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Non-Archimedean stabilities of multiplicative inverse μ-functional inequalities

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Abstract

This study is motivated through the interesting non-Arcchimedean stability results of ρ-inequalities and ρ-equations arising from linear, second power, third power and fourth power mappings. The aim of this paper is to determine the solutions of new multiplicative inverse μ-inequalities and μ-equations arising from multiplicative inverse mapping. Further, their stabilities involved with various superior limits are proved in the context of non-Archimedean complete normed spaces.

1 Introduction & Preliminaries

In many physical problems, investigating correct solution to a mathematical equation is complicated. In such cases, an approximate solution replaces the exact solution of those equations without affecting the nature of the problem. This is an essence to focus on finding approximate solution of an equation. When a mathematical equation has an approximate solution near to its exact solution, then the equation is said to be stable. There are many interesting applications of functional equations in different disciplines such as Economics, Utility theory, Aggregation problems, Tax functions, Electrical engineering, Optics, Electromagnetism, Physics, Medicine, etc.

The investigation of finding approximate mappings is developed through a famous query posed in [30] referring to the approximate homomorphisms

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arising in the theory of groups. The foremost response is provided in [12] with a partial answer to Ulam's query in the setting of Banach spaces. If a functional equation is shown to satisfy the proof provided in [12], then an approximation exists that is close to its exact solution and hence it is said to have stability. This result is acclaimed as Ulam-Hyers (U-H) stability or approximation of functional equation. Further, U-H stability is extended as a general version for additive mappings in [2] and by taking sum of exponents of norms as an upper bound in [22]. Later, Ulam stability is obtained as a general outcome by reinstating the sum of exponents of norms with a generic control function in [8]. The Jensen functional equation

$$J(u+v) = \frac{1}{2}[J(u) + J(v)], \tag{1}$$

has been extensively investigated and was dealt by many mathematicians to find the solution of (1) in [1, 13]. A solution of equation (1) is of the form $J(u) = ku + c$.

The functional equation

$$g(p_1 + p_2) = \frac{g(p_1)g(p_2)}{g(p_1) + g(p_2)}, \tag{2}$$

is introduced to find its stabilities in [24]. The geometrical interpretation and an application of (2) related with electric circuits are discussed in [25]. It is proved that a multiplciative inverse mapping or a reciprocal mapping satisfies equation (2). Hence equation (2) is referred as a multiplicative inverse functional equation or a reciprocal functional equation. The following equation

$$J_r\left(\frac{u+v}{2}\right) = \frac{2J_r(u)J_r(v)}{J_r(u) + J_r(v)}, \tag{3}$$

is motivated through Jensen's functional equation (1) which has a solution of the form $J_r(u) = \frac{1}{u} + c$. The stability of equation (3) is investigated in [27]. Several other functional equations of rational type and cubic form are studied to determine their non-Archimedean stabilities in ([6, 9, 28]). It is proved in [10] that if a mapping g satisfies the ensuing inequality

$$\left\|2g(p) + 2g(q) - g(pq^{-1})\right\| \leq \|g(pq)\|, \tag{4}$$

then g satisfies the equation

$$2g(p) + 2g(q) = g(pq) + g(pq^{-1}).$$

Further, the stabilty results of inequality (4) are obtained in [7, 11]. The Ulam stabilities of additive functional inequalities are available in [20]. Besides, ρ-inequalities and ρ-equations arising from additive functions are defined and determined their stabilities in Archimedean Banach spaces in [18, 19], in non-Archimedean 2-normed spaces in [32]. The stabilities of Cauchy-Jensen kind additive ρ-functional inequalities are studied in [29]. Also, the stabilities of ρ-inequalities and ρ- equations arising from quadratic functions are investigated in [4, 16]. The ρ-inequalities arising from cubic and quartic functions are dealt in [21] to prove their Ulam stabilities.

For instance, there are many functional inequalities having interesting applications in the quotients of geometric functions [3], probability [5], parabolic stochastic partial differential equations containing rotation [14], log convex and reverse log convex properties of several unknown functions [31]. The method of Maggi's equations is applied to realize the assembly of the equations of motion for a planar mechanical systems using finite two-dimensional elements [26]. The inequalities associated with the Jordan-von Neuman functional equation were dealt in [23].

