DOI: 10.2478/auom-2024-0016
An. Şt. Univ. Ovidius Constanţa **Vol. 32(2),2024, 5–18**

A new approach to (dual) Rickart modules via isomorphisms

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Abstract

In the past few decades, researchers have found that studying modules using endomorphisms is a powerful and useful tool. This has led to valuable works in this field. Recently, the study of (dual) Rickart modules has become an important approach as they are deeply connected to endomorphisms. Building on this work, the authors introduce a new perspective on (dual) Rickart modules using isomorphism. We also define virtually (dual) Rickart modules. It is shown that rings with all modules virtually Rickart are semisimple rings. The paper includes various examples to illustrate the concepts presented.

1 Introduction

The rings discussed in this paper will have an identity element and will be associative. Similarly, unless specified otherwise, all modules will be right modules. A submodule L of V is considered essential in V if it does not contain a nonzero element from an arbitrary nonzero submodule U of V. On the other hand, a submodule W of V is considered small in V if V is not equal to the sum of any proper submodule L of V with W. The module V is hollow, provided each proper submodule is small in V. For a module V and $W \leq V$, the notion $W \leq^{\oplus} V$ means W is a direct summand (ds, for short) of V. Note also that $Rad(V)$ is defined to be the sum of all small submodules of V and $Soc(V)$ is equal to sum of all simple submodules of V.

Key Words: annihilator, dual Rickart module, virtually Rickart module, virtually dual Rickart module, endomorphism ring.
2010 Mathematics Subject Classification: Primary 16D10, 16D40; Secondary 16D90.
Received: 08.05.2023
Accepted: 25.10.2023

The Q-module V has a subset called $Z(V)$, which consists of elements x in V such that $xI = 0$ for some essential right ideal I of the ring Q. If $Z(V) = V$, then V is called singular and when $Z(V) = 0$ then V is said to be nonsingular.

The concepts of Rickart and Baer rings have their origins in functional analysis and are closely related to C^*-algebras and von Neumann algebras. The notion of Baer rings was first introduced by Kaplansky in 1955, and later in 1960, he extended it to Rickart rings. In 1967, he added quasi-Baer rings to the list. Rickart rings (also called p.p. rings) and (quasi-)Baer rings are significant for providing a wide range of idempotents, which in turn contributes to the structure theory for rings. There have been several research papers exploring the characteristics of Baer, quasi-Baer, and Rickart rings ([2, 4, 5, 6, 7]). These rings have specific characteristics, such as Baer rings having an idempotent that generates the right annihilator of any nonempty subset. The concept of Baer rings has since been extended to modules ([18]), with a Baer module having the property that the right annihilator of any nonempty subset of the module is a ds. Similarly, Rickart modules have the property that the right annihilator of any single element of the endomorphism ring is a ds. Dual Baer modules have also been studied ([12]), and there is interest in exploring the dual notion of the Rickart property for modules.

The study on modules V, where kernel of f is a ds of V for any $f \in S = End_Q(V)$ (known as Rickart modules), was conducted in [14]. In [14], the authors discuss Rickart modules and provide some characterizations of them. They also explore the properties of Rickart modules and demonstrate that any ds of a Rickart module will also possess the same property. Additionally, they prove that each Rickart module is $\mathcal{K}$-nonsingular and has the Summand Intersection Property (SIP). The authors establish that the class of rings Q in which every right Q-module is Rickart is precisely that of the semisimple artinian rings. Similarly, the class of rings Q in which each free right Q-module is Rickart is exactly that of the right hereditary rings. They also provide examples and results that distinguish the concept of a Rickart module from that of a Baer module.

A module V is dual Rickart provided the image of every endomorphism of V is a direct summand of V ([15]). The property of being a dual Rickart module is inherited by ds but not by direct sums. It is also shown that dual Rickart modules satisfy the Summand Sum Property (SSP). The concept of relative dual Rickart condition is introduced to characterize a dual Rickart module. In [15], it is shown that a ring over which each right module is dual Rickart is semisimple artinian, while a ring where every finitely generated free module is dual Rickart is a von Neumann regular ring (a ring Q is called von Neumann regular if for each $a \in Q$, there is $x \in Q$ such that $a = axa$). Some generalizations of dual Rickart modules have been introduced and studied, recently ([1, 16, 17]).

