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Quasifinite fields of prescribed characteristic and Diophantine dimension

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Abstract

Let $\mathbb{P}$ be the set of prime numbers, $\overline{\mathbb{P}}$ the union $\mathbb{P} \cup \{0\}$, and for any field E, let $\mathrm{char}(E)$ be its characteristic, $\mathrm{ddim}(E)$ the Diophantine dimension of E, $\mathcal{G}_E$ the absolute Galois group of E, and $\mathrm{cd}(\mathcal{G}_E)$ the Galois cohomological dimension $\mathcal{G}_E$. The research presented in this paper is motivated by the open problem of whether $\mathrm{cd}(\mathcal{G}_E) \leq \mathrm{ddim}(E)$. It proves the existence of quasifinite fields $\Phi_q : q \in \mathbb{P}$, with $\mathrm{ddim}(\Phi_q)$ infinity and $\mathrm{char}(\Phi_q) = q$, for each q. It shows that for any integer $m > 0$ and $q \in \overline{\mathbb{P}}$, there is a quasifinite field $\Phi_{m,q}$ such that $\mathrm{char}(\Phi_{m,q}) = q$ and $\mathrm{ddim}(\Phi_{m,q}) = m$. This is used for proving that for any $q \in \overline{\mathbb{P}}$ and each pair $k, \ell \in (\mathbb{N} \cup \{0, \infty\})$ satisfying $k \leq \ell$, there exists a field $E_{k,\ell;q}$ with $\mathrm{char}(E_{k,\ell;q}) = q$, $\mathrm{ddim}(E_{k,\ell;q}) = \ell$ and $\mathrm{cd}(\mathcal{G}_{E_{k,\ell;q}}) = k$. Finally, we show that the field $E_{k,\ell;q}$ can be chosen to be perfect unless $k = 0 \neq \ell$.

1 Introduction

Let F be a field, F_{sep} its separable closure, $\mathrm{Fe}(F)$ the set of finite extensions of F in F_{sep}, and $\mathcal{G}_F$ the absolute Galois group of F, i.e. the Galois group $\mathcal{G}(F_{\mathrm{sep}}/F)$. By definition, the Brauer dimension $\mathrm{Brd}(F)$ of F, introduced in

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[4], is equal to the least integer $n \geq 0$, for which the degree $\deg(D)$ of any finite-dimensional associative central division F-algebra D divides $\exp(D)^n$, where $\exp(D)$ is the exponent of D, i.e. the order of the equivalence class $[D]$ of D as an element of the Brauer group $\mathrm{Br}(F)$; if no such n exists, $\mathrm{Brd}(F)$ is defined to be infinity. The Brauer p-dimension $\mathrm{Brd}_p(F)$ is defined analogously, for each prime number p, by letting D run across the class of central division E-algebras of p-primary degrees. In view of the primary tensor product decomposition theorem for central division algebras over fields (cf. [27], Sect. 14.4), $\mathrm{Brd}(F)$ equals the supremum of $\mathrm{Brd}_p(F)\colon p \in \mathbb{P}$, where $\mathbb{P}$ is the set of prime numbers. As in [7], the supremum $\sup\{\mathrm{Brd}_p(Y)\colon Y \in \mathrm{Fe}(F)\}$, denoted by $\mathrm{abrd}_p(F)$, is called an absolute Brauer p-dimension of F. We say that F is a field of dimension ≤ 1, in the sense of Serre (see [28], Ch. II, 3.1), if $\mathrm{Br}(Y') = \{0\}$, i.e. $\mathrm{Brd}(Y') = 0$, for every algebraic field extension Y'/F; in view of the Albert-Hochschild theorem [28, Ch. II, 2.2] (and well-known general properties of maximal separable subextensions of algebraic extensions, see [22, Ch. V, Sects. 4 and 6]), this holds if and only if $\mathrm{abrd}_p(F) = 0$, for every $p \in \mathbb{P}$.

The cohomological dimension $\mathrm{cd}(\mathcal{G}_F)$ of $\mathcal{G}_F$ (viewed as a profinite group, whence, a compact totally disconnected group with respect to the Krull topology) is defined to be the supremum of the cohomological p-dimensions $\mathrm{cd}_p(\mathcal{G}_F)$, $p \in \mathbb{P}$. By Galois cohomology (cf. [28], Ch. I, Proposition 21), $\mathrm{cd}_p(\mathcal{G}_F) \leq n$, for a given $p \in \mathbb{P}$ and an integer $n \geq 0$, if and only if the cohomology group $H^{n+1}(\mathcal{G}_F, \mathbb{Z}/p\mathbb{Z})$ is trivial, where the field $\mathbb{Z}/p\mathbb{Z}$ (of residue classes of integers modulo p) is viewed as a discrete $\mathcal{G}_F$-module with the trivial action of $\mathcal{G}_F$; also, $\mathrm{cd}_p(\mathcal{G}_F) = n$ if and only if n is minimal with this property. We set $\mathrm{cd}_p(\mathcal{G}_F) = \infty$ if $H^n(\mathcal{G}_F, \mathbb{Z}/p\mathbb{Z}) \neq \{0\}$, for all $n \in \mathbb{N}$. When $p \neq \mathrm{char}(F)$, it follows that $\mathrm{cd}_p(\mathcal{G}_F) \leq n < \infty$ if and only if the cohomology group $H^{n+1}(\mathcal{G}_{F(\mu(p))}, \mu(p))$ is trivial, $F(\mu(p))$ being the extension of F generated by the set (in fact, a multiplicative group) $\mu(p)$ of p-th roots of unity lying in F_{sep} (cf. [28, Ch. I, Proposition 14], and [22, Ch. VI, Sect. 3]). Thus the computation of $\mathrm{cd}_p(\mathcal{G}_F)$ reduces to the special case in which F contains a primitive p-th root of unity unless $p = \mathrm{char}(F)$.

It is well-known (cf. [28, Ch. II, 3.1]) that $\dim(F) \leq 1$ if and only if $\mathrm{Br}(F') = \{0\}$, where F' runs across $\mathrm{Fe}(F)$. For example, $\dim(F_{\mathrm{sep}}) \leq 1$; this follows from the Albert-Hochschild theorem if $\mathrm{char}(F) \neq 0$. When F is perfect, we have $\dim(F) \leq 1$ if and only if $\mathrm{cd}(\mathcal{G}_F) \leq 1$. For example, $\dim(F) \leq 1$ if F is a quasifinite field, i.e. a perfect field which admits a unique extension in F_{sep} of degree n, for each $n \in \mathbb{N}$. It is known that then $\mathcal{G}_F$ is isomorphic to $\mathcal{G}_{\mathbb{F}}$, for any finite field $\mathbb{F}$. The inequality $\mathrm{cd}(\mathcal{G}_F) \leq 1$ holds, since $\mathcal{G}_F$ is isomorphic to the topological group product $\prod_{p\in\mathbb{P}} \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the additive group of p-adic integers, for each p (see [14, Examples 4.1.2]).

We say that the Diophantine dimension $\mathrm{ddim}(F)$ of F is finite and equal to m, if m is the least integer ≥ 0, for which F is a field of type C_m; if no such m exists, we set $\mathrm{ddim}(F) = \infty$. By type C_m (or a C_m-field), for an integer $m \geq 0$, we mean that every F-form (a homogeneous nonzero polynomial with coefficients in F) f of degree $\deg(f)$ in more than $\deg(f)^m$ variables has a nontrivial zero over F. For example, F is a C_0-field, i.e. $\mathrm{ddim}(F) = 0$ if and only if it is algebraically closed. The class of C_m-fields is closed under taking algebraic extensions, and it contains every extension of transcendence degree m over any algebraically closed field (cf. [21]). The question of whether this class consists of fields of Brauer dimensions less than m is presently open; it is known, however, that these fields have absolute Brauer p-dimensions less than p^{m-1}, for all $p \in \mathbb{P}$ (see [23]). When F is a C_m-field and $\mathrm{char}(F) = q > 0$, it is easily verified that $[F\colon F^q] \leq q^m$, where $F^q = \{\alpha^q : \alpha \in F\}$ (the F-form $\sum_{i=1}^{q^{m'}} a_i X_i^q$ does not possess a nontrivial zero over F, provided $[F\colon F^q] = q^{m'}$ and the system $a_i \in F\colon i = 1, \dots, q^{m'}$, is a basis of F over F^q). Since, by Galois cohomology (cf. [28, Ch. I, 3.3]), $\mathrm{cd}(\mathcal{G}_F) = 0$ if and only if $F = F_{\mathrm{sep}}$, these observations prove the following:

(1.1) In order that $\mathrm{cd}(\mathcal{G}_F) = 0$ and $\mathrm{ddim}(F) = m < \infty$, it is sufficient that $F = F_{\mathrm{sep}}$, $\mathrm{char}(F) = q > 0$, $[F\colon F^q] = q^m$, and F is an extension of an algebraically closed field F_0 of transcendence degree m. We have $\mathrm{cd}(\mathcal{G}_F) = 0$ and $\mathrm{ddim}(F) = \infty$ if $F = F_{\mathrm{sep}}$, $\mathrm{char}(F) = q > 0$ and $[F\colon F^q] = \infty$.

The research presented in this paper is motivated by the following open question (arising from an observation made by Serre in [28, Ch. II, 4.5]):

Question 1. Find whether $\mathrm{cd}(\mathcal{G}_F) \leq m$ whenever F is a field of type C_m, for some $m \in \mathbb{N}$. Equivalently, find whether $\mathrm{cd}(\mathcal{G}_F) \leq \mathrm{ddim}(F)$.

Finite fields have type C_1, by the classical Chevalley-Warning theorem, and C_1-fields have dimension ≤ 1 (see [14, Theorem 2.6], and [28, Ch. II, 3.1 and 3.2], respectively). Nontrivially, it follows from Merkur'ev-Suslin's theorem (cf. [29, Corollary 24.9]) that $\mathrm{cd}(\mathcal{G}_E) \leq 2$, for every C_2-field E. Question 1 is open if $m \geq 3$. It has been proved that $\mathrm{cd}_p(\mathcal{G}_{E_m}) < \infty$, $p \in \mathbb{P}$, for every C_m-field E_m; however, in case $m \geq 3$, the sequence $c_p(m)$, $p \in \mathbb{P}$, of best presently known explicit upper bounds on $\mathrm{cd}_p(\mathcal{G}_{E_m})$, found in [20], is unbounded, which does not rule out the possibility that $\mathrm{cd}(\mathcal{G}_{E_m}) = \infty$.

Note that if $\mathrm{char}(F) = q > 0$ and F is a C_m-field, then the q-dimension $\dim_q(F)$ introduced, for example, in [19], is at most equal to m. This theorem, established by Arason and Baeza [3], implies the answer to Question 1 will be affirmative if and only if $\mathrm{ddim}(F) \geq \mathrm{CD}(F)$, for every field F, where $\mathrm{CD}(F)$ is the cohomological dimension of F (in the sense of [17]), defined as follows:

Definition 1. (i) $\mathrm{CD}(F) = \mathrm{cd}(\mathcal{G}_F)$ *if* $\mathrm{char}(F) = 0$.

