is an osculating-type curve if and only if*

$$\overline{r}(s)=\left(\frac{\varrho_2(s)}{\mathcal{K}_1(s)}-\frac{1}{\mathcal{K}_1(s)}\left(\frac{\varrho_3(s)}{\mathcal{K}_2(s)}\right)'\right)\overline{\zeta}_1(s)+\frac{\varrho_3(s)}{\mathcal{K}_2(s)}\overline{\zeta}_2(s), \tag{2.4}$$

where $\varrho_3(s)\neq 0$.

Proof. Suppose that $\overline{r}(s)$ is an osculating-type curve with $\varrho_3(s)\neq 0$. According to the equation (2.3), we get $\omega(s)\neq 0$ and $\mathcal{K}_2(s)\neq 0$. Then, we obtain:

$$\omega(s)=\frac{\varrho_3(s)}{\mathcal{K}_2(s)}$$

and since $\mathcal{K}_1(s) > 0$, we have:

$$\eta(s) = \frac{\varrho_2(s)}{\mathcal{K}_1(s)} - \frac{1}{\mathcal{K}_1(s)}\left(\frac{\varrho_3(s)}{\mathcal{K}_2(s)}\right)'.$$

Therefore, if the equation (2.1) is considered, what is desired is shown. Conversely, assume that the equation (2.4) is provided when $\varrho_3(s) \neq 0$. Then, we obtain:

$$\begin{cases} \left\langle \overline{r}(s), \overline{\zeta}_1(s) \right\rangle = \dfrac{\varrho_2(s)}{\mathcal{K}_1(s)} - \dfrac{1}{\mathcal{K}_1(s)}\left(\dfrac{\varrho_3(s)}{\mathcal{K}_2(s)}\right)', \\ \left\langle \overline{r}(s), \overline{\zeta}_2(s) \right\rangle = \dfrac{\varrho_3(s)}{\mathcal{K}_2(s)}. \end{cases} \tag{2.5}$$

By differentiating of the last equation of equation (2.5), we get:

$$\varrho_2(s) - \mathcal{K}_1(s)\left\langle \overline{r}(s), \overline{\zeta}_1(s) \right\rangle + \mathcal{K}_2(s)\left\langle \overline{r}(s), \overline{\zeta}_3(s) \right\rangle = \left(\frac{\varrho_3(s)}{\mathcal{K}_2(s)}\right)'.$$

By using the equation $\left\langle \overline{r}(s), \overline{\zeta}_1(s) \right\rangle$, since $\mathcal{K}_2(s) \neq 0$, we get $\left\langle \overline{r}(s), \overline{\zeta}_3(s) \right\rangle = 0$. Therefore, $\overline{r}(s)$ is an osculating-type curve. □

Remark 2.2. *In Euclidean space E_3, the curve is an osculating curve if and only if the curve is a planar curve (i.e. $\tau = 0$) for Frenet frame (cf. [14]). However, this is not the case in the case of $\varrho_3(s) \neq 0$ in Myller configuration.*

Theorem 2.3. *Let $\overline{r}(s) : I \to E_3$ be a curve with Frenet-type frame in Myller configuration for Euclidean space E_3 with $\varrho_3(s) \neq 0$. Then, $\overline{r}(s)$ is an osculating-type curve if and only if $\mathcal{K}_1(s)$-curvature, $\mathcal{K}_2(s)$-torsion and the functions $\varrho_1(s), \varrho_2(s), \varrho_3(s)$ satisfy the following relation*

$$\left(\frac{\varrho_2(s)}{\mathcal{K}_1(s)} - \frac{1}{\mathcal{K}_1(s)}\left(\frac{\varrho_3(s)}{\mathcal{K}_2(s)}\right)'\right)' - \frac{\varrho_3(s)}{\mathcal{K}_2(s)}\mathcal{K}_1(s) - \varrho_1(s) = 0. \tag{2.6}$$

Proof. Assume that $\overline{r}(s)$ be an osculating-type curve where $\varrho_3(s) \neq 0$. In the first equation of the equation (2.3), by using $\eta(s)$ and $\omega(s)$, we have the equation (2.6). Conversely, suppose that $\overline{r}$ is a curve satisfying the relation given in the equation (2.6). Then, we have:

$$\frac{d}{ds}\left[\overline{r}(s) - \left(\frac{\varrho_2(s)}{\mathcal{K}_1(s)} - \frac{1}{\mathcal{K}_1(s)}\left(\frac{\varrho_3(s)}{\mathcal{K}_2(s)}\right)'\right)\overline{\zeta}_1(s) - \frac{\varrho_3(s)}{\mathcal{K}_2(s)}\overline{\zeta}_2(s)\right] = 0.$$

