DOI: 10.2478/auom-2024-0009
An. Şt. Univ. Ovidius Constanţa **Vol. 32(1),2024, 155–182**

Several new aspects on q-Horn and related triple functions in the spirit of Karlsson

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Abstract

We first introduce a notation for multiple ($n \geq 3$) q-hypergeometric functions, where negative values of summation indices are allowed. Then we extend the notation for q-Horn functions to include tilde values corresponding to powers of 2. Karlssons reduction formulas are correspondingly q-deformed by using these notations. A formula for sums of inverse q-shifted factorials is used to find further formulas. The second part of the paper is devoted to convergence aspects for q-Horn functions and 'abnormal' q-Horn functions. It turns out that some simple estimates for convergence can be made in the q-case, these are then supplemented with tables of numerical values. It is shown that the convergence regions are significantly increased in the q-case, and we compare with convergence regions in the ordinary case.

1 Introduction

This paper is the second in a series of two papers on q-Horn functions, the first one was [6], where several q-Horn functions were first defined.

The purpose of this paper is also to q-deform the very interesting Karlsson paper [11], with a very useful notation for triple hypergeometric functions. With our notation, the q-analogues are very similar to the original.

Multiple hypergeometric functions have many applications as was outlined in [7]. There are even hypergeometric functions over finite fields. Since Horn

Key Words: q-Horn functions, convergence region, notation
2010 Mathematics Subject Classification: Primary 33D70; Secondary 33C65
Received: 09.02.2023
Accepted: 18.04.2023

functions are solutions to systems of partial differential equations [6], this subject has some practical importance. Similarly, q-Horn functions are solutions to systems of q-difference equations [6]. However, our systems of operator q-difference equations for double functions are totally different from the systems of partial differential equations in [2, p. 233 ff.].

2 Definitions

We now repeat some notation from [3].

Definition 2.1. *Let $\delta > 0$ be an arbitrary small number. We will always use the following branch of the logarithm: $-\pi + \delta < \mathrm{Im}\,(\log q) \leq \pi + \delta$. This defines a simply connected space in the complex plane.*

The power function is defined by

$$q^a \equiv e^{a \log(q)}. \tag{2.1}$$

The following notation is often used when we have long exponents.

$$\mathrm{QE}(x) \equiv q^x. \tag{2.2}$$

The q-shifted factorial [3] is defined by

$$\langle a; q\rangle_n \equiv \prod_{m=0}^{n-1} (1 - q^{a+m}). \tag{2.3}$$

With this notation, q-hypergeometric function- and hypergeometric function equations become very similar.

Sometimes we also use

$$(a; q)_n \equiv \prod_{m=0}^{n-1} (1 - aq^m). \tag{2.4}$$

There are three other types of q-shifted factorials [3]: in the equations (2.8) to (2.13) we assume that $(m, l) = 1$.

Definition 2.2. *In the following, $\dfrac{\mathbb{C}}{\mathbb{Z}}$ will denote the space of complex numbers* $\mathrm{mod}\frac{2\pi i}{\log q}$. *This is isomorphic to the cylinder $\mathbb{R} \times e^{2\pi i \theta}$, $\theta \in \mathbb{R}$. The operator*

$$\widetilde{}: \frac{\mathbb{C}}{\mathbb{Z}} \to \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by the 2-torsion

$$a \mapsto a + \frac{\pi i}{\log q}. \tag{2.5}$$

By (2.5) it follows that

$$\widetilde{\langle a; q\rangle_n} = \prod_{m=0}^{n-1}(1 + q^{a+m}), \tag{2.6}$$

where this time the tilde denotes an involution which changes a minus sign to a plus sign in all the n factors of $\langle a; q\rangle_n$*. Furthermore we define*

$$\widetilde{\langle a; q\rangle_n} \equiv \langle \widetilde{a}; q\rangle_n. \tag{2.7}$$

The generalized tilde operator

$$\widetilde{\tfrac{m}{l}} : \frac{\mathbb{C}}{\mathbb{Z}} \to \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{2\pi i m}{l \log q}. \tag{2.8}$$

We also need another generalization of the tilde operator.

$$_{k}\widetilde{\langle a; q\rangle_n} \equiv \prod_{m=0}^{n-1}(\sum_{i=0}^{k-1} q^{i(a+m)}). \tag{2.9}$$

Formula (2.9) is used in (2.14).

The following, simple congruence rules [3] follow from (2.8).

Theorem 2.1.

$$\widetilde{\tfrac{m}{l}}a \pm b \equiv \widetilde{\tfrac{m}{l}(a \pm b)} \ \left(\mathrm{mod}\ \frac{2\pi i}{\log q}\right), \tag{2.10}$$

$$\sum_{k=1}^{n} \widetilde{\tfrac{1}{n} \pm a_k} \equiv \sum_{k=1}^{n} \pm a_k \ \left(\mathrm{mod}\ \frac{2\pi i}{\log q}\right), \tag{2.11}$$

$$\frac{m}{l} \times \widetilde{a} \equiv \widetilde{\tfrac{m}{2l}\frac{am}{l}} \ \left(\mathrm{mod}\ \frac{2\pi i}{\log q}\right), \tag{2.12}$$

$$\mathrm{QE}(\widetilde{\tfrac{m}{l}a}) = \mathrm{QE}(a)e^{\frac{2\pi i m}{l}}, \tag{2.13}$$

where the second equation is a consequence of the fact that we work $\mathrm{mod}\frac{2\pi i}{\log q}$.

Definition 2.3.

$$\langle\lambda;q\rangle_{kn} \equiv \langle\triangle(q;k;\lambda);q\rangle_n \equiv \prod_{m=0}^{k-1}\langle\frac{\lambda+m}{k};q\rangle_n \times_k \langle\widetilde{\frac{\lambda+m}{k}};q\rangle_n. \tag{2.14}$$

We also use the notation $\triangle(q;k;\lambda)$ as a parameter in q-hypergeometric functions.

If λ is a vector, we mean the corresponding product of vector elements. If λ is replaced by a sequence of numbers, separated by commas, we mean the corresponding product, as in the case of q-factorials.

The last factor in (2.14) corresponds to k^{nk}.

The following definition, like in the one-varible case [8], allows easy limits for parameters to $\pm\infty$.