(ii) *When* $\mathrm{char}(F) = q > 0$, $\mathrm{CD}(F)$ *equals the supremum of* $\dim_q(F)$ *and* $\mathrm{cd}_p(\mathcal{G}_F)\colon p \in \mathbb{P} \setminus \{q\}$*;* $\dim_q(F)$ *is defined to be the minimal integer* u *for which* $[F\colon F^q] \leq q^u$ *and the Kato-Milne cohomology group* $H_q^{u+1}(L)$ *is trivial, for every finite extension* L *of* F *(or to be infinity if such no* u *exists).*

For the definition of the groups $H_q^{n+1}(F)$, $n \geq 0$, in characteristic q, and for more information about them, we refer the reader to [19] and [17]. An interpretation of these groups as the q-part of the Galois cohomology of F has been made in [16], and results on the minimal index n for which $H_q^{n+1}(F) = \{0\}$ can be found, e.g., in [2]. Here we note that $H_q^2(F)$ is isomorphic to the maximal subgroup of $\mathrm{Br}(F)$ of order dividing q (cf. [14, Sect. 9.2] or [18, pages 219-220]). This implies $\dim_q(F) \leq 1$ if and only if $\mathrm{Br}(L)$ does not contain an element of order q, for any finite extension L/F; in addition, it follows that $\mathrm{CD}(F) \leq 1$ if and only if $\dim(F) \leq 1$ and $[F\colon F^q] \leq q$ (see also [14, Theorem 6.1.8]). Using the Arason-Baeza theorem and the former part of (1.1), one computes $\dim_q(F)$ and $\mathrm{CD}(F)$ in the following situation:

(1.2) If F is a field with $\mathrm{char}(F) = q > 0$ and $F = F_{\mathrm{sep}}$, such that $[F\colon F^q] = q^m$, for some $m \in \mathbb{N}$, and F is an extension of an algebraically closed field F_0 of transcendence degree m, then $\mathrm{CD}(F) = \dim_q(F) = m$.

Here it should be pointed out that there exist fields F_m, $m \in \mathbb{N}$, of zero characteristic with $\mathrm{cd}(\mathcal{G}_F) = m < \infty$ and $\mathrm{ddim}(F_m) = \infty$, for each index m; as shown by Ax, F_1 can be chosen to be quasifinite (see [6]). When $m = 2$, one may take as F_2 any finite extension of the field $\mathbb{Q}_{p'}$ of p'-adic numbers (cf. [1], see also Remark 2.3). These results attract interest in the problem of describing all pairs (k, ℓ) which are equal to $(\mathrm{cd}(\mathcal{G}_{E_{k,\ell}}), \mathrm{ddim}(E_{k,\ell}))$, for some field $E_{k,\ell}$. Note that one may take as $E_{\infty,\infty}$ any purely transcendental extension Φ of infinite transcendence degree over another field Φ_0. Since Φ has a subfield Φ_0' such that Φ_0'/Φ_0 is a field extension, Φ_0' is Φ_0-isomorphic to Φ, and Φ/Φ_0' is purely transcendental of transcendence degree 1, the equalities $\mathrm{ddim}(\Phi) = \infty$ and $\mathrm{cd}(\mathcal{G}_\Phi) = \infty$ can be deduced, by assuming the opposite, from the Lang-Nagata-Tsen theorem and Galois cohomology, respectively (see [26] and [28, Ch. II, Proposition 11]). Moreover, it follows from [28, Ch. II, Proposition 11], and the Φ_0-isomorphism $\Phi \cong \Phi_0'$ that $\mathcal{G}_\Phi$ has infinite cohomological p-dimensions $\mathrm{cd}_p(\mathcal{G}_\Phi)$, for all $p \in \mathbb{P}$ different from $\mathrm{char}(\Phi)$. Hence, by [20, Theorem 1.15], and the (topological) group isomorphism $\mathcal{G}_\Phi \cong \mathcal{G}_{\Phi'}$, where Φ' is a perfect closure of Φ (cf. [22, Ch. V, Proposition 6.11, and Ch. VII, Theorem 1.12]), $\mathrm{ddim}(\Phi') = \mathrm{cd}(\mathcal{G}_{\Phi'}) = \infty$. This allows us to restrict ourselves to the case of $(k, \ell) \neq (\infty, \infty)$, i.e. $k \neq \infty$. Also, statements (1.1) (and the equalities $\mathrm{cd}(\mathcal{G}_E) = \mathrm{ddim}(E) = 0$, for every algebraically closed field E) imply we may assume further that $k > 0$.

The present research solves the stated problem affirmatively, for each nonzero pair (k, ℓ) admissible by Question 1, i.e. satisfying $1 \leq k \leq \ell$. It proves that $E_{k,\ell}$ can be chosen to be a perfect field of any prescribed characteristic. In order to facilitate our considerations, we denote by $\mathbb{N}_\infty$ and $\overline{\mathbb{P}}$ the unions $\mathbb{N} \cup \{\infty\}$ and $\mathbb{P} \cup \{0\}$, respectively, and by $\mathbb{P}_q$ the set $\mathbb{P} \setminus \{q\}$, for each $q \in \overline{\mathbb{P}}$.

2 The main results

The main purpose of this paper is to prove the following:

Theorem 2.1. *For each $q \in \overline{\mathbb{P}}$, there exist quasifinite fields $F_{m,q} \colon m \in \mathbb{N}_\infty$, of characteristic q, such that $\mathrm{ddim}(F_{m,q}) = m$, for each m.*

Theorem 2.1 and our next result complement Theorem 1 of [6] as follows:

Theorem 2.2. *Let $q \in \overline{\mathbb{P}}$ and $(k, \ell) \in \mathbb{N} \times \mathbb{N}_\infty$ be a nonzero pair admissible by Question 1. Then there exists a perfect field $E_{k,\ell;q}$ with $\mathrm{char}(E_{k,\ell;q}) = q$, $\mathrm{ddim}(E_{k,\ell;q}) = \ell$ and $\mathrm{cd}(\mathcal{G}_{E_{k,\ell;q}}) = k$.*

It is worth mentioning that $\mathrm{CD}(E) = \mathrm{cd}(\mathcal{G}_E)$, for every perfect field E; in particular, one may write $\mathrm{CD}(E_{k,\ell;q})$ instead of $\mathrm{cd}(\mathcal{G}_{E_{k,\ell;q}})$ in the statement of Theorem 2.2. In characteristic zero, the considered equality holds by definition, and for its proof in case $\mathrm{char}(E) = q > 0$, it is sufficient to show that $\dim_q(E) = \mathrm{cd}_q(\mathcal{G}_E) \leq 1$. The inequality $\mathrm{cd}_q(\mathcal{G}_E) \leq 1$ is a well-known result of Galois cohomology; the same applies to the fact that $\mathrm{cd}_q(\mathcal{G}_E) = 0$ if and only if q does not divide the degree $[E' \colon E]$, for any $E' \in \mathrm{Fe}(E)$ (cf. [28, Ch. I, 3.3, and Ch. II, 2.2]). Observing further that, for any finite extension E' of E, the group $H_q^1(E') = H^1(\mathcal{G}_{E'}, \mathbb{Z}/q\mathbb{Z})$ is isomorphic to the group of continuous homomorphisms of the compact group $\mathcal{G}_{E'}$ into the multiplicative discrete group of complex q-th roots of unity, and using Galois theory and Sylow's theorem (see [22, Ch. I, Sect. 6, and Ch. VI]), one concludes that $\dim_q(E) = 0$ if and only if $\mathrm{cd}_q(\mathcal{G}_E) = 0$. Thus the equality $\mathrm{cd}_q(\mathcal{G}_E) = \dim_q(E)$ reduces to a consequence of the assertion that $\dim_q(E) \leq 1$. The assertion itself is true, since E' is a perfect field, which implies $\mathrm{Br}(E')$ does not contain elements of order q (see [14, Lemma 9.1.7]) and so proves that $H_q^2(E') = \{0\}$. One also sees that $\mathrm{cd}(\mathcal{G}_E) = \overline{\mathrm{CD}}(E)$, where $\overline{\mathrm{CD}}(F)$ is a modification of $\mathrm{CD}(F)$, defined for any field F with $\mathrm{char}(F) = q > 0$, as follows: take as $\dim_q(F)$ in Definition 1 (ii) the minimal index u for which $H_q^{u+1}(L) = \{0\}$, for all finite extensions L/F (cf. [2, page 715]); put $\dim_q(F) = \infty$ if such u does not exist. In addition, it is easily verified that: $\overline{\mathrm{CD}}(F) \leq \mathrm{CD}(F)$; $\overline{\mathrm{CD}}(F) \leq 1$ if and only if $\dim(F) \leq 1$.

Remark 2.3. *Note that* $cd(\mathcal{G}_E) = 2$ *and* $ddim(E) = \infty$ *in the following two cases:* (i) E *is a finite extension of the field* $\mathbb{Q}_p$ *of* p*-adic numbers, for some* $p \in \mathbb{P}$*;* (ii) E *is a totally imaginary number field, i.e. a finite extension of the field* $\mathbb{Q}$ *of rational numbers, which does not embed in the field* $\mathbb{R}$ *of real numbers. The equality* $cd(\mathcal{G}_E) = 2$ *is a well-known result of Galois cohomology (cf. [28, Ch. II, Sects. 4.3, 4.4]), and the equality* $ddim(E) = \infty$ *has been proved in [1], for any finite extension* $E/\mathbb{Q}_p$ *(see also [5], for a proof of the fact that* $ddim(\mathbb{Q}_p) = \infty$*,* $p \in \mathbb{P}$*). The validity of the equality* $ddim(E) = \infty$ *in case* (ii) *can be deduced from its validity in case* (i)*, by assuming the opposite and applying [21, page 379, lemma] (to suitably chosen forms without nontrivial zeroes over the completion* E_v *of* E *with respect to some discrete valuation* v*). Algebraic extensions* $E_0/\mathbb{Q}$ *and* $E_p/\mathbb{Q}_p$*,* $p \in \mathbb{P}$*, such that* $dim(E_{p'}) \leq 1 <$ $\mathrm{ddim}(E_{p'})$*, for each* $p' \in \overline{\mathbb{P}}$*, can be found in [8]. At the same time, the question of whether there exist algebraic extensions* $E_{0,\ell}/\mathbb{Q}$ *and* $E_{p,\ell}/\mathbb{Q}_p$*,* $p \in \mathbb{P}$*, with* $ddim(E_{p',\ell}) = \ell$*, for each* $p' \in \overline{\mathbb{P}}$*, seems to be open, for any integer* $\ell \geq 2$*.*

Theorems 2.1 and 2.2 are proved in Section 4 and 5, respectively. Our proofs rely on general properties of fields with Henselian valuations, particularly, of algebraic extensions of complete discrete valued fields. Preliminaries on Henselian (valued) fields and other related information used in the sequel are presented in Section 3. To prove Theorem 2.1, we show that every algebraically closed field $\mathbb{F}$ possesses extensions F_∞ and F_m, $m \in \mathbb{N}$, which are quasifinite fields such that $\mathrm{ddim}(F_\infty) = \infty$, and for each index m, $\mathrm{ddim}(F_m) = m$ and F_m is a subfield of F_∞. The existence of F_∞ has been established constructively by Ax in the case where $\mathrm{char}(\mathbb{F}) = 0$. When $\mathrm{char}(\mathbb{F}) = q > 0$, F_∞ is defined by modifying Ax's construction. For this purpose, we use implicitly the Mel'nikov-Tavgen' theorem [24] (via Lemma 3.4), and the bridge between Galois theory and the study of tensor products of field extensions, provided by Cohn's theorem (stated below as Lemma 3.3). For the proof of Theorem 2.2, we also need at crucial points Greenberg's theorem [15] (as well as Lemma 3.2). The arithmetic ingredient of our proofs, borrowed from [6], is based on a version of Vinogradov's theorem on the ternary Goldbach problem, for certain prime numbers defined as follows:

Definition 2. *Let* $q \in \overline{\mathbb{P}}$*,* m *be an integer* ≥ 0*, and* α *be a real number such that* $0 < \alpha < 1$*. A number* $p \in \mathbb{P}$ *is said to be* $(m, \alpha; q)$*-representable if* $m = 0$ *or* $m > 0$ *and* $p = p_1 + p_2 + p_3$*, for some* $p_1, p_2, p_3 \in \mathbb{P}$ *with* $q < p^\alpha < p_1 < p_2 < p_3$*, which are* $(m-1, \alpha; q)$*-representable.*

For a proof of the following lemma, we refer the reader to [6, Lemma 2].