Motivated by the interesting applications of several functional inequalities and additive ρ-functional inequalities in [17], in this article, we propose the upcoming multiplicative inverse μ-functional inequalities

$$\left\| g(p+q) - \frac{g(p)g(q)}{g(p)+g(q)} \right\| \leq \left\| \mu \left(\frac{1}{2} g\left(\frac{p+q}{2}\right) - \frac{g(p)g(q)}{g(p)+g(q)} \right) \right\|, \tag{5}$$

and

$$\left\| \frac{2}{g\left(\frac{p+q}{2}\right)} - \frac{1}{g(p)} - \frac{1}{g(q)} \right\| \leq \left\| \mu \left(\frac{1}{g(p+q)} - \frac{1}{g(p)} - \frac{1}{g(q)} \right) \right\|, \tag{6}$$

where μ is a fixed non-Archimedean number with $|\mu| < 1$. We determine the solution of inequalities (5) and (6) and also investigate their non-Archimedean stabilities.

The fundamental definition and other notions related to non-Archimedean field and non-Archimedean space are recalled here to demonostrate our major outcomes.

Definition 1.1. Let us assume that $\mathbb{M}$ as a field. A mapping $|\cdot| : \mathbb{K} \longrightarrow \mathbb{R}^+ \cup \{0\}$ is called a valuation. Then $\mathbb{M}$ is known as a non-Archimedean field if the mapping $|\cdot|$ fulfills the ensuing prerequisites:

(NA1) $|p_1| = 0$ if and only if $p_1 = 0$,

(NA2) $|p_1 p_2| = |p_1| \cdot |p_2|$, and

(NA3) the triangle inequality: $|p_1 + p_2| \leq max\{|p_1|, |p_2|\}$, for all $p_1, p_2 \in \mathbb{M}$.

If a field $\mathbb{M}$ is equipped with a valuation then it is called as a valued field. The conventional absolute values of real and complex numbers are some instances where valuations are applicable. It is clear that $|\pm 1| = 1$ and for all integers $k \geq 0$, $|k| \leq 1$.

Definition 1.2. [15] Let a non-Archimedean field $\mathbb{M}$ be endowed with a valuation $|\cdot|$. Suppose U is a vector space over $\mathbb{M}$. A mapping $\|\cdot\| : U \longrightarrow \mathbb{R}^{+} \cup \{0\}$ is referred to as a non-Archimedean norm if the following circumstances are true:

(i) $\|p_1\| = 0$ if and only if $p_1 = 0$;

(ii) $\|kp_1\| = |k| \, \|p_1\|$, for all $k \in \mathbb{M}$, $p_1 \in U$;

(iii) $\|p_1 + p_2\| \leq \max\{\|p\|, \|q\|\}$, for all $p_1, p_2 \in U$.

Under the above circumstances, the pair $(U, \|\cdot\|)$ is designated as a non-Archimedean normed space.
In the following definitions, let $\{u_p\}$ be a sequence in a non-Archimedean normed space U.

Definition 1.3. The sequence $\{u_p\}$ is termed as Cauchy sequence if for a given $\delta > 0$ there exists an integer $K > 0$ such that $\|u_k - u_\ell\| \leq \delta$ for all $k, \ell \geq K$.

Definition 1.4. If for a given $\delta > 0$, there exists an integer $P > 0$ and an element $u \in U$ such that $\|u_p - u\| \leq \delta$ for all $p \geq P$ for all $p \geq P$, then the sequence $\{u_p\}$ converges to u. If the sequence $\{u_p\}$ converges to $u \in U$, then it is called as its limit and it is symbolized as $\lim\limits_{p \to \infty} u_p = u$.

Definition 1.5. If every Cauchy sequence $\{u_p\}$ is convergent to an element $u \in U$, then U is called as a non-Archimedean Banach space.

In this entire paper, let us presume that $\mathcal{A}$ be a non-Archimedean normed space and $\mathcal{B}$ be a non-Archimedean Banach space. Furthermore, we consider $|2| \neq 1$. Also, we assume that μ be a non-Archimedean number such that $|\mu| < 1$.

2 Solution of multiplicative inverse μ-functional inequalities (5) and (6)

In this section, we assume that $\mathcal{P}$ is a commutative semigroup with division by 2. Then, we solve the inequalities (5) and (6) over non-Archimedean normed spaces. In the following results, we consider $g : \mathcal{P} \longrightarrow \mathcal{B}$ to be a mapping.

