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Stochastic ordering of discrete multivariate distributions. Algorithm in C++ with applications in the comparison of number of claims and extremes order statistics

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Abstract

In this article we present a stochastic ordering verification algorithm between multivariate discrete distributions implemented in the C++ programming language. This algorithm is essential in problems of finding the optimal portfolio when dealing with discrete distributions.

1 Introduction

Let $N = (N_1, ..., N_k)$ a discrete random vector with the probability mass function $p_N(n) = P(N = n)$, $n \in \mathbb{R}^k$.

We denote $\mathbb{Z}_{\geq j} = \{k \in \mathbb{Z} : \ k \geq j\}$ and $A^j = \underbrace{A \times A \times ... \times A}_{j \text{ sets}}$.

Robe-Voinea and Vernic (2016) presented a proof of a formula for probability function of the corresponding multivariate compound distribution. They considered $N = N_1 + ... + N_k$ where:

- N satisfies the Panjer-type recursion $P(N = n) = \left(a + \frac{b}{n}\right) P(N = n - 1)$, $n \in \mathbb{Z}_{\geq 1}$, $a, b \in \mathbb{R}$;

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- $P\left(N_1 = n_1, ..., N_k = n_k \middle|\ N = n\right) = \frac{n!}{\prod\limits_{i=1}^{k} n_k!} \prod\limits_{i=1}^{k} p_i^{n_i},\ (n_1, ..., n_k) \in \mathbb{Z}_{\geq 1}^k,$
$\sum\limits_{i=1}^{k} n_i = n,\ (p_1, ..., p_k) \in (0,1)^k.$

Starting from the second, the first stochastic ordering problem is introduced:

$$P_1 : \textit{When } P\left(N_1 + ... + N_k \geq t\right) \leq P\left(M_1 + ... + M_k \geq t\right)\ \forall\ t \in \mathbb{Z}_{\geq 0}^k \textit{ where we know } p_{(N_1,...,N_k)} \textit{ and } p_{(M_1,...,M_k)}\textit{?}$$

Now, we present another often encountered situation. Catana (2021) generalized Theorems 2.1 and 2.2 from Nadarajah et al. (2017) using a family of multivariate Pareto distributions with the survival function

$P\left(X > x\right) = \left(\sum\limits_{i=1}^{k} \frac{x_i}{b_i} - k + 1\right)^{-a},\ x \in \prod\limits_{i=1}^{k} \left(b_i, \infty\right),\ b \in (0, \infty)^k,\ a \in (0, \infty).$

The following problems was studied:

$$P_2 : \textit{When } P\left(\min\left(X_1, ..., X_k\right) > t\right) \leq P\left(\min\left(Y_1, ..., Y_k\right) > t\right)\ \forall\ t \in \mathbb{R} \textit{ where we know } p_{(N_1,...,N_k)} \textit{ and } p_{(M_1,...,M_k)}\textit{?}$$

$$P_3 : \textit{When } P\left(\max\left(X_1, ..., X_k\right) > t\right) \leq P\left(\max\left(Y_1, ..., Y_k\right) > t\right)\ \forall\ t \in \mathbb{R} \textit{ where we know } p_{(N_1,...,N_k)} \textit{ and } p_{(M_1,...,M_k)}\textit{?}$$

For the studies of stochastic orderings of smallest/largest claim amounts important results were published in: Das and Kayal (2021), Das et al. (2021), Nadeb et al. (2018, 2020). Also, interesting results related to recently discovered stochastic orderings were published in Raducan et al. (2022) and Radulescu et al. (2021).

Important properties of some generalized distributions that can be used in problems with extremes order statistics can be studied in: Bancescu, (2018), Baca and Vernic (2022), Catana (2022), Vernic (2005).

The results discovered so far in the specified works are based on inequalities using the real analysis in $\mathbb{R}^k$ in the case of distributions with F or F^* continuous. However, in the case of discrete multivariate distributions, F and F^* are discontinuous. In addition, there is a large number of inequalities that must be studied.

The structure of the article is as follows: in Section 2 we present some definitions and results regarding the multivariate stochastic ordering that will be used, Section 3 presents the algorithm in C++ for checking stochastic orderings between two discrete multivariate distributions. Also, in this section we present the data structures used, the idea of the algorithm and why it is useful. Section 4 presents applications of the algorithm in the comparison of number of claims and extremes order statistics and Section 5 presents the conclusions of the article.