Lemma 2.4. *For each triple* $q \in \overline{\mathbb{P}}$*,* $m \in \mathbb{N}$*,* $\alpha \in \mathbb{R}$ *with* $0 < \alpha < 1$*, there is* $c = c(m, \alpha; q) \in \mathbb{N}$*, such that every* $p \in \mathbb{P}$*,* $p > c$*, is* $(m, \alpha; q)$*-representable.*

The basic notation, terminology and conventions kept in this paper are standard and virtually the same as in [30], [22] and [28]. Throughout, $\mathbb{Z}$ is the additive group of integers, value groups are written additively, Galois groups are viewed as profinite with respect to the Krull topology, and by a profinite group homomorphism, we mean a continuous one. For any field E, E^* is its multiplicative group, $E^{*n} = \{a^n : a \in E^*\}$, for each $n \in \mathbb{N}$, and $\mu(E)$ is the multiplicative group of all roots of unity lying in E. As usual, for any $p \in \mathbb{P}$, $E(p)$ denotes the maximal p-extension of E (in $E_{\rm sep}$), that is, the compositum of those finite Galois extensions of E in $E_{\rm sep}$, whose Galois groups are p-groups. Given a field extension E'/E, we write $\mathrm{I}(E'/E)$ for the set of intermediate fields of E'/E. When E'/E is a Galois extension, its Galois group is denoted by $\mathcal{G}(E'/E)$; we say that E'/E is cyclic if $\mathcal{G}(E'/E)$ is a cyclic group. By a $\mathbb{Z}_p$-extension, we mean a Galois extension Ψ'/Ψ with $\mathcal{G}(\Psi'/\Psi)$ isomorphic to $\mathbb{Z}_p$. The value group of any discrete valued field is assumed to be an ordered subgroup of the additive group of the field $\mathbb{Q}$; this is done without loss of generality, in view of [13, Theorem 15.3.5], and the fact that $\mathbb{Q}$ is a divisible hull of any of its infinite subgroups (see page 26).

3 Preliminaries on Henselian valuations

For any field K with a (nontrivial) Krull valuation v, $O_v(K) = \{a \in K : v(a) \geq 0\}$ denotes the valuation ring of (K, v), $M_v(K) = \{\mu \in K : v(\mu) > 0\}$ the maximal ideal of $O_v(K)$, $O_v(K)^* = \{u \in K : v(u) = 0\}$ the multiplicative group of $O_v(K)$, $v(K)$ the value group and $\widehat{K} = O_v(K)/M_v(K)$ the residue field of (K, v), respectively; $\overline{v(K)}$ is a divisible hull of $v(K)$. The valuation v is said to be Henselian if it extends uniquely, up-to equivalence, to a valuation v_L on each algebraic extension L of K. When this holds, (K, v) is called a Henselian field. The condition that v is Henselian has the following two equivalent forms (cf. [13, Sect. 18.1], or [30, Theorem A.12]):

(3.1) (a) Given a polynomial $f(X) \in O_v(K)[X]$ and an element $a \in O_v(K)$, such that $2v(f'(a)) < v(f(a))$, where f' is the formal derivative of f, there is a zero $c \in O_v(K)$ of f satisfying the equality $v(c - a) = v(f(a)/f'(a))$;

(b) For each normal extension Ω/K, $v'(\tau(\mu)) = v'(\mu)$ whenever $\mu \in \Omega$, v' is a valuation of Ω extending v, and τ is a K-automorphism of Ω.

Next we recall some facts concerning the case where (K, v) is a real-valued field, i.e. $v(K)$ is embeddable as an ordered subgroup in the additive group $\mathbb{R}$ of real numbers. Fix a completion K_v of K with respect to the topology of v, and denote by $\bar{v}$ the valuation of K_v continuously extending v. Then:

(3.2) (a) (K, v) is Henselian if and only if K has no proper separable (algebraic) extension in K_v (cf. [13, Corollary 18.3.3]); in particular, $(K_v, \bar{v})$ is Henselian.

(b) The topology of K_v as a completion of K is the same as the one induced by $\bar{v}$; also, $\bar{v}(K_v) = v(K)$ and $\widehat{K}$ equals the residue field of $(K_v, \bar{v})$ (cf. [13, Theorems 9.3.2 and 18.3.1]).

When v is Henselian, so is v_L, for any algebraic field extension L/K. In this case, we denote by $\widehat{L}$ the residue field of (L, v_L), and put $O_v(L) = O_{v_L}(L)$, $M_v(L) = M_{v_L}(L)$, $v(L) = v_L(L)$; also, we write v instead of v_L when there is no danger of ambiguity. Clearly, $\widehat{L}/\widehat{K}$ is an algebraic extension and $v(K)$ is an ordered subgroup of $v(L)$, such that $v(L)/v(K)$ is a torsion group; hence, one may assume without loss of generality that $v(L)$ is an ordered subgroup of $\overline{v(K)}$. By Ostrowski's theorem (cf. [13, Theorem 17.2.1]), if $[L\colon K]$ is finite, then it is divisible by $[\widehat{L}\colon \widehat{K}]e(L/K)$, and in case $[L\colon K] \neq [\widehat{L}\colon \widehat{K}]e(L/K)$, the integer $[L\colon K][\widehat{L}\colon \widehat{K}]^{-1}e(L/K)^{-1}$ is a power of $\mathrm{char}(\widehat{K})$ (so $\mathrm{char}(\widehat{K}) \mid [L\colon K]$); here $e(L/K)$ is the ramification index of L/K, i.e. the index of $v(K)$ in $v(L)$. The extension L/K is called defectless if $[L\colon K] = [\widehat{L}\colon \widehat{K}]e(L/K)$. When the valuation v is discrete, i.e. $v(K)$ is infinite cyclic, L/K is defectless in the following two situations:

(3.3) (a) (K, v) is Henselian and L/K is separable (see [13, Sect. 17.4]).

(b) (K, v) is a complete valued field, i.e. $(K, v) = (K_v, \bar{v})$ (cf. [22, Ch. XII, Proposition 6.1]). This holds if K is the Laurent (formal power) series field $K_0((X))$ in a variable X over a field K_0, and v is the standard discrete valuation of K inducing on K_0 the trivial valuation; then $v(K) = \mathbb{Z}$ and K_0 is the residue field of (K, v) (see [13, Examples 4.2.2 and 9.2.2]).

Assume now that (K, v) is a Henselian field and let R be a finite extension of K. We say that R/K is totally ramified if $e(R/K) = [R\colon K]$; R/K is called tamely ramified if it is defectless, $e(R/K)$ is not divisible by $\mathrm{char}(\widehat{K})$, and $\widehat{R}$ is separable over $\widehat{K}$. The extension R/K is said to be inertial if $[R\colon K] = [\widehat{R}\colon \widehat{K}]$ and $\widehat{R}/\widehat{K}$ is separable. Inertial extensions of K are clearly separable. They have a number of useful properties, some of which are presented by the following lemma (for its proof, see [30, Theorem A.23]):

Lemma 3.1. *Let (K, v) be a Henselian field and K_{ur} the compositum of inertial extensions of K in K_{sep}. Then:*

(a) *An inertial extension R'/K is Galois if and only if so is $\widehat{R}'/\widehat{K}$. When this holds, $\mathcal{G}(R'/K)$ and $\mathcal{G}(\widehat{R}'/\widehat{K})$ are canonically isomorphic.*

(b) *$v(K_{\mathrm{ur}}) = v(K)$, K_{ur}/K is a Galois extension and $\mathcal{G}(K_{\mathrm{ur}}/K) \cong \mathcal{G}_{\widehat{K}}$.*

(c) *Finite extensions of K in K_{ur} are inertial, and the natural mapping of $I(K_{\mathrm{ur}}/K)$ into $I(\widehat{K}_{\mathrm{sep}}/\widehat{K})$, by the rule $L \to \widehat{L}$, is bijective.*

(d) *For each* $K_1 \in \mathrm{Fe}(K)$, *the intersection* $K_0 = K_1 \cap K_{\mathrm{ur}}$ *equals the maximal inertial extension of* K *in* K_1*; in addition,* $\widehat{K}_0 = \widehat{K}_1$.

Greenberg's theorem [15] and the next lemma ensure that if $K = K_0((X))$ and K_0 is a field with $\mathrm{ddim}(K_0) < \infty$, then $\mathrm{ddim}(K) = \mathrm{ddim}(K_0) + 1$. They provide some of the basic tools needed to prove Theorems 2.1 and 2.2.

Lemma 3.2. *Let* $K = K_0((X))$ *be the Laurent series field in a variable* X *over a field* K_0*, and let* $f_0(Y_1, \dots, Y_t)$ *be a* K_0*-form of degree* d *in* t *variables. Fix an algebraic closure* $\overline{K}$ *of* K*, denote by* v *the standard discrete valuation of* K *trivial on* K_0*, suppose that* f_0 *is without a nontrivial zero over* K_0*, and let* f *be the* K*-form* $\sum_{i=0}^{d-1} f_0(X_{i,1}, \dots, X_{i,t})X^i$*, where* $X_{i,j}\colon i = 0, \dots, d-1$*;* $j = 1, \dots, t$*, are algebraically independent variables over* K*. Then:*

(a) f *is a* K*-form of degree* d *without a nontrivial zero over* K*; in addition,* f *does not possess such a zero over the perfect closure* $\widetilde{K}$ *of* K *in* $\overline{K}$*, provided that* K_0 *is perfect and* d *is not divisible by* $\mathrm{char}(K_0)$*;*

(b) f_0 *does not possess a nontrivial zero over any totally ramified extension* L *of* K *(with respect to* v*); hence,* $\mathrm{ddim}(K_0) \le \mathrm{ddim}(L)$.