Over the past ten years, there has been a growing interest in using isomorphisms to study modules. A recent paper by Behboodi et al. proposed a new approach to semisimple modules using the concept of virtually (semi)simple modules and rings ([3]). Chaturvedi et al. ([8]) introduced iso-retractable modules as a extension of retractable modules. The present article aims to build on their works by examining Rickart modules and dual Rickart modules.

This paper has two main sections. In Section 2, we introduce the concept of virtually Rickart modules and provide examples of such modules. We say a module V is virtually Rickart if the kernel of each endomorphism of V is isomorphic to a ds of V. Some general properties of such modules are studied and investigated.

Section 3 is devoted to explore the dual notion of virtually Rickart modules, which we call virtually dual Rickart modules. We prove that every virtually dual Rickart module satisfies $GSSP$. Also, it is proven that over a ring Q each projective right Q-module is virtually dual Rickart if and only if Q is a right hereditary ring.

2 virtually Rickart modules

Recently, researchers have become interested in studying module theory concepts via isomorphisms. This is a strong motivation for the authors to focus on some known concepts such as Rickart modules. This section explores the general properties of virtually Rickart modules and their relationship with other known module classes. It also notes that virtually Rickart modules possess $GSIP$. Additionally, we prove that rings where all modules are virtually Rickart are semisimple rings.

Definition 2.1. Let V be a module. We call V virtually Rickart, in case the kernel of each endomorphism of V is isomorphic to a ds of V.

Modules V with each submodule isomorphic to V are called virtually simple in [3]. The $\mathbb{Z}$-module $\mathbb{Z}$ is virtually simple while it is not simple (semisimple).

Recall from [3] that, a module V is said to be *virtually semisimple* provided each submodule of V is isomorphic to a ds of V. Note that each virtually (semi)simple module is virtually Rickart.

Clearly, every Rickart module is virtually Rickart. The converse may not hold.

Example 2.2. Set $V = \mathbb{Z} \oplus \mathbb{Z}_p$ as an $\mathbb{Z}$-module where p is a prime number. Suppose W is a proper submodule of V. Then W is either $a\mathbb{Z} \oplus 0$ or $a\mathbb{Z} \oplus \mathbb{Z}_p$ $(a \in \mathbb{Z})$. In this direction, W is isomorphic to a direct summand of W implying that V is virtually semisimple. In fact, W is virtually Rickart. Note also that by a similar endomorphism to one defined in [14, Example 2.5], we conclude that V is not a Rickart $\mathbb{Z}$-module.

Remark 2.3. Assume V is an indecomposable module. If V is virtually Rickart, then for each endomorphism $\eta : V \to V$, the kernel must be zero or $Ker\eta \cong V$. Note that if V is finite, therefore each endomorphism of V must be a monomorphism.

Definition 2.4. Let V be a module. Then V is said to satisfy T^* provided if $L \leq U \leq V$ and U is a ds of V with $L \cong D$ and D a ds of V, we have L is isomorphic to a ds of U.

It is clear that every semisimple module satisfies T^*.

Recall that a module V is (weak) duo, in case each (ds) submodule W of V is fully invariant (i.e. for each endomorphism η of V, $\eta(W)$ is contained in W).

Proposition 2.5. *Let V be a module such that for each $0 \neq L \leq U \leq V$, there exists an epimorphism from U to L. If V is weak duo, then V satisfies T^*.*

Proof. Let $L \leq U \leq V$, where U is a ds of V. Suppose $L \cong D$ and D is a ds of V. Set $\lambda : L \to D$ be the corresponding isomorphism. From assumption, there exists an epimorphism $f : U \to L$. Now, consider $\theta = j \circ \lambda \circ f \circ \pi_U$. So $\theta \in End_Q(V)$. Being V weak duo implies $\theta(U) \subseteq U$. Note that $\theta(U) = D$. Hence D is contained in U. Now, we are done. □

Proposition 2.6. *Let V be a virtually Rickart module. If V satisfies T^*, then each ds of V is virtually Rickart.*

Proof. Let V be a virtually Rickart module with T^*. Suppose that W is a ds of V with the decomposition $V = W \oplus W'$. Let $\eta : W \to W$ be an arbitrary endomorphism of W. So $h = j \circ \eta \circ \pi_W : V \to V$ is an endomorphism of V. Note that V is virtually Rickart, so that $Kerh$ is isomorphic to a ds D of V. It is not hard to check that $Kerh = Ker\eta \oplus W'$. Now, $Ker\eta$ is isomorphic to a ds of D and hence is isomorphic to a ds of V. Since V satisfies T^*, $Ker\eta$ is isomorphic to a ds of W. This completes the proof. □