Theorem 2.1. The mapping g satisfies the inequality (5) for all $p, q \in \mathcal{P}$

if and only if g is reciprocal.

Proof. Firstly, let us presume that (5) is satisfied by the mapping g. Now, setting $q = p$ in (5), we find that $\left\|g(2p) - \frac{1}{2}g(p)\right\| \leq 0$, which inturn implies

$$g(2p) = \frac{1}{2}g(p), \tag{7}$$

for all $p \in \mathcal{P}$. Utilizing (5) and (7), we obtain that

$$\left\|g(p+q) - \frac{g(p)g(q)}{g(p)+g(q)}\right\| \leq |\mu| \ \left\|g(p+q) - \frac{g(p)g(q)}{g(p)+g(q)}\right\|,$$

and as a consequence, we arrive at $g(p+q) = \frac{g(p)g(q)}{g(p)+g(q)}$, for all $p, q \in \mathcal{P}$. On the contrary, the proof is obvious.

Corollary 2.2. The following equation

$$g(p+q) - \frac{g(p)g(q)}{g(p)+g(q)} = \mu\left(\frac{1}{2}g\left(\frac{p+q}{2}\right) - \frac{g(p)g(q)}{g(p)+g(q)}\right), \tag{8}$$

is satisfied by the mapping g for all $p, q \in \mathbb{U}$ if and only if g is reciprocal.

Theorem 2.3. The inequality (6) is satisfied by the mapping g with the condition that $\frac{1}{g(0)} = 0$, if and only if g is reciprocal.

Proof. Let (6) is satisfied by the mapping g. Letting $q = 0$ in (6), we get $\left\|\frac{2}{g\left(\frac{p}{2}\right)} - \frac{1}{g(p)}\right\| \leq 0$, which yields that

$$g\left(\frac{p}{2}\right) = 2g(p), \tag{9}$$

for all $p \in \mathcal{P}$. Using (6) and (9), we arrive at

$$\begin{aligned}\left\|\frac{1}{g(p+q)} - \frac{1}{g(p)} - \frac{1}{g(q)}\right\| &\leq \left\|\frac{2}{g\left(\frac{p+q}{2}\right)} - \frac{1}{g(p)} - \frac{1}{g(q)}\right\| \\ &\leq |\mu| \ \left\|\frac{1}{g(p+q)} - \frac{1}{g(p)} - \frac{1}{g(q)}\right\|,\end{aligned}$$

which induces $g(p+q) = \frac{g(p)g(q)}{g(p)+g(q)}$, for all $p, q \in \mathcal{P}$. On the contrary, the proof is obvious.

Corollary 2.4. The following equation

$$\frac{2}{h\left(\frac{u+v}{2}\right)} - \frac{1}{h(u)} - \frac{1}{h(v)} = \mu\left(\frac{1}{h(u+v)} - \frac{1}{h(u)} - \frac{1}{h(v)}\right), \tag{10}$$

is satisfied by the mapping g for all $u, v \in \mathbb{U}$ if and only if g is reciprocal.

3 Ulam stabilities of multiplicative inverse μ-functional inequalities (5) and (6)

In the present section, we solve the stability problems concerning the multiplicative inverse μ-functional inequalities (5) and (6) in the framework of non-Archimedean complete Banach spaces. Let us assume that $g : \mathcal{A} \longrightarrow \mathcal{B}$ be a mapping in the following results.

Theorem 3.1. Suppose a function $\varphi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{R}^+$ satisfies the following condition

$$\lim_{n\to\infty} \frac{1}{|2|^n} \varphi\left(\frac{p}{2^n}, \frac{q}{2^n}\right) = 0, \tag{11}$$

for all $u, v \in \mathcal{A}$. If the mapping g satisfies

$$\left\| g(p+q) - \frac{g(p)g(q)}{g(p)+g(q)} \right\| \leq \left\| \mu\left(\frac{1}{2} g\left(\frac{p+q}{2}\right) - \frac{g(p)g(q)}{g(p)+g(q)}\right) \right\| + \varphi(p,q), \tag{12}$$

for all $p, q \in \mathcal{A}$, then a mapping $G : \mathcal{A} \longrightarrow \mathcal{B}$ exists and defined by

$$G(p) = \lim_{n\to\infty} \frac{1}{2^n} g\left(\frac{p}{2^n}\right), \tag{13}$$

for all $p \in \mathcal{A}$. The mapping defined in (13) is unique and satisfies (7) such that

$$\|g(p) - G(p)\| \leq \max\left\{ \frac{1}{|2|^k} \varphi\left(\frac{p}{2^{k+1}}, \frac{p}{2^{k+1}}\right) : k \geq 0 \text{ is an integer} \right\}, \tag{14}$$

for all $p \in \mathcal{A}$.