Proof. Let L be a totally ramified extension of K. We prove Lemma 3.2 (b), by assuming that f_0 has a nontrivial zero over L. Then $f_0(Y_1, \dots, Y_t)$ must have a zero $\alpha = (\alpha_1, \dots, \alpha_t)$, such that $\alpha_i \in O_v(L)$, for $i = 1, \dots, t$, and $v_L(\alpha_{i'}) = 0$, for some index i'. Since $\widehat{L} = K_0$ and f_0 equals its reduction modulo $M_v(L)$, this means that the t-tuple $\hat{\alpha} = (\hat{\alpha}_1, \dots, \hat{\alpha}_t)$ is a nontrivial zero of f_0 over L. The obtained contradiction proves Lemma 3.2 (b).

The proof of Lemma 3.2 (a) relies on the fact that the quotient group $v(K)/dv(K)$ is cyclic of order d, whose elements are the cosets $v(X^i) + dv(K)$, $i = 0, \dots, d-1$. Also, Lemma 3.2 (b) implies that $v(f_0(\lambda_1, \dots, \lambda_t)) \in dv(K)$ whenever $\lambda_1, \dots, \lambda_t \in K$ and $\lambda_{j'} \in K^*$, for at least one index j'. This ensures that $\sum_{i=0}^{d-1} f_0(\lambda_{i,1}, \dots, \lambda_{i,t}) = 0$, where $\lambda_{i,1}, \dots, \lambda_{i,t} \in K$, for each index i, if and only if all $\lambda_{i,j}$ are equal to zero. Thus the former part of Lemma 3.2 (a) is proved. For the proof of the latter one, we may assume that K_0 is perfect, $\mathrm{char}(K_0) = q > 0$ and $q \nmid d$. Then it follows from (3.2) and (3.3) (b) that finite extensions of K in $\widetilde{K}$ are totally ramified of q-primary degrees. Therefore, the natural embedding of K into $\widetilde{K}$ induces canonically a group isomorphism $v(K)/dv(K) \cong v(\widetilde{K})/dv(\widetilde{K})$; hence, $v(\widetilde{K})/dv(\widetilde{K})$ equals the set $\{v(X^i) + dv(\widetilde{K})\colon i = 0, \dots, d-1\}$. Since, by Lemma 3.2 (b), f_0 does not possess a nontrivial zero over $\widetilde{K}$, this allows to prove the latter part of Lemma 3.2 (a) in the same way as the former one. □

The following lemma (for a proof, see [9, Theorem 2.2]) gives a sufficient condition that the tensor product $E_1 \otimes_{E_0} E_2$ is a field, where E_i/E_0, $i = 1, 2$, are field extensions. This lemma considerably simplifies the presentation of the proof of Theorem 2.1, and its valuation-theoretic preparation.

Lemma 3.3. *Assume that E_0 is a field and E_1, E_2 are extensions of E_0 at least one of which is Galois. Then the tensor product $E_1 \otimes_{E_0} E_2$ is a field if and only if no proper extension of E_0 is embeddable as an E_0-subalgebra in E_i, $i = 1, 2$. Moreover, if E_0 has an extension F such that $E_i \in I(F/E_0)$, $i = 1, 2$, then $E_1 \otimes_{E_0} E_2$ is a field if and only if $E_1 \cap E_2 = E_0$; when this holds, $E_1 \otimes_{E_0} E_2$ and the compositum E_1E_2 are isomorphic as E_0-algebras.*

Assuming again that (K, v) is a Henselian field, denote by $K_{\rm tr}$ the compositum of tamely ramified extensions of K in $K_{\rm sep}$. It is well-known (see [30, Theorem A.24]) that $K_{\rm tr}/K$ is a Galois extension, and finite extensions of K in $K_{\rm tr}$ are tamely ramified. Our next lemma presents additional information concerning $K_{\rm tr}/K$. The stated results are known but we prove them here for convenience of the reader. The description of some of them relies on the fact that the quotient $\overline{v(K)}/v(K)$ is an abelian torsion group, whence it is isomorphic to the direct sum of its p-components $(\overline{v(K)}/v(K))_p$, $p \in \mathbb{P}$.

Lemma 3.4. *Let (K, v) be a Henselian field with $\mathrm{char}(\widehat{K}) = q$ and $\mu(\widehat{K}) = \mu(\widehat{K}_{\rm sep})$. Then there is $\Theta \in I(K_{\rm tr}/K)$ with the following properties:*

(a) *$K_{\rm ur} \cap \Theta = K$, $K_{\rm ur}.\Theta = K_{\rm tr}$ and Θ/K is a Galois extension with $\mathcal{G}(\Theta/K)$ isomorphic to the topological group product $\prod_{p \in \mathbb{P}_q} \Gamma_p$, Γ_p being the (continuous) character group of the discrete group $(v(K_{\rm tr})/v(K))_p$, for each index p; in particular, $\mathcal{G}(\Theta/K)$ is abelian.*

(b) *Finite extensions of K in Θ are totally ramified, and Θ equals the compositum of the fields $T_p = \Theta \cap K(p)$, $p \in \mathbb{P}_q$; specifically, if v is discrete, then T_p/K, $p \in \mathbb{P}_q$, are $\mathbb{Z}_p$-extensions.*

(c) *$K_{\rm tr}/K$ is a Galois extension with $\mathcal{G}(K_{\rm tr}/K) \cong \mathcal{G}(K_{\rm ur}/K) \times \mathcal{G}(\Theta/K)$.*

Moreover, if $q > 0$, then there exists $W \in I(K_{\rm sep}/K)$, such that $W \cap K_{\rm tr} = K$ and $W.K_{\rm tr} = K_{\rm sep} = W_{\rm tr}$; also, finite extensions of K in W are of q-primary degrees, $W\Theta \cap K_{\rm ur} = K$, $W\Theta.K_{\rm ur} = (W\Theta)_{\rm ur} = K_{\rm sep}$ and $\widehat{W\Theta}/\widehat{K}$ is a purely inseparable field extension.

Proof. The existence of a field $\Theta \in I(K_{\rm tr}/K)$ with $K_{\rm ur} \cap \Theta = K$ and $K_{\rm ur}\Theta = K_{\rm tr}$, and in case $q > 0$, the existence of $W \in I(K_{\rm sep}/K)$, such that $W \cap K_{\rm tr} = K$ and $W.K_{\rm tr} = K_{\rm sep}$, follow from the Mel'nikov-Tavgen' theorem [24] and Galos theory. This, combined with Lemma 3.1 (d), proves that $K_{\rm tr} = \Theta_{\rm ur}$, $v(\Theta) = v(K_{\rm tr})$, and finite extensions of K in Θ are totally ramified. Note further that these extensions are normal with abelian Galois groups. Indeed, $\mu(\widehat{K}) = \mu(\widehat{K}_{\rm sep})$, which means that $\widehat{K}$ contains a primitive ν-th root of unity, for each $\nu \in \mathbb{N}$ not divisible by q. This enables one to obtain the claimed property of the considered extensions as a consequence of [30, Proposition A.22]. Moreover, it follows that Θ/K is Galois, $\mathcal{G}(\Theta/K)$ is abelian, and because of the equality $K_{\rm ur} \cap \Delta = K$, there are isomorphisms $\mathcal{G}(\Theta/K) \cong \mathcal{G}(K_{\rm tr}/K_{\rm ur})$ and $\mathcal{G}(K_{\rm tr}/K) \cong \mathcal{G}(K_{\rm ur}/K) \times \mathcal{G}(\Theta/K)$ as profinite groups. Hence, by Galois theory, the decomposability of $v(K_{\rm tr})/v(K)$ into the direct sum $\oplus_{p\in\mathbb{P}_q}(v(K_{\rm tr}/v(K))_p$, and [30, Theorem A.24 (v)], $\mathcal{G}(\Theta/K)$ is isomorphic to the product $\prod_{p\in\mathbb{P}_q}\Gamma_p$ defined in Lemma 3.4 (a). Observe that $v(K_{\rm tr})/v(K)$ is divisible. Clearly, $v(K_{\rm tr})/v(K)$ is an abelian torsion group without an element of order q, so it suffices to prove that $v(K_{\rm tr}) = pv(K_{\rm tr})$, for each $p \in \mathbb{P}_q$. Assuming the opposite and taking an element $\pi \in K_{\rm tr}^*$ of value out of $pv(K_{\rm tr})$, for some $p \in \mathbb{P}_q$, one obtains that the extension $K'_{\rm tr}$ of $K_{\rm tr}$ generated by some p-th root $\pi_p \in K_{\rm sep}$ of π must be totally ramified of degree p. This requires that $v(K'_{\rm tr})/v(K)$ is an abelian torsion group without an element of order divisible by q, which leads to the contradiction that $K'_{\rm tr} = K_{\rm tr}$. Thus it turns out that $v(K_{\rm tr}) = pv(K_{\rm tr})$, $p \in \mathbb{P}$, so $v(K_{\rm tr})/v(K)$ is divisible, as claimed. In addition, it is easily verified that the p-group $(v(K_{\rm tr})/v(K))_p$ is nontrivial, for some $p \in \mathbb{P}_q$, if and only if $v(K) \neq pv(K)$. Suppose now that $v(K) = \mathbb{Z}$. Then $v(K_{\rm sep}) \cong \mathbb{Q}$, so it follows from the equality $v(\Theta) = v(K_{\rm tr})$ and the cyclicity of finitely-generated subgroups of $\mathbb{Q}$ that $(v(K_{\rm tr})/v(K))_p$ is a quasi-cyclic p-group, for every $p \in \mathbb{P}_q$. This implies the character group of $(v(K_{\rm tr})/v(K))_p$ is isomorphic to $\mathbb{Z}_p$, which allows to deduce Lemma 3.4 (b) from Galois theory and Lemma 3.4 (a).

It remains for us to complete the proof of the concluding assertion of Lemma 3.4, so we assume that $q > 0$. It follows from Lemma 3.3 and the equalities $W \cap K_{\rm tr} = K$, $W.K_{\rm tr} = K_{\rm sep} = W_{\rm tr}$ and $\Theta \cap K_{\rm ur} = K$, $\Theta.K_{\rm ur} = K_{\rm tr}$ that there exist isomorphisms $K_{\rm sep} \cong W \otimes_K K_{\rm tr}$, $K_{\rm tr} \cong \Theta \otimes_K K_{\rm ur}$, and $W\Theta \cong W \otimes_K \Theta$ as K-algebras. Observing also that $K_{\rm sep}$ is K-isomorphic to $W \otimes_K (\Theta \otimes_K K_{\rm ur})$ and $(W \otimes_K \Theta) \otimes_K K_{\rm ur}$, one obtains that $K_{\rm sep} \cong W\Theta \otimes_K K_{\rm ur}$, i.e. $W\Theta \cap K_{\rm ur} = K$ and $W\Theta.K_{\rm ur} = K_{\rm sep} = (W\Theta)_{\rm ur}$. Hence, by Lemma 3.1 (d), $\widehat{W\Theta}/\widehat{K}$ is a purely inseparable extension.