2 Preliminaries

The inequality in P_1 represents the stochastic ordering between $N_1 + ... + N_k$ and $M_1 + ... + M_k$.

We specify definitions and results that will be used in this article related to the stochastic ordering of multivariate distributions. In the multivariate case, it is necessary to know how we can compare two points. The following definition is often used, although we cannot compare any two points with it.

Definition 2.1. *(Shaked and Shanthikumar (2007)) Let x, $y \in \mathbb{R}^k$. We say x is smaller (greater) than y (and denote $x \leq (\geq) y$) if $x_i \leq (\geq) y_i$ $i = \overline{1,k}$.*

Once the ordering between two points is introduced, we can also define the minimum and maximum between two points, given by Definition 2.2:

Definition 2.2. *(Shaked and Shanthikumar (2007)) Let $x, y \in \mathbb{R}^k$. We denote*

$$\min(x, y) = (\min(x_1, y_1), ..., \min(x_k, y_k))$$

and

$$\max(x, y) = (\max(x_1, y_1), ..., \max(x_k, y_k)) .$$

The following definition will be useful in defining the usual multivariate stochastic ordering between two distributions.

Definition 2.3. *(Shaked and Shanthikumar (2007)) A set $C \subset \mathbb{R}^k$ is called increasing if*

$$\forall\ x \in C\ \forall\ y \in \mathbb{R}^k \text{ then } x \leq y \Longrightarrow y \in C.$$

Unlike the univariate case where the stochastic ordering had only one definition, in the multivariate case due to the fact that any two points on the right are comparable, in the multivariate case three definitions appear, one for the usual stochastic ordering (the one that generalizes the usual ordering) and the others two weaker.

Definition 2.4. *(Shaked and Shanthikumar (2007)) Let X, $Y : \Omega \to \mathbb{R}^k$ be two random vectors. We say that X is said to be smaller than Y in the*

(i) usual stochastic order (written as $X \prec_{st} Y$) if

$$P(X \in C) \leq P(Y \in C)\ \forall\ C \subset \mathbb{R}^k;$$

(ii) *weak stochastic order (written as* $X \prec_{wst} Y$*) if*

$$F_X^*(x) \leq F_Y^*(x)\ \forall\ x \in \mathbb{R}^k;$$

(iii) *dual weak stochastic order (written as* $X \prec_{dwst} Y$*) if*

$$F_X(x) \geq F_Y(x)\ \forall\ x \in \mathbb{R}^k.$$

We now define the increasing functions $u : \mathbb{R}^k \to \mathbb{R}$:

Definition 2.5. *(Shaked and Shanthikumar (2007)) A function* $u : \mathbb{R}^k \to \mathbb{R}$ *is called increasing if* $\forall\, x, y \in \mathbb{R}^k$ *then* $x \leq y \Longrightarrow u(x) \leq u(y)$.

The following theorem presents a characterization of multivariate stochastic ordering:

Theorem 2.6. *(Shaked and Shanthikumar (2007)) Let* $X, Y : \Omega \to \mathbb{R}^k$ *be two random vectors. Then*

$$X \prec_{st} Y \Longleftrightarrow Eu(X) \leq Eu(Y)\ \forall\ u : \mathbb{R}^k \to \mathbb{R} \text{ increasing function.}$$

Theorem 2.7 presents a characterization of multivariate weak stochastic orders:

Theorem 2.7. *(Shaked and Shanthikumar (2007), Theorem 6.G.15., p. 315) Let* $X, Y : \Omega \to \mathbb{R}^k$ *be two positive random vectors. Then*

$$(i)\ X \prec_{wst} Y \Longleftrightarrow \min(\alpha_1 X_1, ..., \alpha_k X_k) \prec_{st} \min(\alpha_1 Y_1, ..., \alpha_k Y_k)$$

$\forall\ \alpha_1, ..., \alpha_k \in (0, \infty)$.

$$(ii)\ X \prec_{dwst} Y \Longleftrightarrow \max(\alpha_1 X_1, ..., \alpha_k X_k) \prec_{st} \max(\alpha_1 Y_1, ..., \alpha_k Y_k)$$

$\forall\ \alpha_1, ..., \alpha_k \in (0, \infty)$.