Definition 2.4. *[3, p. 367 f]. The vectors*

$$(a),(b),(g_i),(h_i),(a'),(b'),(g'_i),(h'_i)$$

have dimensions

$$A,B,G_i,H_i,A',B',G'_i,H'_i.$$

Let

$$1+B+B'+H_i+H'_i-A-A'-G_i-G'_i \geq 0, i=1,\ldots,n.$$

Then the generalized q-Kampé de Fériet function is defined by

$$\Phi^{A+A':G_1+G'_1;\ldots;G_n+G'_n}_{B+B':H_1+H'_1;\ldots;H_n+H'_n}\left[\begin{array}{c}(\hat{a}):(\hat{g_1});\ldots;(\hat{g_n})\\(\hat{b}):(\hat{h_1});\ldots;(\hat{h_n})\end{array}|\vec{q};\vec{x}||\begin{array}{c}(a'):(g'_1);\ldots;(g'_n)\\(b'):(h'_1);\ldots;(h'_n)\end{array}\right]\equiv$$

$$\sum_{\vec{m}}\frac{\langle(\hat{a});q_0\rangle_m(a')(q_0,m)\prod_{j=1}^n(\langle(\hat{g_j});q_j\rangle_{m_j}((g'_j)(q_j,m_j)x_j^{m_j})}{\langle(\hat{b});q_0\rangle_m(b')(q_0,m)\prod_{j=1}^n(\langle(\hat{h_j});q_j\rangle_{m_j}(h'_j)(q_j,m_j)\langle1;q_j\rangle_{m_j})}\times$$

$$(-1)^{\sum_{j=1}^n m_j(1+H_j+H'_j-G_j-G'_j+B+B'-A-A')}\times$$

$$\mathrm{QE}\left((B+B'-A-A')\binom{m}{2},q_0\right)\prod_{j=1}^n\mathrm{QE}\left((1+H_j+H'_j-G_j-G'_j)\binom{m_j}{2},q_j\right), \tag{2.15}$$

where

$$\widehat{a}\equiv a\vee\widetilde{a}\vee\widetilde{\tfrac{m}{l}}a\vee_k\widetilde{a}\vee\triangle(q;l;\lambda). \tag{2.16}$$

It is assumed that there are no zero factors in the denominator. We assume that $(a')(q_0,m),(g'_j)(q_j,m_j),(b')(q_0,m),(h'_j)(q_j,m_j)$ contain factors of the form

$\langle a(\hat{k}); q\rangle_k$, $(s; q)_k, (s(k); q)_k$ *or* QE$\,(f(\vec{m}))$.
The numbers before colon denote the number of q-shifted factorials with index m in numerator and denominator. The numbers after colon denote the number of q-shifted factorials with index m_i in numerator and denominator. The numbers after semicolon denote the number of q-shifted factorials with index n in numerator and denominator. Every ∞ corresponds to multiplication with 1.

In our previous definition of q-Kampé de Fériet functions [3, p. 368], only sums of summation indices and summation indices were allowed for q-Pochhammer symbols. In order to define more general triple functions, Karlsson [11] introduced the notation

$$\mathrm{F}\left[\begin{array}{c} \cdot;\cdot;\cdot \\ \cdot;\cdot;\cdot \end{array}\middle| x;y;z\right] \tag{2.17}$$

with nine blank spaces for the parameters. Each parameter is now written in the space(s) corresponding to its Pochhammer symbol subscript. When a parameter occurs with sums of summation indices, it is written in both places above. When a minus summation index occurs, it is written below. Thus, only $\pm$ signs of summation indices are allowed. In case a parameter occurs as a product, like $(b)_n(b)_p$, the last parameter is written in brackets. For the q-case, a similar notation is used. The following reduction formula for triple q-functions [3, (10.174)] will be used.

Theorem 2.2. *If $\{C_{m,n}\}_{m,n=0}^{\infty}$ is a sequence of bounded complex numbers, then*

$$\begin{gathered}\sum_{m,n,p=0}^{\infty}\frac{C_{m,n+p}\langle b;q\rangle_n\langle b;q\rangle_p x_1^m x_2^n(-x_2)^p}{\langle 1;q\rangle_m\langle 1;q\rangle_n\langle 1;q\rangle_p}= \\ \sum_{m,k=0}^{\infty}\frac{C_{m,2k}\langle b,\widetilde{b};q\rangle_k x_1^m x_2^{2k}}{\langle 1;q\rangle_m\langle 1,\widetilde{1};q\rangle_k}.\end{gathered} \tag{2.18}$$

The Horn functions were studied in a period over fifty years from 1889 to 1939 by their inventor, who determined their convergence regions in terms of cartesian curves in the plane. The Horn method applies to multiple hypergeometric functions of any number of variables. It gives a parametric equation for the convergence region by a limit process for the rational function of the summation indices. This method is more blunt than the standard convergence test by Stirlings formula, which works well for Appell and Lauricella functions. In the whole paper, we give tables of numerical values for $q = .9$; sometimes the function values are large, but the series still converges. This is in accordance with previous investigations of q-Appell functions. We want to point out that similar convergence criteria for multiple q-series were given in [5]. In [3, p. 366] we pointed out that for the four so-called abnormal q-Appell functions, $x_2^{m_2}q^{k\binom{m_2}{2}}$ replaces $x_2^{m_2}$ in the general term. We will see that normal functions (in a broader sense) correspond to the Ward–AlSalam q-addition and abnormal functions, with $k = 1$, (in a broader sense) correspond to the Jackson–Hahn q-addition.

Hopefully, the abnormal functions will have slightly greater convergence region because of the extra q-power.

We define 10 q-analogues of two-variable Horn series.

Definition 2.5.

$$\mathrm{G}_1(a;b;b'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1+m_2}\langle b;q\rangle_{m_2-m_1}\langle b';q\rangle_{m_1-m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}. \tag{2.19}$$

$$\mathrm{G}_2(a;a';b,b'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1}\langle a';q\rangle_{m_2}\langle b;q\rangle_{m_2-m_1}\langle b';q\rangle_{m_1-m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}. \tag{2.20}$$

$$\mathrm{G}_3(a;a'|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{2m_2-m_1}\langle a';q\rangle_{2m_1-m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}. \tag{2.21}$$

$$\mathrm{H}_1(a;b;c;d|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1-m_2}\langle b;q\rangle_{m_1+m_2}\langle c;q\rangle_{m_2}}{\langle 1,d;q\rangle_{m_1}\langle 1;q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}. \tag{2.22}$$

$$\mathrm{H}_2(a;b;c;d;e|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1-m_2}\langle b;q\rangle_{m_1}\langle c,d;q\rangle_{m_2}}{\langle 1,e;q\rangle_{m_1}\langle 1;q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}. \tag{2.23}$$

$$\mathrm{H}_3(a;b;c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{2m_1+m_2}\langle b;q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}} x_1^{m_1}x_2^{m_2}. \tag{2.24}$$

$$\mathrm{H}_4(a;b;c;d|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{2m_1+m_2}\langle b;q\rangle_{m_2}}{\langle 1,c;q\rangle_{m_1}\langle 1,d;q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}. \tag{2.25}$$

$$\mathrm{H}_5(a;b;c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{2m_1+m_2}\langle b;q\rangle_{m_2-m_1}}{\langle 1;q\rangle_{m_1}\langle 1,c;q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}. \tag{2.26}$$

$$\mathrm{H}_6(a;b;c|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{2m_1-m_2}\langle b;q\rangle_{m_2-m_1}\langle c;q\rangle_{m_2}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}. \tag{2.27}$$

$$\mathrm{H}_7(a;b;c;d|q;x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{2m_1-m_2}\langle b,c;q\rangle_{m_2}}{\langle 1,d;q\rangle_{m_1}\langle 1;q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}. \tag{2.28}$$

We have the following alternative definitions with an extra quadratic q-factor; note that these fuctions are different from those in [6].