Let now W_0 be a finite extension of K in W. Since $W.K_{\rm tr} \cong W \otimes_K K_{\rm tr}$, it follows that $[W_0\colon K] = [W_0.K_{\rm tr}\colon K_{\rm tr}]$ (apply [27, Sect. 2, Proposition c]). This, combined with the fact that $K_{\rm sep} = K_{\rm tr}(q)$ (cf. [24]), implies $[W_0\colon K] = [W_0K_{\rm tr}\colon K_{\rm tr}]$ is a q-primary number, so Lemma 3.4 is proved. □

4 Proof of Theorem 2.1

The proof of the first main result of this paper relies on the following sufficient condition for a field F to be quasifinite with $\mathrm{ddim}(F) = \infty$:

Proposition 4.1. *Let $\mathbb{F}$ be an algebraically closed field, $q = \mathrm{char}(\mathbb{F})$, and $\mathbb{P}_q$ the ordered set $\{p_n \in \mathbb{P}, n \in \mathbb{N} \colon p_n \neq q, p_n < p_{n+1}$, for each $n\}$. Assume that F_0 and $F_m \colon m \in \mathbb{N}$, are perfect fields satisfying the following conditions:*

(a) *$F_0 = \mathbb{F}$, provided that $q = 0$; if $q > 0$, then $\mathcal{G}_{F_0} \cong \mathbb{Z}_q$ and $F_0/\mathbb{F}$ is a field extension of transcendence degree* 1*;*

(b) *For each $m \in \mathbb{N}$, F_m is an algebraic extension of the Laurent series field $F_{m-1}((X_m))$, such that $F_{m-1,\mathrm{sep}} \otimes_{F_{m-1}} F_m$ is a field and $\mathcal{G}_{F_m}$ is isomorphic to the topological group product $\mathcal{G}_{F_{m-1}} \times \mathbb{Z}_{p_m}$.*

Then the union F_∞ of F_n, $n \in \mathbb{N}$, is a quasifinite field with $\mathrm{ddim}(F) = \infty$*.*

Proof. The fields F_m, $m \geq 0$, form an ordered set with respect to inclusion which implies F_∞ is a field including F_m as a subfield, for each m. Note that F_∞ is perfect. The assertion is obvious if $q = 0$, and in case $q > 0$, it is easily verified that F_∞^q equals the union $\cup_{m \in \mathbb{N}} F_m^q$. Since the fields F_m are perfect, i.e. $F_m^q = F_m$, for all integers $m \geq 0$, this yields $F_\infty^q = F_\infty$, proving that F_∞ is perfect. We show that F_∞ is quasifinite. Identifying as we can $F_{m,\mathrm{sep}}$, $m \geq 0$ with their isomorphic copies in $F_{\infty,\mathrm{sep}}$, and taking into account that the polynomial ring $F_\infty[X]$ in an indeterminate X over F_∞ equals $\cup_{m \in \mathbb{N}} F_m[X]$, one obtains that $F_{\infty,\mathrm{sep}} = \cup_{m \in \mathbb{N}} F_{m,\mathrm{sep}}$. Moreover, it follows from conditions (a) and (b) of Proposition 4.1 that $F_{m-1,\mathrm{sep}} \cap F_m = F_{m-1}$ and $F_{m-1,\mathrm{sep}} \otimes_{F_{m-1}} F_m$ is F_{m-1}-isomorphic to the compositum $F_{m-1,\mathrm{sep}}.F_m$, for each $m \in \mathbb{N}$. Arguing by induction on k and observing that $F_{m-1,\mathrm{sep}}.F_m$ is an F_m-subalgebra of $F_{m,\mathrm{sep}}$, and in case $k \geq 2$, $F_{m-1,\mathrm{sep}} \otimes_{F_{m-1}} F_{m-1+k}$ and $(F_{m-1,\mathrm{sep}} \otimes_{F_{m-1}} F_m) \otimes_{F_m} F_{m-1+k}$ are isomorphic F_{m-1+k}-algebras (see [27, Sect. 9.4, Corollary a]), one concludes that $F_{m-1,\mathrm{sep}} \otimes_{F_{m-1}} F_{m-1+k}$, $k \in \mathbb{N}$, and $F_{m-1,\mathrm{sep}} \otimes_{F_{m-1}} F_\infty$ are fields. Hence, by Galois theory (cf. [22, Ch. VI, Theorem 1.12]), for each $\mu \in \mathbb{N}$, every irreducible polynomial $f_\mu[X] \in F_\mu[X]$ over F_μ remains irreducible over the fields $F_{\mu'}$, $\mu < \mu' \leq \infty$; in addition, the Galois groups of $f_\mu(X)$ over $F_{\mu'}$, $\mu \leq \mu'$, are isomorphic. It is now clear from the conditions on $\mathcal{G}_{F_\mu}$, $0 \leq \mu < \infty$, that finite extensions of F_∞ are cyclic, and for each $\nu \in \mathbb{N}$, there exists a unique degree ν extension of F_∞ in $F_{\infty,\mathrm{sep}}$; in other words, the perfect field F_∞ is quasifinite, as claimed.

Our next objective is to prove that $\mathrm{ddim}(F_\infty) = \infty$. As a first step towards this goal, we fix an index $\nu \in \mathbb{N}$ and show that if g_ν is an F_ν-form without a nontrivial zero over F_ν, then so is g_ν over F_∞. This amounts to proving that g_ν does not possess a nontrivial zero over $F_{\nu'}$, for any integer $\nu' > \nu$. Proceeding by induction on $\nu' - \nu$, one reduces our proof to the case where $\nu' = \nu + 1$. Denote by $\kappa_{\nu+1}$ the standard valuation of the field $K_{\nu+1} = F_\nu((X_{\nu+1}))$ trivial on F_ν, and by $v_{\nu+1}$ the $\mathbb{Q}$-valued valuation of $F_{\nu+1}$ extending $\kappa_{\nu+1}$. Observing that $F_{\nu,\mathrm{sep}}.K_{\nu+1} = K_{\nu+1,\mathrm{ur}}$ and $F_{\nu,\mathrm{sep}}.K_{\nu+1} \otimes_{K_{\nu+1}} F_{\nu+1}$ is a field, one obtains that $F_{\nu,\mathrm{sep}}.K_{\nu+1} \cap F_{\nu+1} = K_{\nu+1}$. This, combined with (3.3) (b) and Lemma 3.1, implies finite extensions of $K_{\nu+1}$ in $F_{\nu+1}$ are totally ramified, so it follows from Lemma 3.2 (b) that g_ν has no nontrivial zero over $F_{\nu+1}$. Now our assertion about g_ν is obvious, which yields $\mathrm{ddim}(F_\infty) \geq \mathrm{ddim}(F_\nu)$, for every $\nu \in \mathbb{N}$.

It remains to be seen that, for any $m \in \mathbb{N}$, there exists $\mu(m) \in \mathbb{N}$, such that $\mathrm{ddim}(F_{\mu(m)}) > m$, i.e. $F_{\mu(m)}$ is not of type C_m. Evidently, there is a sequence $\beta_n \in \mathbb{R}$, $n \in \mathbb{N}$, satisfying $0 < \beta_n < 1$ and $\sum_{j=0}^{n} \beta_n^j > n$, for each index n. Therefore, our assertion can be deduced from the following lemma which applies, by Lemma 2.4, to any pair $(\kappa, \beta) \in \mathbb{N} \times (0,1)$.

Lemma 4.2. *Assume that $F_\infty/\mathbb{F}$ is a field extension, where $\mathbb{F}$ is algebraically closed,* $\mathrm{char}(\mathbb{F}) = q$, *and F_∞ is quasifinite; also, let F_0 and F_n, $n \in \mathbb{N}$, be intermediate fields of $F_\infty/\mathbb{F}$ satisfying the conditions of Proposition 4.1, and such that $\cup_{n\in\mathbb{N}} F_n = F_\infty$. Take a pair $(\kappa, \beta) \in \mathbb{Z} \times \mathbb{R}$ with $0 < \beta < 1$ and $\kappa \geq 0$, put $\lambda = \sum_{j=0}^{\kappa} \beta^j$, and fix some $\mu \in \mathbb{N}$ so that $p_\mu \in \mathbb{P}_q$ be $(\beta, \kappa; q)$-representable. Then there exists an F_μ-form f_μ of degree p_μ, which depends on at least p^λ variables and does not possess a nontrivial zero over F_μ.*

Lemma 4.2 has been proved inductively by Ax in the case of $q = 0$ (see [6, Lemma 1]). Here we show that Ax's proof remains valid for any $q \in \overline{\mathbb{P}}$. Note first that the field F_ν satisfies condition (b) of Proposition 4.1, for each $\nu \in \mathbb{N}$, so it follows from Galois theory that F_ν has an extension $\widetilde{F}_\nu$ of degree p_ν. Hence, the norm form of $\widetilde{F}_\nu/F_\nu$ with respect to any F_ν-basis of $\widetilde{F}_\nu$ is an F_ν-form of degree p_ν in p_ν variables without a nontrivial zero over F_ν, which proves the assertion of Lemma 4.2 in case $\kappa = 0$. We assume further that $\kappa \geq 1$, $p_\mu = p_{\mu_1} + p_{\mu_2} + p_{\mu_3}$, where $p_{\mu_1}, p_{\mu_2}, p_{\mu_3} \in \mathbb{P}_q$ are $(\beta, \kappa - 1; q)$-representable with $q < p_\mu^\beta < p_{\mu_1} < p_{\mu_2} < p_{\mu_3}$, and the statement of Lemma 4.2 holds for the $(\beta, \kappa - 1; q)$-representable numbers p_{μ_1}, p_{μ_2} and p_{μ_3}. This means that there are F_{μ_i}-forms f_{μ_i} of degree p_{μ_i} $(i = 1, 2, 3)$, depending on $d \geq p_{\mu_1}^{\lambda'} > p_\mu^{\beta\lambda'}$ variables $X_1, \dots, X_d$, where $\lambda' = \sum_{j=0}^{\kappa-1} \beta^j$. The product $\tilde{f}_\mu = f_{\mu_1}.f_{\mu_2}.f_{\mu_3}$ is clearly an $F_{\mu-1}$-form of degree p_μ in variables $X_1, \dots, X_d$. Using Galois theory, Lemma 3.4, and condition (b) of Proposition 4.1, one proves that $v_\mu(F_\mu) \neq p_\mu v_\mu(F_\mu)$, v_μ being the $\mathbb{Q}$-valued valuation of F_μ extending the standard valuation of $F_{\mu-1}((X_\mu))$. Fix an element $\pi \in F_\mu^*$ with $v_\mu(\pi) \notin p_\mu v_\mu(F_\mu)$, and let f_μ be the F_μ-form $\sum_{u=0}^{p_\mu - 1} \tilde{f}_\mu(X_{u,1}, \dots, X_{u,d})\pi^u$ (in $dp_\mu > p_\mu^{1+\beta\lambda'} = p_\mu^\lambda$ variables). It follows from the above-noted properties of F_ν-forms, for $\nu \in \mathbb{N}$, that $\tilde{f}_\mu$ does not possess a nontrivial zero over $F_{\mu-1}$. Since the fulfillment of condition (b) guarantees that finite extensions of $F_{\mu-1}((X_\mu))$ in F_μ are totally ramified, this allows to deduce from Lemma 3.2 (b) that $\tilde{f}_\mu$ is without a nontrivial zero over F_μ. Therefore, one obtains as in the proof of Lemma 3.2 (a) that f_μ has the properties claimed by Lemma 4.2, so Proposition 4.1 is proved. □

Next we show that every algebraically closed field $\mathbb{F}$ has extensions subject to the restrictions of Proposition 4.1. We begin with the following lemma.