Remark 2.8. *(Shaked and Shanthikumar (2007))* $X \prec_{st} Y \Longrightarrow X \prec_{wst} Y$ *and* $X \prec_{st} Y \Longrightarrow X \prec_{dwst} Y$.

We note:

$U_a = \left\{x \in \mathbb{R}^k :\ x \geq a,\ x \neq a\right\}$,

$\overline{U_a} = \left\{x \in \mathbb{R}^k :\ x \geq a\right\}$,

$\mathcal{V} = \left\{U_a :\ a \in \mathbb{R}^k\right\}$,

$\mathcal{V}_\sigma$ the set of all countable reunions of $U_a \in \mathcal{V}$.

Catana (2019) proved a fundamental equivalence of stochastic orderings of multivariate distributions:

Theorem 2.9. *(Catana (2019), Theorem 3.4) Let* $X, Y : \Omega \to \mathbb{R}^k$ *be two random vectors. Then*

$$X \prec_{st} Y \Longleftrightarrow P(X \in C) \le P(Y \in C)\ \forall\ C \in \mathcal{V}_\sigma.$$

These results show us that the verification of multivariate stochastic ordering is reduced to a particular class of increasing sets. The following proposition is a direct application of this theorem in the case of discrete multivariate distributions:

Proposition 2.10. *(Catana (2019), Proposition 3.5) Let the multivariate random vectors* $X, Y : \Omega \to \mathbb{R}^k$, *with* $X \sim \sum_{i=1}^{n} p_i \delta_{a_i}$ *and* $Y \sim \sum_{i=1}^{m} q_i \delta_{b_i}$. *Then*

$$X \prec_{st} Y \Longleftrightarrow P(X \in C) \le P(Y \in C)\ \forall\ C = \cup_{x \in M} \overline{U_x}\ ,$$

with $M \subset \{a_1, ..., a_n, b_1, ..., b_m\}$ *set of mutual not comparable points or singleton.*

3 The algorithm

The algorithm is based on the Proposition 2.10. To represent a point in any dimension we use the vector data structure (the dimension is read). In this algorithm we have implemented a function that decides if $x \le y$, $x, y \in \mathbb{R}^k$.

In the algorithm, after reading the size of the working space (let us consider the dimension$= k$), a point in $\mathbb{R}^k$ will be represented by a list-type vector and a set of points in $\mathbb{R}^k$ will be represented by a list-type vector of vectors.

We want to verify if $X \prec_{st} Y$. The algorithm forms all sets of mutually incomparable points (or singletons) M then for each set it calculates $P(X \in \cup_{x \in M} \overline{U_x})$ and $P(Y \in \cup_{x \in M} \overline{U_x})$ and checks if $P(X \in \cup_{x \in M} \overline{U_x}) \le P(Y \in \cup_{x \in M} \overline{U_x})$.

If for each such set M found $P(X \in \cup_{x \in M} \overline{U_x}) \le P(Y \in \cup_{x \in M} \overline{U_x})$ then $X \prec_{st} Y$.

If it finds a set T such that $P(X \in \cup_{x \in T} \overline{U_x}) > P(Y \in \cup_{x \in T} \overline{U_x})$ then the algorithm stops and $X \not\prec_{st} Y$.

The advantage of using the algorithm based on Proposition 2.10 is the finite and in practice quite small number of checks that the computer makes to determine if $X \prec_{st} Y$.