$$\mathrm{H}_1(a;b;\widetilde{c;d|q};x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1-m_2}\langle b;q\rangle_{m_1+m_2}\langle c;q\rangle_{m_2} q^{\binom{m_1}{2}}}{\langle 1,d;q\rangle_{m_1}\langle 1;q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}. \tag{2.29}$$

$$\mathrm{H}_2(a;b;\widetilde{c;d;e|q};x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{m_1-m_2}\langle b;q\rangle_{m_1}\langle c,d;q\rangle_{m_2} q^{\binom{m_1}{2}}}{\langle 1,e;q\rangle_{m_1}\langle 1;q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}. \tag{2.30}$$

$$\mathrm{H}_3(a;\widetilde{b;c|q};x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{2m_1+m_2}\langle b;q\rangle_{m_2} q^{\binom{m_1}{2}}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\langle c;q\rangle_{m_1+m_2}} x_1^{m_1}x_2^{m_2}. \tag{2.31}$$

$$\mathrm{H}_4(a;b;\widetilde{c;d|q};x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{2m_1+m_2}\langle b;q\rangle_{m_2} q^{\binom{m_1}{2}}}{\langle 1,c;q\rangle_{m_1}\langle 1,d;q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}. \tag{2.32}$$

$$\mathrm{H}_5(a;\widetilde{b;c|q};x_1,x_2) \equiv \sum_{m_1,m_2=0}^{\infty} \frac{\langle a;q\rangle_{2m_1+m_2}\langle b;q\rangle_{m_2-m_1} q^{\binom{m_1}{2}}}{\langle 1;q\rangle_{m_1}\langle 1,c;q\rangle_{m_2}} x_1^{m_1}x_2^{m_2}. \tag{2.33}$$

For the explanation of the importance of multiple q-Horn functions, see the previous explanation of dual q-additions.

3 Reduction formulas for triple q-functions

The notation will be illustrated by extensive proofs of the formulas, where extra tildes are inserted for q-Horn and multiple functions.

Theorem 3.1. *A q-analogue of [11, (3)].*

$$\Phi\left[\begin{matrix} c,d;a,b;a,[b] \\ a;2b;2b \end{matrix}\middle| q;x,y,-y\right] = \mathrm{H}_7\left[\begin{matrix} a;c;d \\ \widetilde{1},b+\frac{1}{2},\widetilde{b+\frac{1}{2}} \end{matrix}\middle| q;y^2,x\right]. \tag{3.34}$$

Proof. Put

$$C_{m,n+p} = \frac{\langle a;q\rangle_{n+p-m}\langle c,d;q\rangle_m}{\langle 2b;q\rangle_{n+p}} \tag{3.35}$$

in (2.18). Then we have

$$\begin{aligned} \text{LHS} &\overset{\text{by}(2.18)}{=} \sum_{m,k=0}^{\infty} \frac{\langle a;q\rangle_{2k-m}\langle c,d;q\rangle_m\langle b,\widetilde{b};q\rangle_k}{\langle 1;q\rangle_m\langle 2b;q\rangle_{2k}\langle 1,\widetilde{1};q\rangle_k} x^m y^{2k} \\ &\overset{\text{by}[3,\ (6.33\text{-}34)]}{=} \sum_{m,k=0}^{\infty} \frac{\langle a;q\rangle_{2k-m}\langle c,d;q\rangle_m}{\langle 1;q\rangle_m\langle 1,\widetilde{1},b+\frac{1}{2},\widetilde{b+\frac{1}{2}};q\rangle_k} x^m y^{2k} = \text{RHS}. \end{aligned} \tag{3.36}$$

□

Theorem 3.2. *A q-analogue of [11, (4)].*

$$\begin{aligned}&\Phi\left[\begin{matrix}a,a;a,b;a,[b]\\ d;2b;2b\end{matrix}\middle|q;x,y,-y\right]\\ &=\Phi^{4:2;3}_{4:1;2}\left[\begin{matrix}a,\widetilde{a},a+\frac{1}{2},\widetilde{a+\frac{1}{2}}:2\infty;3\infty\\ 4\infty:d;b+\frac{1}{2},\widetilde{b+\frac{1}{2}}\end{matrix}\middle|q;x,y^2\right].\end{aligned}\tag{3.37}$$

Proof. Put

$$C_{m,n+p}=\frac{\langle a;q\rangle_{2m+n+p}}{\langle d;q\rangle_m\langle 2b;q\rangle_{n+p}}.\tag{3.38}$$

Then we have

$$\begin{aligned}\text{LHS}&\overset{\text{by(2.18)}}{=}\sum_{m,k=0}^{\infty}\frac{\langle a;q\rangle_{2m+2k}\langle b,\widetilde{b};q\rangle_k}{\langle 1,d;q\rangle_m\langle 2b;q\rangle_{2k}\langle 1,\widetilde{1};q\rangle_k}x^my^{2k}\\ &\overset{\text{by[3, (6.33-34)]}}{=}\sum_{m,k=0}^{\infty}\frac{\langle a;q\rangle_{2m+2k}}{\langle 1,d;q\rangle_m}\langle 1,\widetilde{1},b+\frac{1}{2},\widetilde{b+\frac{1}{2}};q\rangle_kx^my^{2k}=\text{RHS}.\end{aligned}\tag{3.39}$$

□

Theorem 3.3. *A q-analogue of [11, (5)].*

$$\Phi\left[\begin{matrix}a,c;a,b;a,[b]\\ d;2b;2b\end{matrix}\middle|q;x,y,-y\right]=\text{H}_4\left[\begin{matrix}a;c\\ \widetilde{1},b+\frac{1}{2},\widetilde{b+\frac{1}{2}};d\end{matrix}\middle|q;y^2,x\right].\tag{3.40}$$

Proof. Put

$$C_{m,n+p}=\frac{\langle a;q\rangle_{m+n+p}\langle c;q\rangle_m}{\langle d;q\rangle_m\langle 2b;q\rangle_{n+p}}.\tag{3.41}$$

Then we find

$$\begin{aligned}\text{LHS}&\overset{\text{by(2.18)}}{=}\sum_{m,k=0}^{\infty}\frac{\langle a;q\rangle_{m+2k}\langle c;q\rangle_m\langle b,\widetilde{b};q\rangle_k}{\langle 1,d;q\rangle_m\langle 2b;q\rangle_{2k}\langle 1,\widetilde{1};q\rangle_k}x^my^{2k}\\ &\overset{\text{by[3, (6.33-34)]}}{=}\sum_{m,k=0}^{\infty}\frac{\langle a;q\rangle_{m+2k}\langle c;q\rangle_m}{\langle 1,d;q\rangle_m\langle 1,\widetilde{1},b+\frac{1}{2},\widetilde{b+\frac{1}{2}};q\rangle_k}x^my^{2k}=\text{RHS}.\end{aligned}\tag{3.42}$$