Lemma 4.3. *Let $\mathbb{F}$ be an algebraically closed field of characteristic $q > 0$, Π a nonempty subset of $\mathbb{P}$, and $\mathbb{F}_1 = \mathbb{F}(X)$ be the rational function field over $\mathbb{F}$ in a variable X. Then there exists an algebraic extension $F_0/\mathbb{F}_1$, such that F_0 is a perfect field and $\mathcal{G}_{F_0} \cong \prod_{p\in\Pi} \mathbb{Z}_p$.*

Proof. Let $\overline{\mathbb{F}}_1$ be an algebraic closure of $\mathbb{F}_{1,\mathrm{sep}}$, and for each $p \in \mathbb{P}$, let L_p be an extension of $\mathbb{F}_1$ in $\overline{\mathbb{F}}_1$ obtained by adjunction of a root of the polynomial $Y^p - \delta_p Y - X^{-1}$, where $\delta_p = 1$ if $p = q$, and $\delta_p = 0$, otherwise. The field $\mathbb{F}_1$ has a valuation v with $v(\mathbb{F}_1) = \mathbb{Z}$ and $v(X) = 1$ (see [13, Example 4.1.3]); this implies v is trivial on $\mathbb{F}$ and $L_p/\mathbb{F}_1$, $p \in \mathbb{P}$, are cyclic field extensions of degree p. As $\mathbb{F}_1$ contains a primitive h-th root of unity, for each $h \in \mathbb{N}$ not divisible by q, it follows from Kummer theory and Witt's lemma (cf. [12, Sect. 15, Lemma 2]) that $\mathbb{F}_1$ has $\mathbb{Z}_p$-extensions Γ_p in $\overline{\mathbb{F}}_1$, $p \in \mathbb{P}$, such that $L_p \in I(\Gamma_p/\mathbb{F}_1)$, for each p. Note also that the compositum Γ of the fields Γ_p, $p \in \Pi$, is a Galois extension of $\mathbb{F}_1$ with $\mathcal{G}(\Gamma/\mathbb{F}_1) \cong \prod_{p\in\Pi} \mathbb{Z}_p$. Hence, $\mathcal{G}(\Gamma/\mathbb{F}_1)$ is a projective profinite group, in the sense of [28, Ch. I, 5.9], which allows to obtain from Galois theory that there exists $\widetilde{F}_0 \in I(\mathbb{F}_{1,\mathrm{sep}}/\mathbb{F}_1)$ satisfying $\Gamma \cap \widetilde{F}_0 = \mathbb{F}_1$ and $\Gamma.\widetilde{F}_0 = \mathbb{F}_{1,\mathrm{sep}}$. It is proved similarly that if F_0 is the perfect closure of $\widetilde{F}_0$ in $\overline{\mathbb{F}}_1$, then $\Gamma \cap F_0 = \mathbb{F}_1$ and $\Gamma F_0 = \overline{\mathbb{F}}_1$; in particular, F_0 is a perfect field with $\mathcal{G}_{F_0} \cong \mathcal{G}(\Gamma/\mathbb{F}_1) \cong \prod_{p\in\Pi} \mathbb{Z}_p$. □

Let now $\mathbb{F}$ be an algebraically closed field, $F_0/\mathbb{F}$ a field extension satisfying condition (a) of Proposition 4.1, and let F_n, $n \in \mathbb{N}$, be perfect fields with Henselian $\mathbb{Q}$-valued valuations v_n, defined inductively as follows:

(4.1) For each $n \in \mathbb{N}$, F_n is an extension of the Laurent series field $F_{n-1}((X_n)) := K_n$ in an algebraic closure $\overline{K}_n$ of $F_{n-1,\mathrm{sep}}.K_n$, such that:

(a) $(F_n, v_n)/(K_n, \kappa_n)$ is a valued field extension, where κ_n is the standard $\mathbb{Z}$-valued valuation of K_n trivial on F_{n-1}, and v_n is the valuation of F_n extending κ_n; hence, (F_n, v_n) is Henselian with $\mathbb{Z} \subseteq v_n(F_n) \subseteq \mathbb{Q}$;

(b) F_n contains as a subfield a separable extension W_n of K_n in $\overline{K}_n$, such that $W_n \cap K_{n,\mathrm{tr}} = K_n$ and $W_n.K_{n,\mathrm{tr}} = W_{n,\mathrm{tr}}$ equals the separable closure $K_{n,\mathrm{sep}}$ of K_n in $\overline{K}_n$; in particular, if $q = 0$, then $W_n = K_n$;

(c) F_n is the extension of W_n generated by the h_n-th roots of X_n in $\overline{K}_n$, where h_n runs across the set of positive integers not divisible by p_n.

We show that the fields F_n, $n \in \mathbb{N}$, satisfy condition (b) of Proposition 4.1. Proceeding by induction on n, one obtains that it is sufficient to prove our assertion, for a fixed index n, under the hypothesis that F_{n-1} is perfect and satisfies condition (a) or (b) of Proposition 4.1 depending on whether or not $n = 1$. Let $T_{n,p}$ be the extension of K_n in $\overline{K}_n$ generated by all roots of X_n of p-primary degrees, for each $p \in \mathbb{P}_q$, and let T_n be the compositum of the fields $T_{n,p}$, $p \in \mathbb{P}_q$. Clearly, $X_n \notin K_n^{*p}$, for any $p \in \mathbb{P}$, so it follows from Kummer theory and (3.3) that, in case $p \neq q$, $T_{n,p}$ is a $\mathbb{Z}_p$-extension of K_n in $K_{n,\mathrm{tr}}$, and finite extensions of K_n in $T_{n,p}$ are totally ramified of p-primary degrees. As $K_{n,\mathrm{ur}} = F_{n-1,\mathrm{sep}}.K_n$, these facts and Lemma 3.4 (a) show that $F_{n-1,\mathrm{sep}}.K_n \cap T_n = K_n$ and $F_{n-1,\mathrm{sep}}T_n = K_{n,\mathrm{tr}}$. On the other hand, if $q > 0$, then $K_n^q = F_{n-1}((X_n^q))$, which implies $[K_n \colon K_n^q] = q$ and $T_{n,q}$ is the perfect closure of K_n in $\overline{K}_n$. Put $\widetilde{W}_n = W_n T_{n,q}$ if $q > 0$, and $\widetilde{W}_n = K_n$ if $q = 0$. It is easily verified that $\widetilde{W}_{n,\mathrm{ur}} = F_{n-1,\mathrm{sep}}.\widetilde{W}_n$. Using (4.1), one obtains that $F_n = \widetilde{W}_n.\Phi_n$, Φ_n being the compositum of the fields $T_{n,p}$, for $p \in \mathbb{P} \setminus \{q, p_n\}$. Therefore, F_n is perfect, $F_{n,\mathrm{tr}} = \overline{K}_n$, and in case $q = 0$, we have $F_n = \Phi_n$, $F_{n-1,\mathrm{sep}}T_{n,p_n} \cap F_n = K_n$, $F_{n-1,\mathrm{sep}}T_{n,p_n}.F_n = F_{n,\mathrm{sep}}$. When $q > 0$, it follows that $\widetilde{W}_n \cap K_{n,\mathrm{tr}} = K_n$, $\widetilde{W}_n.K_{n,\mathrm{tr}} = \overline{K}_n$, and $\widetilde{W}_n\Phi_n = F_n$. Putting $\Theta_{n,p} = \widetilde{W}_n T_{n,p}$, for each $p \in \mathbb{P}_q$, one obtains that F_n equals the compositum of the fields $\Theta_{n,p}$, $p \in \mathbb{P} \setminus \{q, p_n\}$. Since, by (3.3), (4.1) (b) and Lemma 3.4, finite extensions of K_n in $\widetilde{W}_n$ are totally ramified of q-primary degrees, this allows to deduce from Galois theory and the definition of $T_{n,p}$, $p \in \mathbb{P}_q$, that $\Theta_{n,p}/\widetilde{W}_n$ is a $\mathbb{Z}_p$-extension whose finite subextensions are totally ramified, for each $p \neq q$. One also sees that finite extensions of $\widetilde{W}_n$ in F_n are tamely and totally ramified of degrees not divisible by p_n. Similarly, it is proved that, for any finite extension $\widetilde{W}_n'$ of $\widetilde{W}_n$ in $F_{n-1,\mathrm{sep}}T_{n,p_n}.\widetilde{W}_n$, $e(\widetilde{W}_n'/\widetilde{W}_n)$ is a p_n-primary number. These observations show that $F_{n-1,\mathrm{sep}}\Theta_{n,p_n}$ is a Galois extension of $\widetilde{W}_n$, such that $F_{n-1,\mathrm{sep}}\Theta_{n,p_n} \cap F_n = \widetilde{W}_n$, so it follows from Galois theory that $\mathcal{G}_{F_n} \cong \mathcal{G}(F_{n-1,\mathrm{sep}}\Theta_{n,p_n}/\widetilde{W}_n) \cong \mathcal{G}_{F_{n-1}} \times \mathbb{Z}_{p_n}$. In view of our hypothesis on F_{n-1}, this completes the proof of the assertion that the fields F_n, $n \in \mathbb{N}$, defined by (4.1) satisfy condition (b) of Proposition 4.1. Hence, the union $F_\infty = \cup_{n\in\mathbb{N}} F_n$ is a quasifinite field with $\mathrm{ddim}(F_\infty) = \infty$. As $\mathbb{F}$ is an arbitrary algebraically closed field, the existence of quasifinite fields $F_{\infty,q}$, $q \in \overline{\mathbb{P}}$, such that $\mathrm{char}(F_{\infty,q}) = q$ and $\mathrm{ddim}(F_{\infty,q}) = \infty$, for each q, is now obvious.

We turn to the proof of Theorem 2.1 in the case where $m < \infty$. Fix an algebraically closed field $\mathbb{F}$ of characteristic q as well as an extension F_0 of $\mathbb{F}$ satisfying condition (a) of Proposition 4.1, and a sequence of fields K_n, $\overline{K}_n$, W_n and F_n, $n \in \mathbb{N}$, defined in agreement with (4.1). If $m = 1$ and $q = 0$, then one may put $F_{m,q} = K_1$, and in case $m = 1$ and $q > 0$, $F_{m,q}$ may be defined by applying Lemma 4.3 to the set $\Pi = \mathbb{P}$. Henceforth, we assume that $m \geq 2$. It follows from Greenberg's theorem (and Tsen's theorem, for $n = 0$), that $\mathrm{ddim}(F_n) \leq n + 1$, for every $n \in \mathbb{N}$. Let $\mathrm{char}(F_0) = q$ and μ be the minimal integer for which $\mathrm{ddim}(F_\mu) \geq m$. Using Greenberg's theorem and the closeness of the class of C_n-fields under the formation of algebraic extensions, for each $n \in \mathbb{N}$, one obtains that $\mathrm{ddim}(\Psi_\mu) = m$ whenever $\Psi_\mu \in I(F_\mu/K_\mu)$. Note that Ψ_μ can be chosen to be a quasifinite field. If $\mu = 1$ (which requires that $q > 0$), then one may take as Ψ_μ the perfect closure of W_μ in $\overline{F}_\mu$. Suppose further that $\mu \geq 2$. It can be deduced from Galois theory and Lemma 3.4 that $\mathcal{G}_{W_\mu}$ is isomorphic to the profinite groups $\mathcal{G}(K_{\mu,\mathrm{tr}}/K_\mu)$ and $\mathcal{G}_{F_{\mu-1}} \times \prod_{p \in \mathbb{P}_q} \mathbb{Z}_p$. This implies the existence of a Galois extension Ψ'_μ of W_μ in F_μ with $\mathcal{G}(\Psi'_\mu/W_\mu) \cong \prod_{j=1}^{\mu-1} \mathbb{Z}_{p_j}$. Since F_0 is algebraically closed if $q = 0$, and there are isomorphisms $\mathcal{G}_{F_0} \cong \mathbb{Z}_q$ and $\mathcal{G}_{F_{\mu-1}} \cong \mathbb{Z}_q \times \prod_{j=1}^{\mu-1} \mathbb{Z}_{p_j}$ in case $q > 0$, it is easy to see that $\mathcal{G}_{\Psi'_\mu} \cong \prod_{p \in \mathbb{P}} \mathbb{Z}_p$. Let finally Ψ_μ be the perfect closure of Ψ'_μ in $F_{\mu,\mathrm{sep}} = \overline{K}_\mu$. Then it follows from Galois theory and [22, Ch. V, Proposition 6.11], that $\mathcal{G}_{\Psi_\mu} \cong \mathcal{G}_{\Psi'_\mu}$, which means that Ψ_μ is a quasifinite field. Taking into account that F_μ is perfect, one concludes that $\Psi_\mu \in I(F_\mu/W_\mu)$, so Theorem 2.1 is proved.