Proposition 2.7. *Let $V = V_1 \oplus \ldots \oplus V_n$ be a weak duo module, where $V_i \leq V$. If V_i is virtually Rickart for $i = 1, \ldots, n$, then V is virtually Rickart.*

Proof. Let $f : V \to V$ be an endomorphism of V. Hence $f_i = f\mid_{V_i} : V_i \to V_i$ is an endomorphism of V_i for $i = 1, \ldots, n$. Being V_i virtually Rickart implies there exists a ds D_i of V_i such that $Kerf_i \cong D_i$, for $i = 1, \ldots, n$. As V_i is a fully invariant submodule of V, we have $Kerf = Kerf_1 \oplus \ldots \oplus Kerf_n \cong D_1 \oplus \ldots \oplus D_n$. Note that $D_1 \oplus \ldots \oplus D_n$ is a ds of V. □

The following introduces a condition to ensure us a virtually Rickart module is Rickart.

Theorem 2.8. *Assume V is a virtually Rickart module. Then V is Rickart if and only if for each $\eta : V \to V$ and every isomorphism $\sigma : Ker\eta \to W$ where W is a ds of V, σ can be extended to an endomorphism θ of V.*

Proof. Suppose that V is a Rickart module. Consider an endomorphism $\eta : V \to V$ and an isomorphism $\sigma : Ker\eta \to W$ where W is a ds of V. Being V Rickart implies $Ker\eta \oplus L = V$ where $L \leq V$. It is not hard to verify that $j \circ \sigma \circ \pi_{Ker\eta}$ is an extension of σ. Here $j : W \to V$ is the inclusion and $\pi_{Ker\eta} : V \to Ker\eta$ is the projection on $Ker\eta$.

For the converse, suppose that $\eta : V \to V$ is an endomorphism of V. As V is virtually Rickart, hence there exists an isomorphism $\sigma : Ker\eta \to W$ where W is a ds of V. By assumption σ can be extended to $\theta : V \to V$. Now consider $\xi = \pi_W \circ \theta$, where $\pi_W : V \to W$ is the projection of V on W. Note that for each $t \in Ker\eta$, we have $\xi(t) = \pi_W \circ \theta(t) = \pi_W(\theta(t)) = \pi_W(\sigma(t)) = \sigma(t)$. We next show that $V = Ker\xi \oplus Ker\eta$. To verify, choose $m \in V$ as an arbitrary element of V. Then $\xi(m) = (\pi_W \circ \theta)(m) = \sigma(t) = \xi(t)$ (note that $(\pi_W \circ \theta)(m) \in W$ and σ is onto, so there is an element t of $Ker\eta$ such that $\sigma(t) = (\pi_W \circ \theta)(m)$). It follows that $m - t \in Ker\xi$. This means that $V = Ker\xi + Ker\eta$. Choose $y \in Ker\xi \cap Ker\eta$. Hence $\sigma(y) = \xi(y) = 0$, as $y \in Ker\xi$ and also $y \in Ker\eta$. As σ is one-to-one, $y = 0$. This completes the proof. □

Definition 2.9. ([20]) Let V be a module. Then V satisfies $GSIP$ (generalized SIP) provided the intersection of each two ds of V is isomorphic to a ds of V.

Theorem 2.10. *([20, Theorem 2.3]) An Q-module V satisfies $GSIP$ if and only if for every decomposition $V = A \oplus B$ and each Q-homomorphism $\eta : A \to B$, the kernel of η is isomorphic to a ds of V.*

From [14, Proposition 2.16], each Rickart module satisfies SIP. It is natural to verify an analogue for virtually Rickart modules.