□

Lemma 3.1. *A q-analogue of [2, §2.8(5)]*

$$\frac{1}{2}\left[\frac{1}{(z^{\frac{1}{2}};q)_a}+\frac{1}{(-z^{\frac{1}{2}};q)_a}\right] = {}_4\phi_3\left[\begin{array}{c}\Delta(q;2;a)\\ \frac{1}{2},\widetilde{\frac{1}{2}},\widetilde{1}\end{array}\middle|q;z\right]. \tag{3.43}$$

$$\frac{1}{2}\left[\frac{1}{(z^{\frac{1}{2}};q)_a}-\frac{1}{(-z^{\frac{1}{2}};q)_a}\right] = z^{\frac{1}{2}}\{a\}_q\ {}_4\phi_3\left[\begin{array}{c}\Delta(q;2;a+1)\\ \frac{3}{2},\widetilde{\frac{3}{2}},\widetilde{1}\end{array}\middle|q;z\right]. \tag{3.44}$$

Proof. In both cases, use the q-binomial theorem and add or subtract the corresponding contributions. □

Theorem 3.4. *A q-analogue of [11, (7)].*

$$\begin{aligned}
&\Phi\left[\begin{array}{c}a,a;a,b;a,[b]\\ \frac{1}{2},\widetilde{\frac{1}{2}},\widetilde{1};d;d\end{array}\middle|q;x,y,-y\right]\\
&=\frac{1}{2(x^{\frac{1}{2}};q)_a}\ {}_8\phi_7\left[\begin{array}{c}2\infty,\Delta(q;2;a),b,\widetilde{b}\\ \Delta(q;2;d),\widetilde{1}\end{array}\middle|q;y^2\middle\|\begin{array}{c}\cdot\\ (x^{\frac{1}{2}}q^a;q)_{2k}\end{array}\right]\\
&+\frac{1}{2(-x^{\frac{1}{2}};q)_a}\ {}_8\phi_7\left[\begin{array}{c}2\infty,\Delta(q;2;a),b,\widetilde{b}\\ \Delta(q;2;d),\widetilde{1}\end{array}\middle|q;y^2\middle\|\begin{array}{c}\cdot\\ (-x^{\frac{1}{2}}q^a;q)_{2k}\end{array}\right].
\end{aligned} \tag{3.45}$$

Proof. Put

$$C_{m,n+p}=\frac{\langle a;q\rangle_{2m+n+p}}{\langle \frac{1}{2},\widetilde{\frac{1}{2}},\widetilde{1};q\rangle_m\langle d;q\rangle_{n+p}}. \tag{3.46}$$

Then we find

$$\begin{aligned}
\text{LHS} &\overset{\text{by}(2.18)}{=} \sum_{m,k=0}^{\infty}\frac{\langle a;q\rangle_{2m+2k}\langle b,\widetilde{b};q\rangle_k}{\langle 1,\frac{1}{2},\widetilde{\frac{1}{2}},\widetilde{1};q\rangle_m\langle d;q\rangle_{2k}\langle 1,\widetilde{1};q\rangle_k}x^m y^{2k}\\
&=\sum_{k=0}^{\infty}\frac{\langle a;q\rangle_{2k}\langle b,\widetilde{b};q\rangle_k}{\langle d;q\rangle_{2k}\langle 1,\widetilde{1};q\rangle_k}y^{2k}\ {}_4\phi_3\left[\begin{array}{c}\Delta(q;2;a)\\ \frac{1}{2},\widetilde{\frac{1}{2}},\widetilde{1}\end{array}\middle|q;x\right]\overset{\text{by}(3.43)}{=}\text{RHS}.
\end{aligned} \tag{3.47}$$

□

Theorem 3.5. *A q-analogue of [11, (8)].*

$$\begin{aligned}
&\Phi\left[\begin{array}{c} a, a; d, b; d, [b] \\ \frac{1}{2}, \widetilde{\frac{1}{2}}, \widetilde{1}; a; a \end{array}\middle| q; x, y, -y\right] \\
&= \frac{1}{2(x^{\frac{1}{2}}; q)_a}\ {}_8\phi_7\left[\begin{array}{c} 2\infty, \Delta(q; 2; d), b, \widetilde{b} \\ \Delta(q; 2; 1-a), \widetilde{1} \end{array}\middle| q; y^2 \middle\|\ \begin{array}{c} (x^{\frac{1}{2}} q^a; q)_{2k} \\ \cdot \end{array}\right] \\
&+ \frac{1}{2(-x^{\frac{1}{2}}; q)_a}\ {}_8\phi_7\left[\begin{array}{c} 2\infty, \Delta(q; 2; d), b, \widetilde{b} \\ \Delta(q; 2; 1-a), \widetilde{1} \end{array}\middle| q; y^2 \middle\|\ \begin{array}{c} (-x^{\frac{1}{2}} q^a; q)_{2k} \\ \cdot \end{array}\right].
\end{aligned} \tag{3.48}$$

Proof. Put

$$C_{m,n+p} = \frac{\langle a; q\rangle_{2m-n-p}\langle d; q\rangle_{n+p}}{\langle \frac{1}{2}, \widetilde{\frac{1}{2}}, \widetilde{1}; q\rangle_m}. \tag{3.49}$$

$$\begin{aligned}
&\text{LHS} \overset{\text{by}(2.18)}{=} \sum_{m,k=0}^{\infty} \frac{\langle a; q\rangle_{2m-2k}\langle b, \widetilde{b}; q\rangle_k \langle d; q\rangle_{2k}}{\langle 1, \widetilde{1}, \frac{1}{2}, \widetilde{\frac{1}{2}}; q\rangle_m \langle 1, \widetilde{1}; q\rangle_k} x^m y^{2k} \\
&= \sum_{k=0}^{\infty} \frac{q^{\binom{2k}{2}+2k(1-a)}\langle b, \widetilde{b}; q\rangle_k \langle d; q\rangle_{2k}}{\langle 1-a; q\rangle_{2k}\langle 1, \widetilde{1}; q\rangle_k} y^{2k}\ {}_4\phi_3\left[\begin{array}{c} \Delta(q; 2; a-2k) \\ \frac{1}{2}, \widetilde{\frac{1}{2}}, \widetilde{1} \end{array}\middle| q; x\right] \\
&\overset{\text{by}(3.43)}{=} \frac{1}{2}\sum_{k=0}^{\infty} \frac{q^{\binom{2k}{2}+2k(1-a)}\langle b, \widetilde{b}; q\rangle_k \langle d; q\rangle_{2k}}{\langle 1-a; q\rangle_{2k}\langle 1, \widetilde{1}; q\rangle_k} y^{2k}\left[\frac{1}{(x^{\frac{1}{2}}; q)_{a-2k}} + \frac{1}{(-x^{\frac{1}{2}}; q)_{a-2k}}\right] \\
&\overset{\text{by}[3,\ (6.13)]}{=} \text{RHS}.
\end{aligned} \tag{3.50}$$

□

This section contained both q-Horn functions and general q-hypergeometric functions on the right-hand side. The Srivastava $\triangle$ notation, the tilde and the infinity symbol have been extensively used. We hope that our new notations will be suitable for other functions too, since this subject is far from fully exploited.