Algorithm 3.1.

```
#include <iostream>
#include <stdio.h>
#include <fstream>
#include <vector>
```

```
using namespace std;
ifstream fin("points.in");
ofstream fout("results.out");
vector<vector<double>> A,B,C;
vector<double> a;
double p,x,s1,s2;
int n,m,M[100],dim,q,k,d;
int inequality_less_than(vector<double> k1, vector<double> k2)
{
int z=1;
for(int i=0;i<dim;i++)
{
if(k1.at(i)>k2.at(i))
{
z=0;
break;
}
}
return z;
}
void init()
{
int U = 10;
int T = 100;
fin>>dim;
fin>>n;
fin>>m;
for(int i=0;i<n;i++)
{
for(int j=0;j<dim;j++)
{
fin>>x;
a.push_back(x);
}
fin>>p;
a.push_back(p);
A.push_back(a);
a.clear();
}
for(int i=0;i<m;i++)
{
```

```
for(int j=0;j<dim;j++)
{
fin>>x;
a.push_back(x);
}
fin>>p;
a.push_back(p);
B.push_back(a);
a.clear();
}
for(int i=0;i<n;i++)
C.push_back(A.at(i));
for(int i=n;i<n+m;i++)
C.push_back(B.at(i-n));
for(int i=0;i<n+m;i++)
{
M[i]=i;
}
}
int verify(int data[], int r)
{
int q=1;
if(r>1)
for(int i=0;i<r;i++)
for(int j=i+1;j<r;j++)
if((inequality_less_than(C.at(data[i]),C.at(data[j]))==1)||
(inequality_less_than(C.at(data[j]),C.at(data[i]))==1))
{
q=0;
break;
}
return q;
}
int verify_dominance(int data[],int r)
{
int q;
double s1=0,s2=0;
for(int i=0;i<n;i++)
{
q=0;
for(int j=0;j<r;j++)
```

```
if(inequality_less_than(C.at(data[j]),A.at(i))==1)
{
q=1;
break;
}
if(q==1)
s1=s1+A.at(i).at(dim);
}
for(int i=0;i<m;i++)
{
q=0;
for(int j=0;j<r;j++)
if(inequality_less_than(C.at(data[j]),B.at(i))==1)
{
q=1;
break;
}
if(q==1)
s2=s2+B.at(i).at(dim);
}
fout<<"probability_X="<<s1<<" and "<<"probability_Y="<<s2<<
endl;
if(s1>s2)
k=0;
}
void combinationUtil(int M[], int n, int r, int index, int data[], int i);
void printCombination(int M[], int n, int r)
{
int data[r];
combinationUtil(M, n, r, 0, data, 0);
}
void combinationUtil(int M[], int n, int r, int index, int data[], int i)
{
if (index == r)
{
if(verify(data,r)==1)
{
fout<<"For ";
for (int j = 0; j < r; j++)
{
fout<<"(";
```

```
for(int i=0;i<dim-1;i++)
fout<<C.at(data[j]).at(i)<<",";
fout<<C.at(data[j]).at(dim-1);
fout<<") ";
}
fout<<"we have ";
verify_dominance(data,r);
}
return;
}
if (i >= n)
{
return;
}
data[index] = M[i];
combinationUtil(M, n, r, index + 1, data, i + 1);
combinationUtil(M, n, r, index, data, i + 1);
}
int main()
{
init();
k=1;
for(int r=1;r<=n+m;r++)
{
printCombination(M, n+m, r);
fout<<endl;
}
if(k==1)
fout<<"X is stochastic dominated by Y";
else
fout<<"X is not stochastic dominated by Y";
return 0;
}
```

4 The applications

We now analyze how we can apply the algorithm.

Let $X \sim \sum_{i=1}^{n} p_i\delta_{a_i}$ and $Y \sim \sum_{i=1}^{m} q_i\delta_{b_i}$ k-dimensional random vectors,

$\{a_i\}_{i=\overline{1,n}}$, $\{b_i\}_{i=\overline{1,m}}$, $\{p_i\}_{i=\overline{1,n}}$, $\{q_i\}_{i=\overline{1,m}}$ randomly generated.
We want to determine if $X \prec_{st} Y$.
We simulated two situations to compare with $k, n, m \in [0,9] \cap \mathbb{Z}_{\geq 0}$, $\{a_i\}_{i=\overline{1,n}}$, $\{b_i\}_{i=\overline{1,m}} \subset [0,9] \cap \mathbb{Z}_{\geq 0}$ and $\{p_i\}_{i=\overline{1,n}}$, $\{q_i\}_{i=\overline{1,m}} \subset (0,1)$ with one decimal.

In the first situation we obtain:

$k = 3$, $n = 2$, $m = 3$,
$a_1 = (0,0,1)$, $a_2 = (3,5,2)$,
$b_1 = (1,0,1)$, $b_2 = (2,1,4)$, $b_3 = (4,5,6)$,
$p_1 = 0.5$, $p_2 = 0.5$,
$q_1 = 0.2$, $q_2 = 0.2$, $q_3 = 0.6$.