Proof. Initially, let us put $q = p$ in (12) which yields

$$\left\| g(2p) - \frac{1}{2} g(p) \right\| \leq \varphi(p,p),$$

for all $p \in \mathcal{A}$. Hence, we have

$$\left\| g(p) - \frac{1}{2} g\left(\frac{p}{2}\right) \right\| \leq \varphi\left(\frac{p}{2}, \frac{p}{2}\right), \tag{15}$$

for all $p \in \mathcal{A}$. Therefore using (15),we we have

$$\begin{aligned}
&\left\|\frac{1}{2^r} g\left(\frac{p}{2^r}\right) - \frac{1}{2^s} g\left(\frac{p}{2^s}\right)\right\| \\
&\quad \leq \max\left\{\left\|\frac{1}{2^r} g\left(\frac{p}{2^r}\right) - \frac{1}{2^{r+1}} g\left(\frac{p}{2^{r+1}}\right)\right\|, \dots, \left\|\frac{1}{2^{s-1}} g\left(\frac{p}{2^{s-1}}\right) - \frac{1}{2^s} g\left(\frac{p}{2^s}\right)\right\|\right\} \\
&\quad = \max\left\{\frac{1}{|2|^r}\left\|g\left(\frac{p}{2^r}\right) - \frac{1}{2} g\left(\frac{p}{2^{r+1}}\right)\right\|, \dots, \frac{1}{|2|^{s-1}}\left\|g\left(\frac{p}{2^{s-1}}\right) - \frac{1}{2} g\left(\frac{p}{2^s}\right)\right\|\right\} \\
&\quad \leq \max\left\{\frac{1}{|2|^r}\varphi\left(\frac{p}{2^{r+1}}, \frac{p}{2^{r+1}}\right), \dots, \frac{1}{|2|^{s-1}}\varphi\left(\frac{p}{2^s}, \frac{p}{2^s}\right)\right\} \\
&\quad = \frac{1}{|2|^r}\varphi\left(\frac{p}{2^{r+1}}, \frac{p}{2^{r+1}}\right) \quad \to 0 \quad \text{as} \quad r \to \infty.
\end{aligned} \tag{16}$$

By (16), it is evident that the sequence $\left\{\frac{1}{2^n} g\left(\frac{p}{2^n}\right)\right\}$ transforms into a Cauchy sequence. Owing to the completeness of $\mathcal{B}$, the sequence $\left\{\frac{1}{2^n} g\left(\frac{p}{2^n}\right)\right\}$ converges to a mapping $G : \mathcal{A} \longrightarrow \mathcal{B}$ defined by(13). Moreover, putting $r = 0$ and letting the limit $s \to \infty$ in (16), we arrive at (14). Next is to show that G is the unique reciprocal mapping satisfying (14). For this, let us assume that there occurs an additional mapping $H : \mathcal{A} \longrightarrow \mathcal{B}$ which satisfies (7) and (14). Then, we obtain

$$\begin{aligned}
&\|G(p) - H(p)\| \\
&\quad = \left\|\frac{1}{2^m} G\left(\frac{p}{2^m}\right) - \frac{1}{2^m} H\left(\frac{p}{2^m}\right)\right\| \\
&\quad \leq \max\left\{\left\|\frac{1}{2^m} G\left(\frac{p}{2^m}\right) - \frac{1}{2^m} g\left(\frac{p}{2^m}\right)\right\|, \left\|\frac{1}{2^m} g\left(\frac{p}{2^m}\right) - \frac{1}{2^m} H\left(\frac{p}{2^m}\right)\right\|\right\} \\
&\quad \leq \frac{1}{|2|^m}\varphi\left(\frac{p}{2^{m+1}}, \frac{p}{2^{m+1}}\right) \to 0 \quad \text{as} \quad m \to \infty,
\end{aligned}$$