4 Convergence regions and numerical values

4.1 Convergence regions for Horn functions

Figures (1)-(6) show convergence regions in the first quadrant for the six Horn functions H_1, H_2, H_3, H_4, H_6 and H_7. The convergence region for H_5 is more complicated, see [2]. We remind that convergence regions are independent of

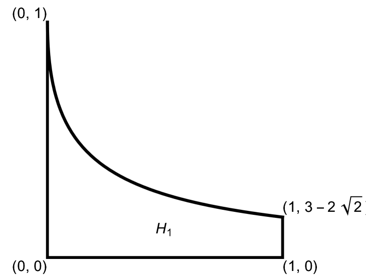

Figure 1: Convergence region for H_1

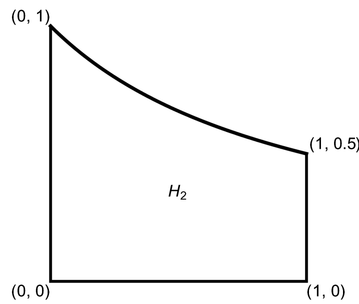

Figure 2: Convergence region for H_2

the parameters. We illustrate convergence region computations with the q-Stirling formula in one example. Later we show that Horns method leads to the same result. We adhere to the terminology in Horn [9]. In the whole paper, A_{m_1,m_2} denotes the coefficient of $x_1^{m_1}x_2^{m_2}$ for the respective function. For the function G_1 we have

$$A_{m_1,m_2} = \Gamma_q \left[\begin{array}{c} a+m_1+m_2, b-m_1+m_2, b'+m_1-m_2, 1, 1 \\ a, b, b', 1+m_1, 1+m_2 \end{array} \right]. \tag{4.51}$$

We will use the following equivalent approximation for the $q-$Stirling formula [4]:

$$\Gamma_q(z) \sim \{z\}_q^{\,z-\frac{1}{2}}. \tag{4.52}$$

Then we find $\lim_{m_1,m_2\to\infty}$

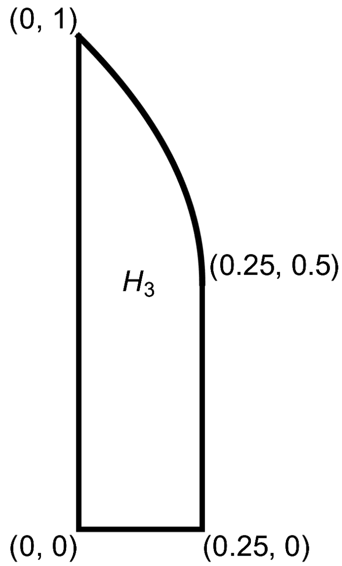

Figure 3: Convergence region for H$_3$

$$A_{m_1,m_2} \sim \Gamma_q \left[\begin{array}{c} - \\ a, b, b' \end{array} \right] \lim_{m_1,m_2 \to \infty} \{m_1+m_2\}_q^{a-1} \{m_1\}_q^{b'+m_1-m_2-1} \{m_2\}_q^{b-m_1+m_2-1} \binom{m_1+m_2}{m_1}_q . \tag{4.53}$$

The series converges for $(|x_1| \oplus_q |x_2|)^r < 1$ and G$_1$ converges in the same region.

Definition 4.1. *Let* lim *denote that all parameters and* $1 \to 0$.

$$\Phi(m_1, m_2) \equiv \lim \frac{A_{m_1+1,m_2}}{A_{m_1,m_2}}. \tag{4.54}$$

$$\Psi(m_1, m_2) \equiv \lim \frac{A_{m_1,m_2+1}}{A_{m_1,m_2}}. \tag{4.55}$$

The positive quantities $\{r_i\}_{i=1}^2$ are called the associated radii of convergence for the double series

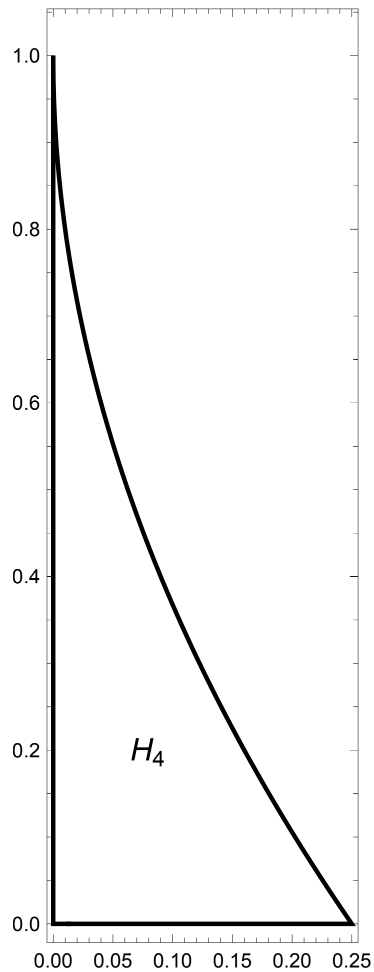

Figure 4: Convergence region for H_4

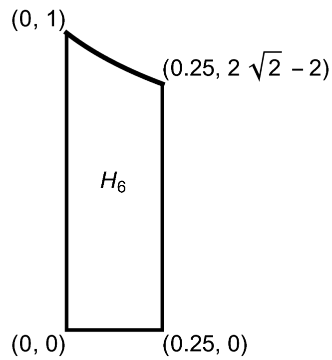

Figure 5: Convergence region for H_6

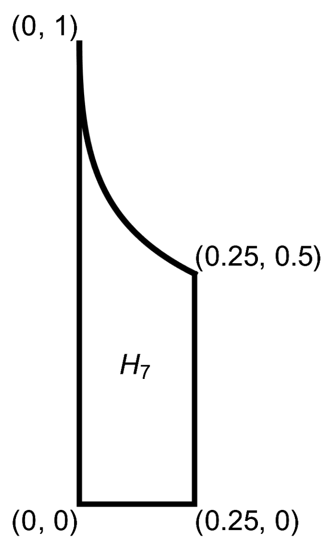

Figure 6: Convergence region for H_7

$$\sum_{m_1,m_2} A_{m_1,m_2} x_1^{m_1} x_2^{m_2}. \tag{4.56}$$

The convergence region, a q-deformed hypersurface in $\mathbb{R}^2$, has the parametric representation

$$r_1 = |\Phi(m_1, m_2)|^{-1},\ r_2 = |\Psi(m_1, m_2)|^{-1} \tag{4.57}$$

In the following, we follow the notation in [7, p. 37-39]. Let $\xi \equiv |x_1|$ and $\eta \equiv |x_2|$.