Corollary 4.4. *Let F_∞ be a quasifinite field defined in accordance with Proposition 4.1. Then* $\mathrm{ddim}(F'_\infty) = \infty$*, for every finite extension F'_∞/F_∞.*

Proof. By definition, F_∞ possesses subfields $\mathbb{F}$, F_0 and F_n, $n \in \mathbb{N}$, such that $\mathbb{F}$ is algebraically closed, F_0 and F_n, $n \in \mathbb{N}$, satisfy conditions (a) and (b) of Proposition 4.1, respectively, and $F_\infty = \cup_{n \in \mathbb{N}} F_n$. We first show that the fields $K_n = F_{n-1}((X_n))$, $n \in \mathbb{N}$, given in condition (b), can be chosen so that $p_n \nmid [\Lambda_n \colon K_n]$, for any finite extension Λ_n of K_n in F_n (and any $n \in \mathbb{N}$). Proceeding by induction on n, one obtains that there exist elements $Y_n \in F_n$, $n \in \mathbb{N}$, such that Y_n is a root of X_n of p_n-primary degree and the degrees of finite extensions of $K_n(Y_n)$ in F_n are not divisible by p_n, for any n. Since $K_n(Y_n) = F_{n-1}((Y_n))$, $n \in \mathbb{N}$, this yields the desired reduction.

Take a finite extension F'_∞ of F_∞ in $F_{\infty,\mathrm{sep}}$, put $\theta = [F'_\infty \colon F_\infty]$, and for each $n \in \mathbb{N}$, let θ_n be the greatest p_n-primary divisor of θ; put $\theta_0 = 1$ if $q = 0$, and let θ_0 be the greatest q-primary divisor of θ in case $q > 0$. Fix a primitive element λ of F'_∞/F_∞, and denote by $f(X)$ the minimal polynomial of λ over

F_∞. Clearly, the coefficients of $f(X)$ lie in F_m, for some $m < \infty$. Consider the fields F'_n, $n \geq 0$, defined as follows: $F'_n = F_n(\lambda)$ if $n \geq m$; $F'_n = F_{n,\mathrm{sep}} \cap F'_m$ if $n < m$. It follows from condition (b) that $[F'_n \colon F_n] \leq \theta$, for all $n \geq 0$, and equality holds if $n \geq m$. Since $\mathbb{Z}_p$ is isomorphic to its open subgroups, for each $p \in \mathbb{P}$, this means that F'_0 satisfies condition (a) of Proposition 4.1. Similarly, it follows that $\mathcal{G}_{F'_n} \cong \mathcal{G}_{F_n}$, for every $n \in \mathbb{N}$.

Note further that, by Lemma 3.3 and condition (b), $F_{n-1,\mathrm{sep}} \cap F_n = F_{n-1}$ and $F_{n-1,\mathrm{sep}} \otimes_{F_{n-1}} F_n \cong F_{n-1,\mathrm{sep}}.F_n$ as an F_{n-1}-algebra. These observations and our choice of the fields K_n, $n \in \mathbb{N}$, enable one to deduce from Galois theory that $[F'_n \colon F_n] = \prod_{j=0}^{n} \theta_j$, for each $n < m$, and when $n > 0$, $F'_n = F'_{n-1}.F_n(Z_n)$ and $F'_{n-1}.K_n(Z_n) = F'_{n-1}((Z_n))$, where $Z_n \in F'_n$ is a θ_n-th root of X_n. This ensures that $F'_n = F_n.F'_{n-1}((Z_n))$ and finite extensions of $F'_{n-1}((Z_n))$ in F'_n are totally ramified. It is therefore clear from (3.3) (b) and the equality $F_{n-1,\mathrm{sep}}.F'_{n-1}((Z_n)) = F'_{n-1}((Z_n))_{\mathrm{ur}}$ that $F_{n-1,\mathrm{sep}}.F'_{n-1}((Z_n)) \cap F'_n = F'_{n-1}((Z_n))$, which in turn implies $F_{n-1,\mathrm{sep}} \cap F'_n = F_{n-1,\mathrm{sep}} \cap F'_{n-1}((Z_n)) = F'_{n-1}$. Applying now Lemma 3.3, one concludes that $F_{n-1,\mathrm{sep}}.F'_n \cong F_{n-1,\mathrm{sep}} \otimes_{F'_{n-1}} F'_n$ as F'_{n-1}-algebras, for every $n > 0$. Thus the extensions $F'_n/F'_{n-1}((Z_n))$, $n \in \mathbb{N}$ (where $Z_n = X_n$ if $n > m$), satisfy condition (b) of Proposition 4.1, so $\mathrm{ddim}(F'_\infty) = \infty$, as required. □

Remark 4.5. *Let $F_{m,q}$, $(m,q) \in \mathbb{N}_\infty \times \mathbb{P}$, be quasifinite fields such that char$(F_{m,q}) = q$ and ddim$(F_{m,q}) = m$, for each pair (m,q). Then, by Witt's theorem (cf. [13, Theorem 12.4.1]), there are complete discrete valued fields $(K_{m,q}, v_{m,q})$ with char$(K_{m,q}) = 0$ and $\widehat{K}_{m,q} = F_{m,q}$, for all $(m,q) \in \mathbb{N}_\infty \times \mathbb{P}$. Also, it follows from Lemma 3.4 that, for each (m,q), there exists an extension $T_{m,q}$ of $K_{m,q}$ in $K_{m,q,\mathrm{sep}}$, such that $K_{m,q,\mathrm{ur}}.T_{m,q} = K_{m,q,\mathrm{sep}}$ and $K_{m,q,\mathrm{ur}} \cap T_{m,q} = K_{m,q}$. This ensures that $T_{m,q,\mathrm{ur}} = K_{m,q,\mathrm{sep}}$ and $\widehat{T}_{m,q} = F_{m,q}$, so Lemma 3.1 (b) implies $T_{m,q}$ is a quasifinite field, ddim$(T_{m,q}) \geq m$ (and equality holds if $m = \infty$). Note that $T_{m,q}$, $(m,q) \in \mathbb{N}_\infty \times \mathbb{P}$, are pairwise non-isomorphic fields. Since the valuation of $T_{m,q}$ extending $v_{m,q}$ is Henselian with $v_{m,q}(T_{m,q}) = \mathbb{Q}$, for every (m,q), this can be proved by assuming the opposite, and using the non-existence of a field with a pair of Henselian real-valued valuations whose residue fields are quasifinite and non-isomorphic. The noted fact follows from the validity of Schmidt's Uniqueness Theorem in the case of Henselian real-valued valuations on a field (see [13, Corollary 21.1.2]). It would be of interest to know whether ddim$(T_{m,q}) = m$, for every $(m,q) \in \mathbb{N} \times \mathbb{P}$.*

5 Perfect fields of characteristic $q \in \overline{\mathbb{P}}$ and prescribed Galois cohomological and Diophantine dimensions

In this Section we present a proof of Theorem 2.2. Our starting point is the following lemma which proves Theorem 2.2 in case $q = 0$:

Lemma 5.1. *Let $\ell \in \mathbb{N}_\infty$ and k be an integer satisfying $1 \leq k \leq \ell$. Then there exists a field $E_{k,\ell}$ with $\mathrm{ddim}(E_{k,\ell}) = \ell$ and $\mathrm{cd}(E_{k,\ell}) = k$. Furthermore, for each $q \in \overline{\mathbb{P}}$, $E_{k,\ell}$ can be chosen so that $\mathrm{char}(E_{k,\ell}) = q$.*

Proof. Our assertion is contained in Theorem 2.1 in case $k = 1$, so we assume that $k \geq 2$. If $\ell \in \mathbb{N}$, then it follows from Galois cohomology (see [6, pages 1219-1220], and [28, Ch. II, 2.2 and 4.3]), Greenberg's theorem and Lemma 3.2 that one may take as $E_{k,\ell}$ the iterated Laurent formal power series field $F_{\ell-k+1,q}((X_1))\dots((X_{k-1}))$, where $F_{\ell-k+1,q}$ has the properties required by Theorem 2.1, for $m = \ell - k + 1$ and any $q \in \overline{\mathbb{P}}$. Similarly, if $\ell = \infty$ and $F_{\infty,q}$ is a quasifinite field with $\mathrm{char}(F_{\infty,q}) = q$ and $\mathrm{ddim}(F_{\infty,q}) = \infty$, then one may put $E_{k,\infty} = F_{\infty,q}((X_1))\dots((X_{k-1}))$. □

Theorem 2.1 and Lemma 5.1 allow to assume in the rest of the proof of Theorem 2.2 that $q > 0$ and $k \geq 2$. Retaining notation as in the proof of Lemma 5.1, denote by $E'_{k,\ell}$ the perfect closure of $E_{k,\ell}$ in its algebraic closure $\overline{E}_{k,\ell}$. It follows from Galois theory and [22, Ch. V, Proposition 6.11], that $\mathcal{G}_{E_{k,\ell}} \cong \mathcal{G}_{E'_{k,\ell}}$, so it suffices to prove that the quasifinite constant field of $E_{k,\ell}$ can be chosen so that $\mathrm{ddim}(E'_{k,\ell}) = \ell$. Suppose first that $\ell = \infty$ and $F_{\infty,q}$ is a quasifinite field with $\mathrm{char}(F_{\infty,q}) = q$ and $\mathrm{ddim}(F_{\infty,q}) = \infty$, defined as in Proposition 4.1. Then there exist $q_n \in \mathbb{P}_q$, $n \in \mathbb{N}$, such that for each n, there is an $F_{\infty,q}$-form t_n of degree q_n in at least q_n^n variables, which does not possess a nontrivial zero over $F_{\infty,q}$. Proceeding by induction on $k-1$ and using Lemma 3.2, one obtains that t_n is without a nontrivial zero over $E'_{k,\infty}$. This yields $\mathrm{ddim}(E'_{k,\infty}) = \infty$, which completes our proof in case $\ell = \infty$.