We run the program and we obtain:

For (0,0,1) we have probability_X=1 and probability_Y=1
For (3,5,2) we have probability_X=0.5 and probability_Y=0.6
For (1,0,1) we have probability_X=0.5 and probability_Y=1
For (2,1,4) we have probability_X=0 and probability_Y=0.8
For (4,5,6) we have probability_X=0 and probability_Y=0.6
For (3,5,2) (2,1,4) we have probability_X=0.5 and probability_Y=0.8
X is stochastic dominated by Y

Thus $X \prec_{st} Y$.

In the second situation we obtain:

$k = 8$, $n = 4$, $m = 2$,
$a_1 = (5,3,1,5,2,3,7,9)$,
$a_2 = (0,4,5,0,0,1,0,0)$,
$a_3 = (1,0,1,2,2,9,8,5)$,
$a_4 = (3,1,4,4,5,5,5,9)$,
$b_1 = (4,5,6,5,0,1,0,1)$,
$b_2 = (5,8,9,9,9,5,1,1)$,
$p_1 = 0.2$, $p_2 = 0.4$, $p_3 = 0.2$, $p_4 = 0.2$,
$q_1 = 0.6$, $q_2 = 0.4$.

We run the program and we obtain:

For (5,3,1,5,2,3,7,9) we have probability_X=0.2 and probability_Y=0
For (0,4,5,0,0,1,0,0) we have probability_X=0.4 and probability_Y=1

For (1,0,1,2,2,9,8,5) we have probability_X=0.2 and probability_Y=0
For (3,1,4,4,5,5,5,9) we have probability_X=0.2 and probability_Y=0
For (4,5,6,5,0,1,0,1) we have probability_X=0 and probability_Y=1
For (5,8,9,9,9,5,1,1) we have probability_X=0 and probability_Y=0.4
For (5,3,1,5,2,3,7,9) (0,4,5,0,0,1,0,0) we have probability_X=0.6 and probability_Y=1
For (5,3,1,5,2,3,7,9) (1,0,1,2,2,9,8,5) we have probability_X=0.4 and probability_Y=0
For (5,3,1,5,2,3,7,9) (3,1,4,4,5,5,5,9) we have probability_X=0.4 and probability_Y=0
For (5,3,1,5,2,3,7,9) (4,5,6,5,0,1,0,1) we have probability_X=0.2 and probability_Y=1
For (5,3,1,5,2,3,7,9) (5,8,9,9,9,5,1,1) we have probability_X=0.2 and probability_Y=0.4
For (0,4,5,0,0,1,0,0) (1,0,1,2,2,9,8,5) we have probability_X=0.6 and probability_Y=1
For (0,4,5,0,0,1,0,0) (3,1,4,4,5,5,5,9) we have probability_X=0.6 and probability_Y=1
For (1,0,1,2,2,9,8,5) (3,1,4,4,5,5,5,9) we have probability_X=0.4 and probability_Y=0
For (1,0,1,2,2,9,8,5) (4,5,6,5,0,1,0,1) we have probability_X=0.2 and probability_Y=1
For (1,0,1,2,2,9,8,5) (5,8,9,9,9,5,1,1) we have probability_X=0.2 and probability_Y=0.4
For (3,1,4,4,5,5,5,9) (4,5,6,5,0,1,0,1) we have probability_X=0.2 and probability_Y=1
For (3,1,4,4,5,5,5,9) (5,8,9,9,9,5,1,1) we have probability_X=0.2 and probability_Y=0.4
For (5,3,1,5,2,3,7,9) (0,4,5,0,0,1,0,0) (1,0,1,2,2,9,8,5) we have probability_X=0.8 and probability_Y=1
For (5,3,1,5,2,3,7,9) (0,4,5,0,0,1,0,0) (3,1,4,4,5,5,5,9) we have probability_X=0.8 and probability_Y=1
For (5,3,1,5,2,3,7,9) (1,0,1,2,2,9,8,5) (3,1,4,4,5,5,5,9) we have probability_X=0.6 and probability_Y=0
For (5,3,1,5,2,3,7,9) (1,0,1,2,2,9,8,5) (4,5,6,5,0,1,0,1) we have probability_X=0.4 and probability_Y=1
For (5,3,1,5,2,3,7,9) (1,0,1,2,2,9,8,5) (5,8,9,9,9,5,1,1) we have probability_X=0.4 and probability_Y=0.4
For (5,3,1,5,2,3,7,9) (3,1,4,4,5,5,5,9) (4,5,6,5,0,1,0,1) we have probability_X=0.4 and probability_Y=1
For (5,3,1,5,2,3,7,9) (3,1,4,4,5,5,5,9) (5,8,9,9,9,5,1,1) we have