The convergence curve C has the parametric equations

$$\xi \equiv |(\Phi(m_1, m_2))^{-1}|,\ \eta \equiv |(\Phi(m_1, m_2))^{-1}|. \tag{4.58}$$

Consider the function H_1. We have

$$\begin{aligned} \frac{A_{m_1+1,m_2}}{A_{m_1,m_2}} &= \frac{\langle a - m_2 + m_1, b + m_1 + m_2; q\rangle_1}{\langle d + m_1, 1 + m_1; q\rangle_1}, \\ \Phi(m_1, m_2) &= \frac{\{m_1 + m_2\}_q \{m_1 - m_2\}_q}{\{m_1\}_q^2}. \end{aligned} \tag{4.59}$$

On the other hand,

$$\frac{A_{m_1,m_2+1}}{A_{m_1,m_2}} = \frac{\langle c + m_2, b + m_1 + m_2; q\rangle_1}{\langle 1 + m_2, a - m_2 + m_1 - 1; q\rangle_1},\ \Psi(m_1, m_2) = \frac{\{m_1 + m_2\}_q}{\{m_1 - m_2\}_q}. \tag{4.60}$$

The convergence border C has the parametric equations

$$\xi = \frac{\{m_1\}_q^2}{\{m_1 + m_2\}_q \{m_1 - m_2\}_q},\ \eta = \frac{\{m_1 - m_2\}_q}{\{m_1 + m_2\}_q}. \tag{4.61}$$

For $q = 1$, the convergence region is bounded by the curve $4\xi\eta = (\eta + 1)^2$.

All the numerical values were computed by *Mathematica*.

The following table exemplifies the largely extended convergence region for

$\mathrm{H}_1(.7; .66; -.88; .44|.9; x_1, x_2)$

x_1	x_2	$\mathrm{H}_1(.7; .66; -.88; .44\|.9; x_1, x_2)$
.6	.28	0.574532
.25	.45	0.70123
.25	.5	0.666567
.25	.55	0.63151
.25	.6	0.596016
.25	.7	0.523497
.25	.8	0.448412
.25	.9	0.369667
.25	0.95	0.328091
.4	.99	-0.484385
.6	.96	-7.71041
.8	.999	≈ -59299.8
.9	.99	$-116622.$
.95	.99	$-845949.$
.99	.8	-340.604
.995	.8	-378.912
.999	.8	-412.902
1.	.19	0.524829
.9	1.01	∞
1.01	.8	-524.518
1.05	.8	-1304.2
1.05	.9	$-108294.$
1.1	.8	-4550.01
1.1	.85	-46646.7
1.15	.85	$-463725.$
1.17	.85	-3.24×10^6
1.18	.85	∞

We remark that $1.17 \times .85 \approx 0.9945$, $1.18 \times .85 \approx 1.003$.
The following table exemplifies the largely extended convergence region for

$\mathrm{H}_1(\widetilde{|.9; x_1}, x_2)$.

x_1	x_2	$\mathrm{H}_1(.7; .66; \widetilde{-.88}; .44 \vert .9; x_1, x_2)$
.6	.28	0.577146
.25	.45	0.702179
.25	.5	0.667825
.25	.55	0.633152
.25	.6	0.598134
.25	.7	0.526937
.25	.8	0.453923
.25	.9	0.378612
.25	0.95	0.339816
1.	.19	0.528465
.4	.99	-0.161551
.6	.96	-0.979853
.8	.999	≈ -20.4
.9	.99	-10.339
.95	.99	-15.1764
9	.9	-3.45471
.99	.8	-2.785
.995	.8	-2.84405
.999	.8	-2.89225
1.	.19	0.528465
1.	.22	0.449196
.9	1.01	∞
1.01	.8	-3.02941
1.05	.8	-3.59088
1.05	.9	-7.6383
1.1	.8	-4.45941
1.1	.85	-6.51831
1.15	.85	-8.3796
1.17	.85	-3.24×10^6
1.18	.85	∞

Consider the function H_2. We have

$$\frac{A_{m_1+1,m_2}}{A_{m_1,m_2}} = \frac{\langle a - m_2 + m_1, b + m_1; q\rangle_1}{\langle e + m_1, 1 + m_1; q\rangle_1},\ \Phi(m_1, m_2) = \frac{\{m_1 - m_2\}_q}{\{m_1\}_q}, \quad (4.62)$$

On the other hand,

$$\frac{A_{m_1,m_2+1}}{A_{m_1,m_2}} = \frac{\langle c + m_2, d + m_2; q\rangle_1}{\langle 1 + m_2, a - m_2 + m_1 - 1; q\rangle_1},\ \Psi(m_1, m_2) = \frac{\{m_2\}_q}{\{m_1 - m_2\}_q}. \quad (4.63)$$

The boundary of the convergence region C has the parametric equations

$$\xi = \frac{\{m_1\}_q}{\{m_1 - m_2\}_q}, \ \eta = \frac{\{m_1 - m_2\}_q}{\{m_2\}_q}. \tag{4.64}$$

For $q = 1$, the convergence region is bounded by the curve $-\xi + \eta^{-1} = 1$.

The following table exemplifies the divergence values for $\mathrm{H}_2(.7; .66; .22; -.88; .44; x_1, x_2)$:

x_1	x_2	$\mathrm{H}_2(.7; .66; .22; -.88; .44; x_1, x_2)$
1.17	.85	∞
1.10	.85	∞
.85	1.05	∞
.25	1.1	∞
.7	1.	∞
.92	.99	∞

The following table exemplifies the largely extended convergence region for $\mathrm{H}_2(|.9|q; x_1, x_2)$.

x_1	x_2	$\mathrm{H}_2(.7; .66; .22; -.88; .44\|.9; x_1, x_2)$
1.17	.85	0.766257
1.10	.85	0.943808
.85	1.05	∞
.25	1.1	∞
.7	1.	0.960477
.92	.99	0.934475

The following table exemplifies the largely extended convergence region for $\mathrm{H}_2(\widetilde{|.9}; x_1, x_2)$.

x_1	x_2	$\mathrm{H}_2(.7; .66; .22; \widetilde{-.88}; .44\|.9; x_1, x_2)$
1.17	.85	0.944452
1.10	.85	0.947817
.85	1.05	∞
.25	1.1	∞
.7	1.	0.960676
.92	.99	0.948944

Consider the function H_3. We have

$$\frac{A_{m_1+1,m_2}}{A_{m_1,m_2}} = \frac{\langle a + 2m_1 + m_2; q\rangle_2}{\langle c + m_1 + m_2, 1 + m_1; q\rangle_1}, \ \Phi(m_1, m_2) = \frac{\{2m_1 + m_2\}_q^2}{\{m_1 + m_2\}_q \{m_1\}_q}. \tag{4.65}$$

On the other hand,

$$\frac{A_{m_1,m_2+1}}{A_{m_1,m_2}} = \frac{\langle a+2m_1+m_2, b+m_2; q\rangle_1}{\langle c+m_1+m_2, 1+m_2; q\rangle_1},\ \Psi(m_1,m_2) = \frac{\{2m_1+m_2\}_q}{\{m_1+m_2\}_q}. \tag{4.66}$$

The convergence border C has the parametric equations

$$\xi = \frac{\{m_1+m_2\}_q\{m_1\}_q}{\{2m_1+m_2\}_q^2},\ \eta = \frac{\{m_1+m_2\}_q}{\{2m_1+m_2\}_q}. \tag{4.67}$$

For $q = 1$, the convergence region is bounded by the curve

$$\xi + \left(\eta - \frac{1}{2}\right)^2 = \frac{1}{4}.$$

The following table exemplifies values where $\mathrm{H}_3(.7; .66; -.88; x_1, x_2)$ diverges :

x_1	x_2	$\mathrm{H}_3(.7; .66; -.88; x_1, x_2)$
.25	.8	∞
.25	.95	∞
.4	.6	∞
.8	.3	∞