Proposition 2.11. *Each virtually Rickart module satisfies $GSIP$.*

Proof. Suppose that $V = A \oplus B$ is a decomposition of V. Assume $\eta : A \to B$ is an Q-homomorphism. Therefore $h = j \circ \eta \circ \pi_A : V \to V$ is an endomorphism of V where $j : B \to V$ is the inclusion and $\pi_A : V \to A$ is the projection of V on A. Now consider the kernel of h. It is not hard to check that $Kerh =$

$Ker(\eta \circ \pi_A)$. As V is virtually Rickart, we can say $Kerh$ is isomorphic to a ds of V. Next, we show that $Kerh = Ker\eta \oplus B$. To prove it, suppose that $x \in Kerh$. So $x = a + b$ where $a \in A$ and $b \in B$. As $\eta \circ \pi_A(x) = \eta(a) = 0$, we conclude that $a \in Ker\eta$. It follows that $Kerh \subseteq Ker\eta \oplus B$. Now consider $x = a+b \in Ker\eta \oplus B$. Now, $\eta \circ \pi_A(x) = \eta(a) = 0$. Hence, $Kerh = Ker\eta \oplus B$. By assumption, $Ker\eta \oplus B$ is isomorphic to a ds D of V. Now, it is easy to verify that $Ker\eta$ is isomorphic to a ds of D and hence is isomorphic to a ds of V. □

Remark 2.12. The converse of Proposition 2.11 may not hold. Consider the $\mathbb{Z}$-module $V = \mathbb{Z}_{p^n}$ where $n > 1$. Verify that the kernel of $\eta : V \to V$ with $\eta(x) = px, \forall x \in V$ is $pV \neq 0$. Therefore, V is not virtually Rickart. Note that V is indecomposable, so satisfies $GSIP$.

Remark 2.13. Note that if the injective envelope of a module V over a commutative Noetherian ring is (virtually) Rickart, then $E(V)$ is a Baer module. This is a consequence of [14, Proposition 2.15] and [20, Corollary 2.10].

Definition 2.14. Let V_1 and V_2 be two Q-modules. In this direction, we say V_1 is virtually V_2-Rickart provided for each Q-homomorphism $\eta : V_1 \to V_2$, the kernel of η is isomorphic to a ds of V_1.

Note that if V_1 is V_2-Rickart then V_1 is virtually V_2-Rickart. Also, for a semisimple module V, every submodule W of V is virtually Rickart relative to each submodule U of V.

Proposition 2.15. *Assume that V is a module and $V_i, V_j \leq V$. If V satisfies $GSIP$ and $V_i \oplus V_j$ is a ds of V for $i \neq j$, then for each $\xi : V_i \to V_j$, the kernel of ξ is isomorphic to a ds of V.*

Proof. Let $\eta : V_i \to V_j$ be an arbitrary Q-homomorphism. We define $W = \{m_i + \eta(m_i) \mid m_i \in V_i\}$. Consider $m_i + \eta(m_i) = m_j \in W \cap V_j$. So $m_i = m_j - \eta(m_i)$. It follows that $m_i = 0$, as $V_i \cap V_j = 0$. So $W \cap V_j = 0$. It is clear that $W \oplus V_j \subseteq V_i \oplus V_j$. For the other side, assume $m_i + m_j \in V_i \oplus V_j$. Hence $m_i + m_j = m_i + \eta(m_i) + m_j - \eta(m_i) \in W \oplus V_j$. It follows that $W \oplus V_j = V_i \oplus V_j$. Therefore, W is a ds of V. Satisfying V in $GSIP$ implies that $W \cap V_i$ is isomorphic to a ds of V. This means $Ker\eta$ is isomorphic to a ds of V. □

Corollary 2.16. *If $V \oplus V$ satisfies $GSIP$, then the kernel of each endomorphism of V is isomorphic to a ds of $V \oplus V$.*

Proposition 2.17. *Let V be a module. Consider the following:*

(1) *$E(V)$ is virtually Rickart;*

(2) *For each two submodules A, B of V, $E(A \cap B) = E(A) \cap E(B)$.*

Then (1) $\Rightarrow$ (2). The converse holds in case $E(V) \oplus E(V)$ satisfies SIP.

Proof. (1) $\Rightarrow$ (2) Follows from [20, Corollary 2.8] and Proposition 2.11.
(2) $\Rightarrow$ (1) If (2) holds, then by [14, Corollary 2.23], $E(V)$ is Rickart. □

Theorem 2.18. *Let V be a right Q-module. Then the following are equivalent:*
(1) *V is virtually Rickart;*
(2) *For each submodule W of V, each ds L of V and each Q-homomorphism $\xi : L \to W$, the kernel of ξ is isomorphic to a ds of V;*
(3) *For each ds W, L of V and each Q-homomorphism $\eta : V \to W$, the kernel of $\eta\mid_L$ is isomorphic to a ds of V.*