To prove Theorem 2.2 in case $\ell < \infty$ we need the following two lemmas.

Lemma 5.2. *Let $\mathbb{F}$ be an algebraically closed field, $q = \mathrm{char}(\mathbb{F})$, m an integer ≥ 2, and $\Pi_i\colon i = 0, \dots, m-1$, be nonempty subsets of $\mathbb{P}$, such that $q \in \Pi_0$ in case $q > 0$, $\cup_{i=0}^{m-1} \Pi_i = \mathbb{P}$, and $\Pi_{i'} \cap \Pi_{i''} = \emptyset$, provided that $0 \leq i' < i'' \leq m-1$. Then there exist perfect fields $F_i\colon i = 0, \dots, m-1$, with the following properties:*

(a) *$\mathcal{G}_{F_0} \cong \prod_{p \in \Pi_0} \mathbb{Z}_p$ and $F_0/\mathbb{F}$ is an extension of transcendence degree 1;*

(b) *For each* $i > 0$, F_i *is an algebraic extension of the Laurent series field* $F_{i-1}((X_i))$, *such that* $F_{i-1,\mathrm{sep}} \otimes_{F_{i-1}} F_i$ *is a field and* $\mathcal{G}_{F_i} \cong \mathcal{G}_{F_{i-1}} \times \prod_{q_i \in \Pi_i} \mathbb{Z}_{q_i}$; *in particular,* F_{m-1} *is a quasifinite field with* $\mathrm{ddim}(F_{m-1}) \leq m$.

Proof. The existence of F_0 follows from Lemma 4.3, and the equality $\mathrm{ddim}(F_0) = 1$ is implied by Tsen's theorem and the fact that $F_0 \neq F_{0,\mathrm{sep}}$. Suppose that $i > 0$ and F_{i-1} has been defined in accordance with Lemma 5.2 (a) or (b) depending on whether or not $i = 1$. Put $K_i = F_{i-1}((X_i))$, fix an algebraic closure $\overline{K}_i$ of $K_{i,\mathrm{sep}}$, and denote by κ_i the standard discrete valuation of K_i trivial on F_{i-1}. Considering the Henselian field (K_i, κ_i), let W_i be an extension of K_i in $\overline{K}_i$, such that $W_i.K_{i,\mathrm{tr}} = \overline{K}_i = W_{i,\mathrm{sep}}$ and $W_i \cap K_{i,\mathrm{tr}} = K_i$. Take as F_i the extension of W_i in $\overline{K}_i$ generated by the h_i-th roots of X_i, when h_i runs across the set $\{\nu \in \mathbb{N} \colon \nu$ is not divisible by any $q_i \in \Pi_i\}$. The assertion that F_i is perfect and has the properties claimed by Lemma 5.2 (b) is proved in the same way as the fact that the fields defined by (4.1) satisfy the conditions of Proposition 4.1. Since $\cup_{i=0}^{m-1} \Pi_i = \mathbb{P}$ and $\Pi_i \cap \Pi_{i'} = \emptyset$, $i \neq i'$, this implies F_{m-1} is quasifinite. Observing finally that $\mathrm{ddim}(F_i) \leq \mathrm{ddim}(K_i) \leq i + 1$, for $i = 1, \dots, m-1$ (the latter inequality follows from Greenberg's theorem), one completes the proof. □

Lemma 5.3. *Let* m *be an integer* ≥ 2, *and* β *a real number such that* $\sqrt[m+1]{1 - (m+1)^{-2}} < \beta < 1$. *Assume that* $q \in \overline{\mathbb{P}}$ *and* $\Pi_i \colon i = 0, \dots, m-1$, *are nonempty subsets of* $\mathbb{P}$ *satisfying the following two conditions:*

(c) $q \notin \Pi_{m-1}$, $\cup_{i=0}^{m-1} \Pi_i = \mathbb{P}$, *and* $\Pi_{i'} \cap \Pi_{i''} = \emptyset$, *for each pair of indices* $i' \neq i''$; *in addition,* Π_i *is finite unless* $i = m-1$;

(cc) *If* $0 < i < m-1$, *then* Π_i *consists of* $(\beta, i; q)$*-representable numbers, and for each* $q_i \in \Pi_i$, *we have* $q_i = q'_{i-1} + q''_{i-1} + q'''_{i-1}$, *for some* q'_{i-1}, q''_{i-1}, $q'''_{i-1} \in \Pi_{i-1}$ *with* $q < q_i^{\beta} < q'_{i-1} < q''_{i-1} < q'''_{i-1}$; *also,* Π_{m-1} *contains an element* q_{m-1} *equal to* $q'_{m-2} + q''_{m-2} + q'''_{m-2}$, *where* q'_{m-2}, q''_{m-2} *and* q'''_{m-2} *are pairwise distinct elements of* Π_{m-2}, *and* $q < q_{m-1}^{\beta} < \min\{q'_{m-2}, q''_{m-2}, q'''_{m-2}\}$.

Then any quasifinite field F_{m-1} *singled out by Lemma 5.2 admits a form* f_{m-1} *of degree* q_{m-1} *in more than* q_{m-1}^{m-1} *variables, which does not possess a nontrivial zero over* F_{m-1}; *in particular,* $\mathrm{ddim}(F_{m-1}) = m$.

Proof. Since, by Lemma 5.1 and the choice of F_{m-1}, $\mathrm{ddim}(F_{m-1}) \leq m$, the latter part of our assertion follows from the former one. Arguing by the method of proving Lemma 4.2, one obtains from conditions (c) and (cc) the existence of an F_{m-1}-form f_{m-1} of degree q_{m-1} in more than q_{m-1}^{λ} variables, and without a nontrivial zero over F_{m-1}, where $\bar{\lambda} = \sum_{i=0}^{m-1} \beta^i$. The assumptions on β show that $\bar{\lambda} > m - 1$, so Lemma 5.3 is proved. □

It is now easy to complete the proof of Theorem 2.2. Assume that $\ell < \infty$ and put $m = \ell - k + 1$. Applying Lemma 4.3 (to the case where $\Pi = \mathbb{P}$) if $m = 1$, and Lemma 5.3 when $m \geq 2$, one obtains that there is a quasifinite field $F_{m,q}$ with $\mathrm{char}(F_{m,q}) = q$ and $\mathrm{ddim}(F_{m,q}) = m$, which admits an $F_{m,q}$-form $f_{m,q}$ of degree $q_{m-1} \in \mathbb{P}_q$ in at least $1 + q_{m-1}^{m-1}$ variables, possessing no nontrivial zero over F_{m-1}. Arguing by induction on $k - 1$, and using Lemma 3.2, one can associate with $f_{m,q}$ an $E'_{k,\ell}$-form $y_{m,q}$ of degree q_{m-1} in more than $q_{m-1}^{\ell-1}$ variables, and without a nontrivial zero over $E'_{k,\ell}$. Therefore, $\mathrm{ddim}(E'_{k,\ell}) \geq \ell$, and since $\mathrm{ddim}(E'_{k,\ell}) \leq \mathrm{ddim}(E_{k,\ell}) = \ell$ (apply repeatedly Greenberg's theorem), we have $\mathrm{ddim}(E'_{k,\ell}) = \ell$, so Theorem 2.2 is proved.

As explained in Section 2, every perfect field E satisfies $\mathrm{CD}(E) = \mathrm{cd}(\mathcal{G}_E)$. Generally, this does not apply to arbitrary fields (see (1.1) and (1.2)). However, the equality holds for interesting classes of imperfect fields, such as the iterated Laurent series fields in n variables, for any fixed $n \in \mathbb{N}$, over quasifinite fields of type C_1 and nonzero characteristic. To demonstrate it, consider a field $E_n = E_0((X_1))\dots((X_n))$ of this kind over a quasifinite C_1-field E_0 of characteristic $q > 0$. Then $\mathrm{char}(E_n) = q$ and $[E_n : E_n^q] = q^n$, which allows, similarly to the proof of Lemma 5.1, to deduce from Galois cohomology, combined with Kato-Milne cohomology [18, Theorem 3 (3)], Greenberg's theorem and the Arason-Baeza theorem, that $\mathrm{cd}_q(\mathcal{G}_{E_n}) = 1$, and the dimensions $\mathrm{cd}(\mathcal{G}_{E_n})$, $\mathrm{CD}(E_n)$, $\mathrm{ddim}(E_n)$ and $\dim_q(E_n)$ (as well as $\overline{\mathrm{CD}}(E_n)$, see page 23) are equal to $n + 1$.

Remark 5.4. *Arguing as in the proof of Corollary 4.4, one obtains that the class of quasifinite fields with the property described by Lemma 5.3 is closed under taking finite extensions. This implies in conjunction with Corollary 4.4 that the fields $E_{k,\ell;q}$ singled out by Theorem 2.2 can be chosen so that their finite extensions have Diophantine dimension ℓ, for any $q \in \overline{\mathbb{P}}$ and each pair $(k, \ell) \neq (0, 0)$ admissible by Question 1. It is well-known that the open subgroups of $\mathcal{G}_{E_{k,\ell;q}}$, namely, the absolute Galois groups of the considered fields have cohomological dimension k (see [28, Ch. I, Proposition 14]).*

To conclude with, we note that our proof of Theorem 2.1 strongly depends on the fact that given a quasifinite field F, the Sylow pro-p-subgroup of $\mathcal{G}_F$ is isomorphic to $\mathbb{Z}_p$, for every $p \in \mathbb{P}$. The existence of perfect fields E_p, $p \in \mathbb{P}_2$, with $\dim(E_p) \leq 1$, $\mathrm{ddim}(E_p) \geq 2$ and $\mathcal{G}_{E_p}$ a pro-p-group with infinitely many open subgroups of index p, solving [19, Problem 2], has been established in [11] and [10], by a different method. Both results (as well as Merkur'ev's counterexample to the Kato-Kuzumaki C_2^0 conjecture, see [19] and [25, Theorem 4 and Proposition 5]) leave open the following question.

Question 2. Find whether there exists a perfect field E such that $\dim(E) = \infty$ and some of the following two conditions holds:

(a) $\dim(E) \leq 1$ and the Sylow pro-p-subgroups of $\mathcal{G}_E$ are not isomorphic to $\mathbb{Z}_p$, for any $p \in \mathbb{P}$ (with, possibly, finitely many exceptions);

(b) E is the fixed field of a Sylow pro-p-subgroup G_p of $\mathcal{G}_{\mathbb{Q}_1}$, for some $p \in \mathbb{P}$, where $\mathbb{Q}_1$ is a totally imaginary number field.

As noted at the end of Remark 2.3, nothing seems to be known about the Diophantine dimensions of the fixed fields of Sylow's subgroups of $\mathcal{G}_{\mathbb{Q}_1}$, for any totally imaginary number field $\mathbb{Q}_1$. Against this background, Question 2 (b) makes interest in its own right. A negative answer to it, for some $p \in \mathbb{P}$, would show that $\mathrm{abrd}_p(\mathbb{Q}') < \infty$, for all finitely-generated extensions $\mathbb{Q}'/\mathbb{Q}$. This follows from the Lang-Nagata-Tsen theorem, the main results of [23], and [14, Corollary 4.5.11].

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