The following table exemplifies the largely extended convergence region for $\mathrm{H}_3(|.9; x_1, x_2)$:

x_1	x_2	$\mathrm{H}_3(.7; .66; -.88\vert .9; x_1, x_2)$
.25	.8	-782.114
.25	.95	-11550.8
.4	.6	-1270.47
.8	.3	-4.41834×10^6

The following table exemplifies the largely extended convergence region for $\mathrm{H}_3(|.9; x_1, x_2)$.

x_1	x_2	$\mathrm{H}_3(.7; .66; -.88\vert .9; x_1, x_2)$
.25	.8	-565.037
.25	.95	-8356.2
.4	.6	-375.945
.8	.3	-4810.81

We infer that the only important variables are q, x_1, x_2. We also computed some function values with other parameters, which are listed below (the parameters are the same when x_1, x_2 are the same):

x_1	x_2	$\mathrm{H}_3(\|.9; x_1, x_2)$
.85	.3	-2.93996×10^7
.87	.3	-9.36162×10^7
.87	.8	-3.28457×10^8
.87	.95	-1.61249×10^9
.95	.95	-2.67052×10^{10}
.96	.96	-5.39639×10^{10}
.98	.98	-3.6039×10^{11}
.99	.99	-1.85456×10^{12}

x_1	x_2	$\mathrm{H}_3(\widetilde{\|.9}; x_1, x_2)$
.85	.3	-10895.3
.87	.3	-20906.1
.87	.8	$-276763.$
.87	.95	-4.16117×10^6
.95	.95	-7.34531×10^6
.96	.96	-1.11771×10^7
.98	.98	-3.38574×10^7
.99	.99	-8.44023×10^7

Consider the function H_4. We have

$$\frac{A_{m_1+1,m_2}}{A_{m_1,m_2}} = \frac{\langle a+2m_1+m_2; q\rangle_2}{\langle 1+m_1, c+m_1; q\rangle_1},\ \Phi(m_1, m_2) = \frac{\{2m_1+m_2\}_q^2}{\{m_1\}_q^2}. \tag{4.68}$$

On the other hand,

$$\frac{A_{m_1,m_2+1}}{A_{m_1,m_2}} = \frac{\langle a+2m_1+m_2, b+m_2; q\rangle_1}{\langle 1+m_2, d+m_2; q\rangle_1},\ \Psi(m_1, m_2) = \frac{\{2m_1+m_2\}_q}{\{m_2\}_q}. \tag{4.69}$$

The convergence border C has the parametric equations

$$\xi = \frac{\{m_1\}_q^2}{\{2m_1+m_2\}_q^2},\ \eta = \frac{\{m_2\}_q}{\{2m_1+m_2\}_q}. \tag{4.70}$$

For $q = 1$, the convergence region is bounded by the curve

$$4\xi = (\eta - 1)^2. \tag{4.71}$$

The following table exemplifies the largely extended convergence region for

$\mathrm{H}_4(|.9; x_1, x_2)$.

x_1	x_2	$\mathrm{H}_4(.7; .66; -.88; .34\|.9; x_1, x_2)$
.36	.1	-497.01
.37	.1	-609.267
.38	.11	-838.891
.39	.12	-1157.09
.45	.13	-4605.76
.45	.18	-8299.61
.47	.20	-16276.2
.49	.23	-36273.9
.52	.25	-91169.9
.54	.15	-42621.8
.55	.13	-41678.5
.56	.18	-97537.7
.56	.2	$-124725.$
.56	.22	$-159401.$
.56	.32	$-549200.$
.6	.32	-1.43368×10^6
.66	.32	-6.33515×10^6
.66	.34	-8.2101×10^6
.7	.34	-2.30365×10^7
.7	.37	-3.4246×10^7
.75	.36	-1.15495×10^8
.8	.34	-3.70151×10^8
.8	.4	-8.3214×10^8
.8	.44	-1.45112×10^9
.8	.5	-3.44418×10^9
.8	.55	-7.31888×10^9
.85	.45	-7.92947×10^9
.85	.5	-1.63856×10^{10}
.85	.55	-3.49169×10^{10}
.85	.6	-7.72059×10^{10}
.85	.7	-4.35934×10^{11}
.8	.8	-6.75591×10^{11}
.85	.85	-1.06145×10^{13}
.9	.9	-2.45497×10^{14}
.95	.95	-1.24604×10^{16}
.96	.96	-3.34707×10^{16}
.97	.97	-1.03729×10^{17}
.98	.98	-4.13221×10^{17}
.99	.99	-2.978×10^{18}

The following table exemplifies the largely extended convergence region for $\mathrm{H}_4(\widetilde{|.9; x_1}, x_2)$.

x_1	x_2	$\mathrm{H}_4(.7; .66; \widetilde{-.88; .34}\|.9; x_1, x_2)$
.36	.1	-99.7646
.37	.1	-113.209
.38	.11	-141.525
.39	.12	-176.748
.45	.13	-402.489
.45	.18	-661.881
.47	.20	-1025.15
.49	.23	-1755.84
.52	.25	-3059.3
.54	.15	-1374.96
.55	.13	-1245.31
.56	.18	-2338.93
.56	.2	-2877.25
.56	.22	-3539.46
.56	.32	-10129.1
.6	.32	-15949.1
.66	.32	-30771.7
.66	.34	-38437.2
.7	.34	-58849.6
.7	.37	-82841.8
.75	.36	$-124248.$
.8	.34	$-163178.$
.8	.4	$-330246.$
.8	.44	$-537389.$
.8	.5	-1.15111×10^6
.8	.55	-2.24811×10^6
.85	.45	-1.00264×10^6
.85	.5	-1.9063×10^6
.85	.55	-3.74115×10^6
.85	.6	-7.6255×10^6
.85	.7	-3.66905×10^7
.8	.8	-1.38071×10^8
.85	.85	-7.07608×10^8
.9	.9	-4.385×10^9
.95	.95	-4.01401×10^{10}
.96	.96	-6.91494×10^{10}
.97	.97	-1.27941×10^{11}
.98	.98	-2.68361×10^{11}
.99	.99	-7.57045×10^{11}

We infer that the only important variables (within normal ranges) are

q, x_1, x_2.