probability_X=0.4 and probability_Y=0.4
For (0,4,5,0,0,1,0,0) (1,0,1,2,2,9,8,5) (3,1,4,4,5,5,5,9) we have
probability_X=0.8 and probability_Y=1
For (1,0,1,2,2,9,8,5) (3,1,4,4,5,5,5,9) (4,5,6,5,0,1,0,1) we have
probability_X=0.4 and probability_Y=1
For (1,0,1,2,2,9,8,5) (3,1,4,4,5,5,5,9) (5,8,9,9,9,5,1,1) we have
probability_X=0.4 and probability_Y=0.4
For (5,3,1,5,2,3,7,9) (0,4,5,0,0,1,0,0) (1,0,1,2,2,9,8,5) (3,1,4,4,5,5,5,9)
we have probability_X=1 and probability_Y=1
For (5,3,1,5,2,3,7,9) (1,0,1,2,2,9,8,5) (3,1,4,4,5,5,5,9) (4,5,6,5,0,1,0,1)
we have probability_X=0.6 and probability_Y=1
For (5,3,1,5,2,3,7,9) (1,0,1,2,2,9,8,5) (3,1,4,4,5,5,5,9) (5,8,9,9,9,5,1,1)
we have probability_X=0.6 and probability_Y=0.4
X is not stochastic dominated by Y

Thus $X \not\prec_{st} Y$.

If $X \prec_{st} Y$ then:

1) from Theorem 2.6 (taking $u(x) = (x_1 + x_2 + ... + x_k) \cdot 1_{x \in [0,\infty)^k}(x)$) we obtain $X_1 + ... + X_n \prec_{st} Y_1 + ... + Y_m$. Thus

$$P(N_1 + ... + N_k \geq t) \leq P(N_1 + ... + M_k \geq t) \ \forall\ t \in \mathbb{Z}^k_{\geq 0};$$

2) from Theorem 2.7 (taking $u(x) = \min(\alpha_1 x_1, ..., \alpha_k x_k)$, $\alpha_1, ..., \alpha_k \in (0,\infty)$) we obtain
$\min(\alpha_1 X_1, ..., \alpha_k X_k) \prec_{st} \min(\alpha_1 Y_1, ..., \alpha_k Y_k)$. Thus

$$P(\min(\alpha_1 X_1, ..., \alpha_k X_k) > t) \leq P(\min(\alpha_1 Y_1, ..., \alpha_k Y_k) > t) \ \forall\ t \in \mathbb{R};$$

3) from Theorem 2.7 (taking $u(x) = \max(\alpha_1 x_1, ..., \alpha_k x_k)$, $\alpha_1, ..., \alpha_k \in (0,\infty)$) we obtain
$\max(\alpha_1 X_1, ..., \alpha_k X_k) \prec_{st} \max(\alpha_1 Y_1, ..., \alpha_k Y_k)$. Thus

$$P(\max(\alpha_1 X_1, ..., \alpha_k X_k) > t) \leq P(\max(\alpha_1 Y_1, ..., \alpha_k Y_k) > t) \ \forall\ t \in \mathbb{R}.$$

If $X \not\prec_{st} Y$ then we cannot give an answer to the problems P_1, P_2, P_3 on this way but one can analyze the univariate ordering in P_1, the weak and dual weak orderings in P_2 and P_3.

5 Conclusions

In this article, the problem of stochastic ordering of discrete multivariate distributions was treated using an algorithm implemented in C++. Due to the number of increasing sets for which the inequality describing the stochastic ordering should be verified, it was necessary to implement an algorithm. This algorithm can simplify many analysis problems of F_X^* maximization at a point or $Eu(X)$ (increasing u) according to the parameters of the random vector X.

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