for all $p \in \mathcal{A}$. Hence, we arrive at $G(p) = H(p)$ for all $p \in \mathcal{A}$, which completes the uniquess of G. Therefore, using (12), one can obtain that

$$\begin{aligned}
\left\|G(p+q) - \frac{G(p)G(q)}{G(p) + G(q)}\right\| &= \lim_{n\to\infty}\left\|\frac{1}{2^n}\left(g\left(\frac{p+q}{2^n}\right) - \frac{g\left(\frac{p}{2^n}\right) g\left(\frac{q}{2^n}\right)}{g\left(\frac{p}{2^n}\right) + g\left(\frac{q}{2^n}\right)}\right)\right\| \\
&\leq \lim_{n\to\infty}\left\|\frac{1}{2^n}\mu\left(\frac{1}{2} g\left(\frac{p+q}{2^{n+1}}\right)\right) - \frac{g\left(\frac{p}{2^n}\right) g\left(\frac{q}{2^n}\right)}{g\left(\frac{p}{2^n}\right) + g\left(\frac{q}{2^n}\right)}\right\| \\
&\qquad + \lim_{n\to\infty}\frac{1}{2^n}\varphi\left(\frac{p}{2^n}, \frac{q}{2^n}\right)
\end{aligned}$$

$$= \left\| \mu \left(\frac{1}{2} G \left(\frac{p+q}{2} \right) - \frac{G(p)G(q)}{G(p)+G(q)} \right) \right\|,$$

for all $p, q \in \mathcal{A}$. Therefore, we obtain

$$\left\| G(p+q) - \frac{G(p)G(q)}{G(p)+G(q)} \right\| \leq \left\| \mu \left(\frac{1}{2} G \left(\frac{p+q}{2} \right) - \frac{G(p)G(q)}{G(p)+G(q)} \right) \right\|,$$

for all $p, q \in \mathcal{A}$. Hence by Theorem 2.1, the mapping $G : \mathcal{A} \longrightarrow \mathcal{B}$ is reciprocal.

The following theorem is the other stability result of the inequality (5). The proof follows through idential arguments as in Theorem 3.1 and hence we exclude it.

Theorem 3.2. Let a function $\varphi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{R}^+$ satisfies the following condition

$$\lim_{n\to\infty} |2|^n \varphi(2^n p, 2^n q) = 0,$$

for all $p, q \in \mathcal{A}$. If the mapping g satisfies the inequality (12), then, a mapping $G : \mathcal{A} \longrightarrow \mathcal{B}$ exists which is unique and satisfies (7) with the condition that

$$\|g(p) - G(p)\| \leq \max \left\{ |2|^k \varphi\left(2^k p, 2^k p\right) : k \geq 0 \text{ is an integer} \right\},$$

for all $p \in \mathcal{A}$.

Corollary 3.3. Let $s \neq -1$ and $\lambda > 0$. Let the mapping g satisfies the following inequality

$$\left\| g(p+q) - \frac{g(p)g(q)}{g(p)+g(q)} \right\| \\ \leq \left\| \mu \left(\frac{1}{2} g \left(\frac{p+q}{2} \right) - \frac{g(p)g(q)}{g(p)+g(q)} \right) \right\| + \lambda \left(\|p\|^s + \|q\|^s \right),$$

for all $p, q \in \mathcal{A}$. Then, a mapping $G : \mathcal{A} \longrightarrow \mathcal{B}$ exists which is unique and satisfies (7) such that

$$\|g(p) - G(p)\| \leq \begin{cases} \frac{2\lambda}{|2|^s |2|^{k(s+1)}} \|p\|^s & \text{for } s < -1 \\ 2\lambda |2|^{k(s+1)} \|p\|^s & \text{for } s > -1, \end{cases}$$

for all $p \in \mathcal{A}$.