Consequently, $\overline{r}(s)$ is an osculating-type curve. □

Since $\varrho_3(s) \neq 0$, some special solutions of differential equation given in the equation (2.6) where $\varrho_1^2(s) + \varrho_2^2(s) + \varrho_3^2(s) = 1$ are as follows:

Corollary 2.1. *Let $\overline{r}(s) : I \to E_3$ be an osculating-type curve with Frenet-type frame in Myller configuration for Euclidean space E_3 with $\varrho_3(s) \neq 0$. If $\varrho_1(s) = 0$ and $\varrho_2^2(s) + \varrho_3^2(s) = 1$, we get:*

$$\left(-\frac{1}{\mathcal{K}_1(s)}\left(\frac{\varrho_3(s)}{\mathcal{K}_2(s)}\right)'\right)' - \frac{\varrho_3(s)}{\mathcal{K}_2(s)}\mathcal{K}_1(s) = 0. \tag{2.7}$$

Equation (2.7) *is a homogeneous differential equation with variable coefficients.*

Corollary 2.2. *Let $\overline{r}(s) : I \to E_3$ be an osculating-type curve with Frenet-type frame in Myller configuration for Euclidean space E_3 with $\varrho_3(s) \neq 0$. If $\varrho_2(s) = 0$ and $\varrho_1^2(s) + \varrho_3^2(s) = 1$, we have:*

$$\left(-\frac{1}{\mathcal{K}_1(s)}\left(\frac{\varrho_3(s)}{\mathcal{K}_2(s)}\right)'\right)' - \frac{\varrho_3(s)}{\mathcal{K}_2(s)}\mathcal{K}_1(s) = \varrho_1(s).$$

Since $\varrho_3(s) \neq 0$, assume that $\dfrac{\varrho_3(s)}{\mathcal{K}_2(s)} = y(s)$ and $\dfrac{1}{\mathcal{K}_1(s)} = p(s)$, we have:

$$(p(s)y'(s))' + \frac{y(s)}{p(s)} = -\varrho_1(s).$$

Hence, an inhomogeneous differential equation is obtained from the second order. This differential equation is first solved with respect to the homogeneous part, then special solutions are generated depending on the function $\varrho_1(s)$.

Corollary 2.3. *Let $\overline{r}(s) : I \to E_3$ be an osculating-type curve with Frenet-type frame in Myller configuration for Euclidean space E_3 with $\varrho_3(s) \neq 0$. If $\varrho_1(s) = 0$, $\varrho_2(s) = 0$ and $\varrho_3(s) = 1$, we get:*

$$\left(-\frac{1}{\mathcal{K}_1(s)}\left(\frac{1}{\mathcal{K}_2(s)}\right)'\right)' - \frac{\mathcal{K}_1(s)}{\mathcal{K}_2(s)} = 0.$$

Let us assume that $\dfrac{1}{\mathcal{K}_2(s)} = y(s)$ and $\dfrac{1}{\mathcal{K}_1(s)} = p(s)$, we have:

$$(p(s)y'(s))' + \frac{y(s)}{p(s)} = 0.$$

If we apply the $t = \int \frac{1}{p(s)} ds$, *we get:*

$$\frac{d^2y}{dt^2} + y = 0. \tag{2.8}$$

The solution of the differential equation (2.8) *is* $y = c_1 \cos t + c_2 \sin t$ *where* c_1 *and* c_2 *are constants. Since,* $\frac{1}{\mathcal{K}_2(s)} = y(s)$ *and* $t = \int \frac{1}{p} ds$, *we can write:*

$$\frac{1}{\mathcal{K}_2(s)} = c_1 \cos\left(\int \mathcal{K}_1(s) ds\right) + c_2 \sin\left(\int \mathcal{K}_1(s) ds\right).$$

Therefore,

$$\begin{cases} \omega(s) = \dfrac{1}{\mathcal{K}_2(s)} = c_1 \cos\left(\displaystyle\int \mathcal{K}_1(s) ds\right) + c_2 \sin\left(\displaystyle\int \mathcal{K}_1(s) ds\right), \\ \eta(s) = -\dfrac{1}{\mathcal{K}_1(s)} \left(\dfrac{1}{\mathcal{K}_2(s)}\right)' \\ \qquad = c_1 \sin\left(\displaystyle\int \mathcal{K}_1(s) ds\right) - c_2 \cos\left(\displaystyle\int \mathcal{K}_1(s) ds\right). \end{cases} \tag{2.9}$$

According to the equation (ürkür), *we get:*

$$\langle \overline{r}(s), \overline{r}(s) \rangle = \eta^2(s) + \omega^2(s) = c_1^2 + c_2^2,$$

where c_1 *and* c_2 *are constants. Obviously, the osculating-type curve* $\overline{r}(s)$ *is a spherical curve when* $\varrho_1(s) = 0$, $\varrho_2(s) = 0$ *and* $\varrho_3(s) = 1$.