Proof. (1) $\Rightarrow$ (2) Suppose that $W \leq V$ and L is a ds of V. Let $\xi : L \to W$ be an Q-homomorphism. So $h = \xi \circ \pi_L : V \to W$ is a homomorphism where $\pi_L : V \to L$ is the projection on L. Consider $Ker(\xi \circ \pi_L)$. It can be verified that $Kerh = Ker\xi \oplus L'$ where $L \oplus L' = V$. By (1), we can say $Kerh \cong D$ where D is a ds of V. So the kernel of ξ will be isomorphic to a ds of V, as required.
(2) $\Rightarrow$ (3) Obvious.
(3) $\Rightarrow$ (1) Take W and L to be V. □

The rings with all modules virtually Rickart are precisely semisimple rings, as the following shows.

Theorem 2.19. *Let Q be a ring. Then the following statements are equivalent:*
(1) *Every right Q-module is virtually Rickart;*
(2) *Each injective right Q-module is virtually Rickart;*
(3) *Q is semisimple;*
(4) *Each right Q-module satisfies $GSIP$.*

Proof. (1) $\Rightarrow$ (2) Obvious.
(2) $\Rightarrow$ (3) Since for an injective right Q-module, being virtually Rickart coincide with being Rickart, so by [14, Theorem 2.25] Q is semisimple.
(3) $\Rightarrow$ (1) Clear.
(3) $\Leftrightarrow$ (4) It follows from [20, Corollary 2.9].
□

3 virtually dual Rickart modules

After introducing virtually Rickart modules, it is expected we define an analogue for dual Rickart modules. By the way, we call a module V virtually dual Rickart if the image of any endomorphism of V is isomorphic to a ds of V. We will see that a direct summand of a virtually dual Rickart module does not inherit the property. We shall provide a condition to fix this issue. Every virtually dual Rickart module has been proved to satisfy $GSSP$. If V is a uniserial module and $V \oplus V$ is virtually dual Rickart, then it is shown that V is virtually dual Rickart.

Let's start with the key definition.

Definition 3.1. Let Q be a ring and V be an Q-module. Then we call V virtually dual Rickart in case for each endomorphism η of V, the submodule $Im\eta$ is isomorphic to a ds of V.

The following is a direct consequence of the definitions.

Proposition 3.2. *Assume V is a module. Then the following statements are equivalent:*

(1) *V is virtually dual Rickart;*

(2) *For each $\eta \in End_Q(V)$, there is a decomposition $V = D \oplus D'$ where $Im\eta \cong D$ such that the sequence $0 \to D \to V \to D'$ splits where $j : D \to V$ is the inclusion and $\pi : V \to D'$ is the projection of V on D'.*

Recall that a module V satisfies C_2 condition if for $W \leq V$, we have $W \cong D \leq^{\oplus} V$, then W itself is a ds of V. Each quasi-injective module satisfies C_2. Also V satisfies D_2 provided $V/W \cong D \leq^{\oplus} V$ implies W is a ds of V. It is not hard to verify that any quasi-projective module satisfies D_2.

By the definition, we can say that each dual Rickart module is virtually dual Rickart while the converse may not hold. Although, if V satisfies C_2 and V is virtually dual Rickart then V is dual Rickart.

Remark 3.3. For a quasi-injective module V two concepts virtually dual Rickart and dual Rickart, coincide. This follows from the fact that each quasi-injective module is injective relative to its each ds.

Recall from [11] that a module V is called anti-Hopfian in case V is not simple and for each proper submodule W of V we have $V \cong V/W$. Note that these modules were called co-isosimple in [10].

Let V be a module. Then V is called noncosingular (cosingular) if $\overline{Z}(V) = \bigcap\{Ker\eta \mid \eta : V \to Y, Y \in \mathcal{S}\}$ is equal to V ($\{0\}$). Here $\mathcal{S}$ denotes the class of all small right Q-modules ([19]). Each homomorphic image of a noncosingular module is noncosingular. Note that if a module V is both cosingular and noncosingular, then $V = \{0\}$.

We provide some examples of virtually dual Rickart modules. In fact, we include some virtually dual Rickart modules that are not dual Rickart.

Example 3.4. (1) Each virtually simple module is obviously virtually dual Rickart. Although, there exist virtually simple modules which are not dual Rickart. For instance, the $\mathbb{Z}$-module $\mathbb{Z}$ is not dual Rickart.