Corollary 3.4. Let $a, b \in \mathbb{R}$ with $s = a + b \neq -1$ and $\lambda > 0$. If the mapping g satisfies the following inequality

$$\left\| g(p+q) - \frac{g(p)g(q)}{g(p)+g(q)} \right\| \leq \left\| \mu \left(\frac{1}{2} g\left(\frac{p+q}{2} \right) - \frac{g(p)g(q)}{g(p)+g(q)} \right) \right\| + \lambda \left(\|p\|^a \|q\|^b \right),$$

for all $p, q \in \mathcal{A}$, then, a mapping $G : \mathcal{A} \longrightarrow \mathcal{B}$ exists which is unique with the condition that

$$\|g(p) - G(p)\| \leq \begin{cases} \frac{\lambda}{|2|^s |2|^{k(s+1)}} \|p\|^s & \text{for } \ s < -1 \\ \lambda |2|^{k(s+1)} \|p\|^s & \text{for } \ s > -1, \end{cases}$$

for all $p \in \mathcal{A}$.

Theorem 3.5. Let a function $\varphi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{R}^+$ with the condition $\varphi(u, v) \neq 0$ satisfying

$$\lim_{n\to\infty} |2|^n \frac{1}{\varphi\left(\frac{p}{2^n}, \frac{q}{2^n}\right)} = 0, \tag{17}$$

for all $p, q \in \mathcal{A}$. If the mapping g with $g(0) = \infty$ satisfies the inequality

$$\left\| \frac{2}{g\left(\frac{p+q}{2}\right)} - \frac{1}{g(p)} - \frac{1}{g(q)} \right\| \leq \left\| \mu \left(\frac{1}{g(p+q)} - \frac{1}{g(p)} - \frac{1}{g(q)} \right) \right\| + \frac{1}{\varphi(p,q)}, \tag{18}$$

for all $p, q \in \mathcal{A}$, then, a mapping $G : \mathcal{A} \longrightarrow \mathcal{B}$ defined by

$$G(p) = \lim_{n\to\infty} 2^n \frac{1}{g\left(\frac{p}{2^n}\right)}, \tag{19}$$

exists for all $p \in \mathcal{A}$. The mapping defined in (19) is unique and satisfies (8) with the condition that

$$\|g(p) - G(p)\| \leq \max \left\{ |2|^k \frac{1}{\varphi\left(\frac{p}{2^k}, \frac{p}{2^k}\right)} : k \geq 0 \text{ is an integer} \right\}, \tag{20}$$

for all $p \in \mathcal{A}$.

Proof. Initially, we put $q = 0$ in (18) to obtain

$$\left\| \frac{2}{g\left(\frac{p}{2}\right)} - \frac{1}{g(p)} \right\| \leq \frac{1}{\varphi(p,0)}, \tag{21}$$

for all $p \in \mathcal{A}$. Therefore, using (21), we have

$$\begin{aligned}
&\left\| \frac{2^k}{g\left(\frac{p}{2^k}\right)} - \frac{2^\ell}{g\left(\frac{1}{2^\ell}\right)} \right\| \\
&\qquad \leq \max\left\{ \left\| \frac{2^k}{g\left(\frac{p}{2^k}\right)} - \frac{2^{k+1}}{g\left(\frac{p}{2^{k+1}}\right)} \right\|, \ldots, \left\| \frac{2^{\ell-1}}{g\left(\frac{p}{2^{\ell-1}}\right)} - \frac{2^\ell}{g\left(\frac{p}{2^\ell}\right)} \right\| \right\} \\
&\qquad = \max\left\{ |2|^k \left\| \frac{1}{g\left(\frac{p}{2^k}\right)} - \frac{2}{g\left(\frac{p}{2^{k+1}}\right)} \right\|, \ldots, |2|^{\ell-1} \left\| \frac{1}{g\left(\frac{p}{2^{\ell-1}}\right)} - \frac{2}{g\left(\frac{p}{2^\ell}\right)} \right\| \right\} \\
&\qquad \leq \max\left\{ |2|^k \frac{1}{\varphi\left(\frac{p}{2^k}, 0\right)}, \ldots, |2|^{\ell-1} \frac{1}{\varphi\left(\frac{p}{2^{\ell-1}}, 0\right)} \right\} \\
&\qquad \leq |2|^k \frac{1}{\varphi\left(\frac{p}{2^k}, 0\right)} \quad \to 0 \quad \text{as} \quad k \to \infty. \qquad (22)
\end{aligned}$$

From the above, we notice that the sequence $\left\{2^n \frac{1}{g\left(\frac{p}{2^n}\right)}\right\}$ emerges as Cauchy for all $p \in \mathcal{A}$. Since $\mathcal{B}$ is complete, the sequence $\left\{2^n \frac{1}{g\left(\frac{p}{2^n}\right)}\right\}$ converges to a mapping $G : \mathcal{A} \longrightarrow \mathcal{B}$ defined in (19). On the other hand, by taking $k = 0$ and taking the limit $\ell \to \infty$ in (22), we arrive at (20). The enduring part of the proof is achieved via analogous reasoning as in Theorem 3.1.