Now let us give the characterizations for the case of $\varrho_3(s) = 0$. According to the equation (2.3), we get $\omega(s) = 0$ or $\mathcal{K}_2(s) = 0$.

Proposition 2.1. *Let* $\overline{r}(s) : I \to E_3$ *is a curve with Frenet-type frame in Myller configuration for Euclidean space* E_3. *If* $\overline{r}(s)$ *is an osculating-type curve where* $\varrho_3(s) = 0$, *the functions* $\eta(s) = \left\langle \overline{r}(s), \overline{\zeta}_1(s) \right\rangle$, $\omega(s) = \left\langle \overline{r}(s), \overline{\zeta}_2(s) \right\rangle$ *and* $\varrho_1(s), \varrho_2(s), \varrho_3(s)$ *satisfy*

$$\eta^2(s) + \omega^2(s) = 2 \int \left(\eta(s)\varrho_1(s) + \omega(s)\varrho_2(s)\right) ds, \tag{2.10}$$

where $\mathcal{K}_2(s) = 0$ *or* $\omega(s) = 0$.

Proof. Let $\overline{r}(s)$ be a curve with $\varrho_3(s) = 0$ and we get $\omega(s) = 0$ or $\mathcal{K}_2(s) = 0$. Then, from the equation (2.3), we have:

$$\begin{cases} \eta'(s) - \omega(s)\mathcal{K}_1(s) = \varrho_1(s), \\ \omega'(s) + \eta(s)\mathcal{K}_1(s) = \varrho_2(s). \end{cases} \tag{2.11}$$

If each side of the first equation of (2.11) is multiplied by $\eta(s)$, and each side of the second equation is multiplied by $\omega(s)$, then

$$\eta(s)\eta'(s)+\omega(s)\omega'(s)=\eta(s)\varrho_1(s)+\omega(s)\varrho_2(s). \tag{2.12}$$

By integrating both sides of the equation (2.12), the equation (2.10) is obtained. □

Proposition 2.2. *Let $\overline{r}(s): I \to E_3$ is a curve with Frenet-type frame in Myller configuration for Euclidean space E_3. If*

$$\eta^2(s)+\omega^2(s)=2\int\left(\eta(s)\varrho_1(s)+\omega(s)\varrho_2(s)\right)ds \tag{2.13}$$

one of the expressions

(i) $\mathcal{K}_2(s)=0$, *that is* $\varrho_3(s)=0$

(ii) $\omega(s)=0$, *that is* $\varrho_3(s)=0$ *and* $\overline{r}(s)$ *is a rectifying-type curve*

(iii) $\overline{r}(s)$ *is a osculating-type curves*

is provided.

Proof. Assume that $\overline{r}(s)$ be a curve with the equation (2.13). Then, we have:

$$\eta(s)\eta'(s)+\omega(s)\omega'(s)=\eta(s)\varrho_1(s)+\omega(s)\varrho_2(s).$$

By using the following equations

$$\begin{cases} \eta(s)=\left\langle \overline{r}(s),\overline{\zeta}_1(s)\right\rangle, \\ \eta'(s)=\varrho_1(s)+\left\langle \overline{r}(s),\mathcal{K}_1(s)\overline{\zeta}_2(s)\right\rangle, \\ \omega(s)=\left\langle \overline{r}(s),\overline{\zeta}_2(s)\right\rangle, \\ \omega'(s)=\varrho_2(s)+\left\langle \overline{r}(s),-\mathcal{K}_1(s)\overline{\zeta}_1(s)+\mathcal{K}_2(s)\overline{\zeta}_3(s)\right\rangle, \end{cases}$$

we get:

$$\mathcal{K}_2(s)\left\langle \overline{r}(s),\overline{\zeta}_2(s)\right\rangle\left\langle \overline{r}(s),\overline{\zeta}_3(s)\right\rangle=0. \tag{2.14}$$

That is, $\mathcal{K}_2(s)=0$ or $\left\langle \overline{r}(s),\overline{\zeta}_2(s)\right\rangle=0$ or $\left\langle \overline{r}(s),\overline{\zeta}_3(s)\right\rangle=0$. If $\mathcal{K}_2(s)=0$, then $\varrho_3(s)=0$. If $\left\langle \overline{r}(s),\overline{\zeta}_2(s)\right\rangle=0$, $\overline{r}(s)$ is a rectifying-type curve with $\varrho_3(s)=0$ [24]. However, from the definition of osculating-type curves, $\left\langle \overline{r}(s),\overline{\zeta}_2(s)\right\rangle=0$ is a contradiction. So, $\left\langle \overline{r}(s),\overline{\zeta}_2(s)\right\rangle\neq 0$. If $\left\langle \overline{r}(s),\overline{\zeta}_3(s)\right\rangle=0$, $\overline{r}(s)$ is an osculating-type curve. □