(2) Every anti-Hopfian (co-isosimple) module is virtually dual Rickart. Suppose $\eta : V \to V$ is a nonzero endomorphism of V. Then $Imf \cong V/Kerf$. As V is anti-Hopfian, so $Imf \cong V$. Hence, V is virtually dual Rickart. Note that the converse may not hold. Assume that Q is a semisimple ring. Therefore each right Q-module is virtually dual Rickart as well as dual Rickart. Note that by [11, Remark 6], an Artinian ring does not admit an anti-Hopfian module. Also, if Q is a commutative ring, then each anti-Hopfian module must be quasi-injective ([11, Theorem 10]) while over such rings there are virtually dual Rickart non-quasi-injective modules (for example $\mathbb{Z}_{\mathbb{Z}}$).

(3) Each Rickart module V is virtually dual Rickart. The converse holds if V satisfies D_2.

(4) Each noncosingular hollow module is virtually dual Rickart. This follows from the fact that for each nonzero endomorphism $f \in End_Q(V)$, the image of f is either small in V or equal to V.

Example 3.5. (1) Any submodule of the $\mathbb{Z}$-module $V = \mathbb{Q}$ is a Rickart module ([13, Example 2.2(iv)]). So that any submodule of V is both virtually dual Rickart and virtually Rickart.

(2) Let Q be a Dedekind domain and V be a direct sum of finitely generated torsion- free Q-modules of rank one. So each submodule of V is a Rickart module ([13, Example 2.2(v)]). Therefore, such modules are both virtually dual Rickart and virtually Rickart.

(3) Each finitely generated free (projective) module over a right semihereditary ring is a Rickart module, so that is both virtually dual Rickart and virtually Rickart ([13, Example 2.2(vi)]).

Assume V is an indecomposable module. Note that the converse of Example 3.4(1), holds in case each proper submodule of V is an image of an endomorphism of V.

It is clear that, each virtually semisimple module is virtually dual Rickart.

Note that by $Soc_r(Q)$ for a ring Q, we mean the socle of the right Q-module Q. The left socle is defined in a same manner.

Example 3.6. ([3, Example 2.7])

Assume $Q = \mathbb{Z}_4$ and $W = Soc(Q) = \{0, 2\}$. Set $U = \bigoplus_{i=1}^{\infty} W$. As W is semisimple, U is a semisimple Q-module. Consider the Q-module $V = Q \oplus U$. Then $Soc(V) = Soc(Q) \oplus Soc(U) = W \oplus U \cong U$. Suppose that $f : V \to V$ is an endomorphism of V. We consider two cases. First, suppose that $Imf \subseteq Soc(V)$. It follows that Imf is a ds of $Soc(V)$. In fact, Imf is isomorphic to a ds of U and hence is isomorphic to a ds of V. Otherwise, $Imf \nsubseteq Soc(V)$. So, $Imf \cong Q \oplus D$, where D is a ds of U. In this case, Imf is also isomorphic to a ds of V. Hence, V is virtually dual Rickart. Note that, Q as a ds of V can not be virtually dual Rickart. Consider, the endomorphism $g : Q \to Q$

defined as $g(x) = 2x$ for each $x \in Q$. It is clear that Img is not isomorphic to a ds of Q.

Via Example 3.6, a ds of a virtually dual Rickart module need not be virtually dual Rickart. It is expected to provide a condition to make it possible.

Proposition 3.7. *Let V be a virtually dual Rickart module satisfying T^*. Then each ds of V is virtually dual Rickart.*

Proof. Let V be virtually dual Rickart, $V = W \oplus W'$ and $\eta \in End_Q(W)$. So $h = j \circ \eta \circ \pi_W \in End_Q(V)$. Note also that $Imh = Im\eta$. By assumption $Imh \cong D \leq^{\oplus} V$. As $Imh \leq W \leq^{\oplus} V$ and V satisfies T^*, then Imh is isomorphic to a ds of W, as required. □

Remark 3.8. It is easy to check that each virtually dual Rickart module with D_2 is Rickart. So, we can say that each ds of a virtually dual Rickart module with D_2 is virtually dual Rickart. This follows from the fact that any ds of a Rickart module inherits the property.

A module V satisfies $GSSP$ (generalized SSP) provided the sum of each two ds of V is isomorphic to a ds of V ([21]). Note that each SSP module satisfies $GSSP$. Also every dual Rickart module satisfies SSP (see [15, Proposition 2.11]). We next try to present an analogue for the virtually dual Rickart modules.