The subsequent result is the other stability of the inequality (6). Since the proof is akin to Theorem 3.5, we furnish only the statement.

Theorem 3.6. Let a function $\varphi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathbb{R}^+$ with $\varphi(p, q) \neq 0$ satisfies the condition

$$\lim_{n\to\infty} \frac{1}{|2|^{n+1}} \frac{1}{\varphi\left(\frac{p}{2^n}, \frac{q}{2^n}\right)} = 0,$$

for all $p, q \in \mathcal{A}$. If the mapping g with $g(0) = \infty$ satisfies (18), then, a unique mapping $G : \mathcal{A} \longrightarrow \mathcal{B}$ exists and satisfies *(8)* with the condition that

$$\|g(p) - G(p)\| \leq \max\left\{ \frac{1}{|2|^{k+1}} \frac{1}{\varphi\left(2^k p, 2^k p\right)} : k \geq 0 \text{ is an integer} \right\},$$

for all $p \in \mathcal{A}$.

Corollary 3.7. Let $s \neq -1$ and $\lambda > 0$. If the mapping g satisfies the following inequality

$$\left\|\frac{2}{g\left(\frac{p+q}{2}\right)}-\frac{1}{g(p)}-\frac{1}{g(q)}\right\| \leq \left\|\mu\left(\frac{1}{g(p+q)}-\frac{1}{g(p)}-\frac{1}{g(q)}\right)\right\| + \lambda\left(\|p\|^s+\|q\|^s\right),$$

for all $p,q \in \mathcal{A}$, then, a mapping $G : \mathcal{A} \longrightarrow \mathcal{B}$ exists which is unique such that

$$\|g(p)-G(p)\| \leq \begin{cases} \frac{2\lambda}{|2|^s|2|^{k(s+1)}}\|p\|^s & \text{for } s<-1 \\ \frac{2\lambda}{|2||2|^{k(s+1)}}\|p\|^s & \text{for } s>-1, \end{cases}$$

for all $p \in \mathcal{A}$.

Corollary 3.8. Let $a, b \in \mathbb{R}$ with $s = a + b \neq -1$ and $\lambda > 0$. Let the mapping g satisfies the following inequality

$$\left\|\frac{2}{g\left(\frac{p+q}{2}\right)}-\frac{1}{g(p)}-\frac{1}{g(q)}\right\| \leq \left\|\mu\left(\frac{1}{g(p+q)}-\frac{1}{h(p)}-\frac{1}{g(q)}\right)\right\| + \lambda\left(\|p\|^a\|q\|^b\right),$$

for all $p,q \in \mathcal{A}$. Then, a unique mapping $G : \mathcal{A} \longrightarrow \mathcal{B}$ exists with the condition that

$$\|g(p)-G(p)\| \leq \begin{cases} \frac{\lambda}{|2|^s|2|^{k(s+1)}}\|p\|^s & \text{for } s<-1 \\ \frac{2\lambda}{|2||2|^{k(s+1)}}\|p\|^s & \text{for } s>-1, \end{cases}$$

for all $p \in \mathcal{A}$.

4 Conclusion

We close this study with a conclusion of validity of stability results of inequalities (5) and (6) and the equations associated with them. The stability results pertaining to various μ-functional inequalities arising from additive, quadratic, cubic and quartic mappings are obtained by many mathematicians. This study is the foremost attempt that we have concerned with new multiplicative inverse μ-functional inequalities and multiplicative inverse μ-functional equations to investigate their several stabilities pertinent to the theory of Ulam's approximation. It is interesting to observe from the outcomes attained in this study that the stabilities of inequalities (5) and (6) are still valid over non-Archimedean spaces. The results obtained in this study indicate that the functional inequalities (5) and (6) can be used to approximate the solutions of equations (2) and (3). This would pave a different and fascinating direction to investigate several forms of μ-functional inequalities and equations for their stabilities.

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