Consider the function H_5. We have

$$\Phi(m_1, m_2) = \frac{\{2m_1 + m_2\}_q^2}{\{m_2 - m_1\}_q \{m_1\}_q}, \ \Psi(m_1, m_2) = \frac{\{2m_1 + m_2\}_q \{m_2 - m_1\}_q}{\{m_2\}_q^2}. \tag{4.72}$$

The convergence border C has the parametric equations

$$\xi = \frac{\{m_2 - m_1\}_q \{m_1\}_q}{\{2m_1 + m_2\}_q^2}, \ \eta = \frac{\{m_2\}_q^2}{\{2m_1 + m_2\}_q \{m_2 - m_1\}_q}. \tag{4.73}$$

The following table exemplifies the convergence region for H_5:

x_1	x_2	$\mathrm{H}_5(.7; .66; -.88; x_1, x_2)$
.02	.8	-184.354
.02	.9	-1441.68
.02	.95	-14996.4
.02	.96	-39258.3
.02	.97	$-210261.$
.02	.98	∞
.03	.91	-3078.93
.03	.92	-4804.47
.03	.93	-8281.86
.04	.55	-14.8307
.04	.58	-19.7882
.04	.6	-24.0614
.05	.4	-3.59709
.05	.455	-6.38679
.05	.5	∞
.11	.095	0.620182
.13	.06	0.650251
.13	.07	∞

The following table exemplifies the largely extended convergence region for

$\mathrm{H}_5(|.9; x_1, x_2)$:

x_1	x_2	$\mathrm{H}_5(.7; .66; -.88|.9; x_1, x_2)$
.6	.28	-4156.92
.25	.6	-808.574
.25	.7	-2291.71
.25	.8	-6351.34
.25	.9	$-18336.$
.25	.95	-35810.7
.4	.6	-32055.4
.4	.7	$-132590.$
.6	.46	$-218212.$
.7	.3	-37480.4
.8	.3	$-261695.$
.64	.8	-4.44298×10^8
$.85^2$	.85	-1.36057×10^{10}
$.87^2$	.87	-6.16019×10^{10}
$.9^2$	.9	-7.29556×10^{11}
$.925^2$	.925	-7.42529×10^{12}
$.95^2$	.95	-1.10175×10^{14}
$.97^2$	.97	-1.58031×10^{15}
$.98^2$	.965	-4.41885×10^{15}

The following table exemplifies the largely extended convergence region for $\widetilde{\mathrm{H}_5}(|.9; x_1, x_2)$:

x_1	x_2	$\widetilde{\mathrm{H}_5}(.7; .66; -.88|.9; x_1, x_2)$
.6	.28	-4156.92
.25	.6	-228.492
.25	.7	-530.859
.25	.8	-1290.79
.25	.9	-4074.42
.25	.95	$-10745.$
.4	.6	-2175.84
.4	.7	-5660.8
.6	.46	-5299.9
.7	.3	-1233.93
.8	.3	-2845.42
.64	.8	$-442256.$
$.85^2$	.85	-2.14316×10^6
$.87^2$	.87	-4.06241×10^6
$.9^2$	.9	-1.06891×10^7
$.925^2$	.925	-2.41201×10^7
$.95^2$	.95	-5.4886×10^7
$.97^2$	.97	-1.07131×10^8

Consider the function H_6. We have

$$\Phi(m_1,m_2)=\frac{\{2m_1-m_2\}_q^2}{\{m_2-m\}_q\{m_1\}_q},\ \Psi(m_1,m_2)=\frac{\{m_2-m_1\}_q}{\{2m_1-m_2\}_q}. \tag{4.74}$$

The convergence border C has the parametric equations

$$\xi=\frac{\{m_2-m_1\}_q\{m_1\}_q}{\{2m_1-m_2\}_q^2},\ \eta=\frac{\{2m_1-m_2\}_q}{\{m_2-m_1\}_q}. \tag{4.75}$$

For $q=1$, the convergence region is bounded by the curve $\xi\eta^2+\eta=1$.
The following table exemplifies the convergence region for $\mathrm{H}_6(x_1,x_2)$:

x_1	x_2	$\mathrm{H}_6(.7;.66;-.88\vert q;x_1,x_2)$
.22	.64	1.74196
.22	.8	2.02235
.23	.7	1.83741
.24	.59	1.63362
.25	.25	1.14405
.251	.251	∞

The following table exemplifies the largely extended convergence region for $\mathrm{H}_6(\vert .9;x_1,x_2)$:

x_1	x_2	$\mathrm{H}_6(.7;.66;-.88\vert .9;x_1,x_2)$
.6	.28	0.942578
.25	.6	0.878244
.25	.7	0.851269
.25	.8	0.823884
.25	.9	0.796053
.25	.95	0.781956
.25	1.	0.76771
.83	1.	0.03
.4	.6	0.824688
.6	.46	0.834755
.7	.3	0.931807
.8	.3	0.961308
.85	.3	1.02644

Consider the function H_7. We have

$$\Phi(m_1,m_2)=\frac{\{2m_1-m_2\}_q^2}{\{m_1\}_q^2},\ \Psi(m_1,m_2)=\frac{\{m_2\}_q}{\{2m_1-m_2\}_q}. \tag{4.76}$$

The convergence border C has the parametric equations

$$\xi = \frac{\{m_1\}_q^2}{\{2m_1 - m_2\}_q^2}, \ \eta = \frac{\{2m_1 - m_2\}_q}{\{m_2\}_q}. \tag{4.77}$$

For $q = 1$, the convergence region is bounded by the curve

$$4\xi = \left(\frac{1}{\eta} - 1\right)^2. \tag{4.78}$$

The following table exemplifies the convergence region for H_7:

x_1	x_2	$\mathrm{H}_7(.7; .66; -.88, .44; x_1, x_2)$
.25	.5	1.93661
.25	.6	∞
.8	.2	1.37966
.6	.28	1.52952
.4	.99	∞
.02	.999	∞

The following table exemplifies the largely extended convergence region for $\mathrm{H}_7|.9; x_1, x_2)$:

x_1	x_2	$\mathrm{H}_7(.7; .66; -.88; .44\|.9; x_1, x_2)$
.25	.5	0.956562
.25	.6	0.944654
.25	.7	0.931147
.25	.8	0.915462
.25	.9	0.896338
.25	1.	≈ 0.45
.25	1.1	∞
.4	.99	0.580361
.5	.99	-1.51711
.9	.99	-84046.1
.8	.999	$\approx -57768.$

5 Conclusion

The convergence region has increased considerably, and we can expect an even greater increment when the number of variables increases. Since the q-Horn functions are more complicated than the q-Appell functions, the time to compute numerical values near the convergence border increases and we have only given approximate values in these cases. We have shown numerically that if the q-Horn function contains a q-shifted factorial with index $m_1 - m_2$

in the numerator, we can only expect convergence in regions where $|x_1| > 1$ if $|x_2| < 1$. We have also shown that the tilde q-Horn functions converge slightly faster than the corresponding q-Horn functions. We can summarize the results of the convergence investigations as follows: Out of the seven H-functions, H_3 and H_4 are best in class with parabolas as convergence region boundaries and $\mathrm{H}_3|q; x_1, x_2)$ and $\mathrm{H}_4|q; x_1, x_2)$ with simple convergence boundaries. We could have seen this at once by looking at the q-shifted factorial indices of the two functions, which contain no minuses.

6 Discussion

This investigation could be continued by considering higher order q-Horn functions, even though the computations would take more time. The scientific contribution of the paper is the q-analogues of Karlsson's formulas and the new notation for multiple q-series together with the knowledge that also Horn functions can be q-deformed.

7 Conflict of interest

The manuscript contains no conflicts of interest. There is no funding.

8 Acknowledgement

Special thanks to Karl-Heinz Fieseler who has contributed with sound advice.

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