Proposition 3.9. *Each virtually dual Rickart module satisfies $GSSP$.*

Proof. Assume that $V = A \oplus B$ is an arbitrary decomposition of V and $\eta : A \to B$ is an Q-homomorphism. So $h = j \circ \eta \circ \pi_A : V \to V$ is an endomorphism of V where $j : B \to V$ is the inclusion and $\pi_A : V \to A$ is the projection of V on A. As $Imh = Im\eta$, so $Im\eta$ is isomorphic to a ds U of V. Now, the result follows from [21, Proposition 2.6]. □

We next provide a condition to ensure us a virtually dual Rickart module is dual Rickart.

Theorem 3.10. *Let V be a virtually dual Rickart module. Then V is dual Rickart if and only if for each endomorphism η of V and each isomorphism $\sigma : Im\eta \to D$ where D is a ds of V, there exists $\theta : V \to V$ such that $\theta |_{Im\eta} = \sigma$.*

Proof. Assume V is dual Rickart and $\eta : V \to V$ is an endomorphism of V. Let $\sigma : Im\eta \to D$ be an isomorphism where D is a ds of V. Being V dual Rickart, there is a direct decomposition $V = Im\eta \oplus L$, for $L \leq V$. Now consider $\theta = j \circ \sigma \circ \pi_{Im\eta}$ where $j : D \to V$ is the inclusion map and $\pi_{Im\eta} : V \to Im\eta$ is the projection on $Im\eta$. Then, θ is the required Q-homomorphism.

Conversely, suppose that V satisfies stated property and $\eta : V \to V$ is an endomorphism. As V is virtually dual Rickart, there is a direct summand D of V and an isomorphism $\sigma : Im\eta \to D$. By assumption, σ can be extended to $\theta : V \to V$. Set $\xi = \pi_D \circ \theta$. Let $y \in Im\eta$ be arbitrary. Now, $\xi(y) = (\pi_D \circ \theta)(y) = \pi_D(\sigma(y)) = \sigma(y)$. Consider an arbitrary $m \in V$. So $\xi(m) \in D$. As σ is onto, there exists a $y \in Im\eta$ such that $\xi(m) = \sigma(y)$. Note also that $\sigma(y) = \theta(y)$. It follows that $m - y \in Ker\xi$. In fact, we prove $V = Im\eta + Ker\xi$. For each $x \in Im\eta \cap Ker\xi$, we have $\xi(x) = 0 = \sigma(x)$. As σ is one-to-one, $x = 0$. Therefore, $V = Im\eta \oplus Ker\xi$, as required. □

Definition 3.11. Let V_1 and V_2 be two Q-modules. We say V_1 is virtually V_2-dual Rickart provided for each Q-homomorphism $\eta : V_1 \to V_2$, the image of η is isomorphic to a ds of V_2.

By [15], a module V_1 is V_2-dual Rickart if the image of any Q-homomorphism from V_1 to V_2 is a ds of V_2. So, it is obvious that relatively dual Rickartness implies relatively virtually dual Rickartness.

Example 3.12. Consider the $\mathbb{Z}$-modules $V_1 = \mathbb{Z}_p$ and $V_2 = \mathbb{Z}_{p^\infty}$. Note that both of V_1 and V_2 are virtually dual Rickart while V_1 is not virtually dual Rickart relative to V_2 as the image of each nonzero $\mathbb{Z}$-homomorphisms from V_1 to V_2 is finite and can not be equal to V_2. Note also that, a non-virtually dual Rickart module can be virtually dual Rickart relative to a module. Consider the $\mathbb{Z}$-modules $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_t$ where t is a square-free natural number.

Proposition 3.13. *Let V be a virtually dual Rickart module relative to W. If $W' \leq W$ and $W' \cong V' \leq^{\oplus} V$, then W' is isomorphic to a ds of W.*

Proof. Suppose that $W' \leq W$ and $W' \cong V' \leq^{\oplus} V$. Assume $\xi : V' \to W'$ is the corresponding isomorphism. Hence $h = j \circ \xi \circ \pi_{V'} : V \to W$ is an Q-homomorphism where $\pi_{V'} : V \to V'$ is the projection of V on V' and $j : W' \to W$ is the inclusion. As V is virtually W-dual Rickart, Imh is isomorphic to a ds of W, namely U. Notice that $Imh = W'$. This completes the proof. □

Proposition 3.14. *([21, Proposition 2.6]) Let V be a module and $V_i, V_j \leq V$. If V satisfies $GSSP$ and $V_i \oplus V_j$ is a ds of V for $i \neq j$, then the image of each element of $Hom_Q(V_i, V_j)$ is isomorphic to a ds of V.*

Corollary 3.15. *Let V be a module such that $V \oplus V$ satisfies $GSSP$. Then the image of each endomorphism of V, is isomorphic to a ds of $V \oplus V$.*

Recall that a module V is called uniserial if the lattice of submodules of V is linearly ordered by inclusion.