Special Case 2.4. *Let $\overline{r}(s) : I \to E_3$ is an osculating-type curve with Frenet-type frame in Myller configuration for Euclidean space E_3. Therefore, If $\varrho_1(s) = 1$, $\varrho_2(s) = \varrho_3(s) = 0$, we have the equation* (2.10). *That is,*

$$\eta^2(s) + \omega^2(s) = 2 \int \eta(s) ds. \tag{2.15}$$

The expression (2.15) *is a characterization of osculating curves in Euclidean space according to Frenet frame [14]. Then, we get:*

(i) *If $\eta(s) = 0$, since $\mathcal{K}_2(s) = 0$ and $\mathcal{K}_1(s) =$constant, $\overline{r}(s)$ is a circle.*

(ii) *If $\omega(s) = 0$, since $\mathcal{K}_1(s) = 0$, $\overline{r}(s)$ is a line.*

By inspiring the example of the study [30], we construct the following example:

Example 2.5. *Let us consider versor fields and invariants as following*

$$\begin{cases} \overline{\zeta}_1(s) = \left(-\frac{4}{5}\sin(s), -\cos(s), \frac{3}{5}\sin(s) \right), \\ \overline{\zeta}_2(s) = \left(-\frac{4}{5}\cos(s), \sin(s), \frac{3}{5}\cos(s) \right), \\ \overline{\zeta}_3(s) = \left(-\frac{3}{5}, 0, -\frac{4}{5} \right), \end{cases}$$

and

$$\begin{cases} \mathcal{K}_1(s) = 1, \\ \mathcal{K}_2(s) = 0. \end{cases}$$

By choosing $\varrho_1(s) = \sin(s)$, $\varrho_2(s) = \cos(s)$, $\varrho_3(s) = 0$, we have:

$$\overline{r}(s) = \left(-\frac{4s}{5}, 1, \frac{3s}{5} \right) \tag{2.16}$$

and

$$\begin{aligned} \frac{d\overline{r}}{ds} &= \left(-\frac{4}{5}, 0, \frac{3}{5} \right) \\ &= \sin(s)\left(-\frac{4}{5}\sin(s), -\cos(s), \frac{3}{5}\sin s \right) + \cos(s)\left(-\frac{4}{5}\cos(s), \sin(s), \frac{3}{5}\cos(s) \right) \\ &= \varrho_1(s)\overline{\zeta}_1(s) + \varrho_2(s)\overline{\zeta}_2(s). \end{aligned}$$

Therefore, we get:

$$\overline{r}(s) = \eta(s)\overline{\zeta}_1(s) + \omega(s)\overline{\zeta}_2(s), \tag{2.17}$$

where

$$\begin{cases} \eta(s) = s\sin(s) - \cos(s), \\ \omega(s) = s\cos(s) + \sin(s). \end{cases}$$

Consequently, $\overline{r}(s)$ is a straight line in E_3, and $\overline{r}(s)$ is an osculating-type curve in Myller configuration.

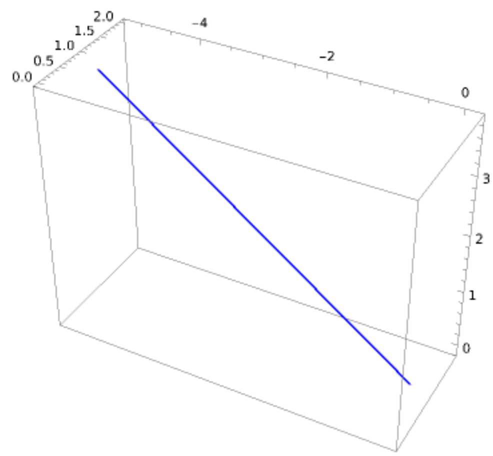

Figure 1: The curve with parametric representation written in the equation (2.16)

It should be noted that the Figure 1 is drawn by using the Wolfram Mathematica (Wolfram Cloud).

3 Conclusion

In this paper, we determine osculating-type curves with Frenet-type frame in Myller configuration for Euclidean space E_3. We present some necessary characterizations for a curve to be an osculating-type curve. Then, we get some new results that are not valid for osculating curves with Frenet frame in E_3. Also, we construct a numerical example with a figure. As one can say, the osculating curves with Frenet frame in E_3 are one of the special cases for the generalized osculating-type curves because of the fact that the geometry of versor fields along a curve with Myller configuration in E_3 is a generalization of the classical theory of curves in E_3.

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