Proposition 3.16. *Let V be a uniserial module. If $V \oplus V$ is virtually dual Rickart, then V is virtually dual Rickart.*

Proof. Assume $\eta : V \to V$ is an endomorphism of V. So $\lambda : V \oplus V \to V \oplus V$ by $\lambda(x, y) = (\eta(x), 0)$ is an endomorphism of $V \oplus V$ such that $Im\lambda \cong Im\eta$. As $V \oplus V$ is virtually dual Rickart, $Im\eta$ is isomorphic to a ds of $V \oplus V$. Now $Im\eta$ will be isomorphic to a ds of V from [9, Proposition 2.4]. This completes the proof. □

Let V be a module. Then V is said to be *(co-)Hopfian* in case each (epimorphism) monomorphism $\eta : V \to V$ is an isomorphism.

Remark 3.17. Let V be an indecomposable module. Therefore V is virtually dual Rickart if and only if the image of any nonzero endomorphism of V is isomorphic to V. In other words, if V is a finite indecomposable module, then V is virtually dual Rickart if and only if any nonzero endomorphism of V is isomorphism. In this case, V is Hopfian and also co-Hopfian.

Proposition 3.18. *Let $V = V_1 \oplus \ldots \oplus V_n$ be a weak duo module, where $V_i \leq V$. If V_i is virtually dual Rickart for $i = 1, \ldots, n$, then V is virtually dual Rickart.*

Proof. Assume $f : V \to V$ is an endomorphism of V. Hence $f_i = f |_{V_i}: V_i \to V_i$ is an endomorphism of V_i for $i = 1, \ldots, n$. Being V_i virtually dual Rickart implies there exists a ds D_i of V_i such that $Imf_i \cong D_i$, for $i = 1, \ldots, n$. As V_i is a fully invariant submodule of V, we have $Imf = Imf_1 \oplus \ldots \oplus Imf_n \cong D_1 \oplus \ldots \oplus D_n$. Note that $D_1 \oplus \ldots \oplus D_n$ is a ds of V. □

The following characterizes modules V, for which their injective envelopes are (virtually) dual Rickart.

Proposition 3.19. *Let V be an Q-module. Consider the following:*

(1) *$E(V)$ is dual Rickart;*

(2) *$E(V)$ is virtually dual Rickart;*

(3) *For each two submodules A and B of V, we have $E(A + B) = E(A) + E(B)$.*

Then (1) ⇔ (2) ⇒ (3).

They are equivalent if $E(V) \oplus E(V)$ satisfies SSP.

Proof. As $E(V)$ is injective (1) ⇔ (2) holds.

(2) ⇒ (3) Suppose that $E(V)$ is virtually dual Rickart. Then, $E(V)$ is dual Rickart, so that $E(V)$ satisfies SSP. Now, the result follows from [21, Theorem 2.10].

(3) ⇒ (1) Let (3) holds. Then by [15, Corollary 2.17], $E(V)$ is dual Rickart. □

Theorem 3.20. *Assume Q is a ring. Consider the following statements:*
(1) *Each right Q-module is virtually dual Rickart;*
(2) *Each projective right Q-module is virtually dual Rickart;*
(3) *Q is right hereditary;*
(4) *Each injective right Q-module is virtually dual Rickart.*
Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

Proof. (1) $\Rightarrow$ (2) Obvious.
(2) $\Rightarrow$ (3) Let V be a an arbitrary projective right Q-module. By assumption, V is virtually dual Rickart. So V is Rickart. Now, by [14, Theorem 2.26] Q is a hereditary ring.
(3) $\Leftrightarrow$ (4) It follows from [15, Theorem 2.29].
□

It remains open to characterize a ring Q such that each right Q-module is virtually dual Rickart. They might be semisimple ones.

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