DOI: 10.2478/auom-2024-0005
An. St. Univ. Ovidius Constanta **Vol. 32(1),2024, 85–104**

Closure operators on hoops

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Abstract

In this article, we study relationships between closure operators and hoops. We investigate the properties of closure operators and hoop-homomorphism on hoops. We show that the image of a closure operator on a hoop is isomorphic to a quotient hoop. In addition, we define the notion of closure operator on ideals of hoop and investigate some properties of it and some related results are proved. We define proper closer operators on ideals of hoop and we show that the set of all proper closure operators on hoops makes a bounded lattice by some operations.

1 Introduction

In recent years, due to the development of artificial intelligence and the use of logical algebraic structures in this field, the study of these structures has become particularly important and has attracted the attention of many mathematicians. On the other hand, considering the types of logical algebraic structures and their relationship with each other, the similarities and limitations of each of them have caused each mathematician to study different concepts on these algebraic structures according to their interest. For example, one of the topics of interest to mathematicians is the subject of specific sub-algebras in logical algebraic structures, including filters and ideals. In logical algebraic structures, the concept of filters has been studied more and ideals have been less considered by mathematicians. One of the logical algebras that is more

Key Words: Hoop, ideal, maximal and prime ideals, closure operators.
2010 Mathematics Subject Classification: Primary 06B99; Secondary 03G25.
Received: 28.10.2022
Accepted: 15.03.2023

considered these days due to its relationship with MV-algebras and L-algebras is hoop.

Hoops are introduced by Bosbach in [8, 9]. In recent years, many mathematicians have studied this algebraic structure from different perspectives such as ideals, filters, relationships with other algebraic structures, etc., and good results have been achieved in this regard which can be found in [2, 4, 8, 9, 14]. Given the importance of the concept of ideals and the results obtained in this field on other algebraic structures, in [1], Aaly and Borzooei defined and characterized the notions of (implicative, maximal, prime) ideal in hoops. They investigated the relation between them and proved that every maximal implicative ideal of a $\vee$-hoop (a hoop with join operation) with (DNP) ($p^{\sim\sim} = p$, where $p \to 0$) is a prime one. Also, they defined a congruence relation on hoops by ideals and studied the quotient that is made by it. In addition, they showed that an ideal is maximal if and only if the quotient hoop is a simple MV-algebra. In [5], Borzooei and Aaly, defined the notion of minimal prime ideals of hoops and investigated some properties of them. Then, by using the notion of annihilators, they studied the relation between minimal prime ideals and annihilators. In [6], Borzooei and et al., by using the notion of ideal of hoop, defined a new congruence relation on hoop and made a new quotient structure. Also, they defined the notion of product ideals in hoops and investigated some properties of it and studied some theorems about prime, maximal and product of them. By using these notions, they defined the concept of nilpotent ideals on hoop and by example they showed that intersection of all maximal ideals is not nilpotent. Then they provided a condition for intersection of all maximal ideals to be nilpotent.

In this article, we study relationships between closure operators and hoops. We investigate the properties of closure operators and hoop-homomorphism on hoops. We show that the image of a closure operator on a hoop is isomorphic to a quotient hoop. In addition, we define the notion of closure operator on ideals of hoop and investigate some properties of it and some related results are proved. We define proper closer operators on ideals of hoop and we show that the set of all proper closure operators on hoops makes a bounded lattice by some operations.

2 Preliminaries

An algebra $(\mathcal{A}, \odot, \to, 1)$ is said to be a **hoop**, where $(\mathcal{A}, \odot, 1)$ is a commutative monoid and for any $p, q, r \in \mathcal{A}$ we have

(Hoop1) $p \to p = 1$,

(Hoop2) $p \odot (p \to q) = q \odot (q \to p)$,

(Hoop3) $p \to (q \to r) = (p \odot q) \to r$.

Define an order $\leq$ on $\mathcal{A}$ by $p \leq q$ iff $p \to q = 1$, which $(\mathcal{A}, \leq)$ is a Poset. A ***bounded hoop***, is a hoop with the least element 0 where for all $p \in \mathcal{A}$, $0 \leq p$, then we can define a unary operation $\sim$ where $p^{\sim} = p \to 0$, for $p \in \mathcal{A}$ and we said that $\mathcal{A}$ has (DNP) property if $p^{\sim\sim} = p$, for all $p \in \mathcal{A}$. Also, we can define a binary operation $\vee$ on a hoop $\mathcal{A}$ such that for any $p, q \in \mathcal{A}$, $p \vee q = ((p \to q) \to q) \wedge ((q \to p) \to p)$. If $\vee$ is a join operation on $\mathcal{A}$, then $\mathcal{A}$ is a $\vee$***-hoop***, where $(\mathcal{A}, \wedge, \vee)$ is a distributive lattice (see [8, 9]).

Note. From now on, we set $\mathcal{A}$ as a hoop such as $(\mathcal{A}, \odot, \to, 1)$.

Proposition 2.1. [10] *We have the next properties, for all $p, q, r \in \mathcal{A}$:*
(i) $(\mathcal{A}, \leq)$ is a meet-semilattice, with $p \wedge q = p \odot (p \to q)$;
(ii) $p \odot q \leq r$ iff $p \leq q \to r$;
(iii) $p \odot q \leq p, q$ and $p \leq q \to p$;
(iv) $p \to 1 = 1$ and $1 \to p = p$;
(v) $p \leq q \to (p \odot q)$;
(vi) $p \to q \leq (q \to r) \to (p \to r)$, $p \to q \leq (r \to p) \to (r \to q)$ and $p \to q \leq (p \odot r) \to (q \odot r)$;
(vii) $p \leq q$ implies $p \odot r \leq q \odot r$, $r \to p \leq r \to q$ and $q \to r \leq p \to r$;
(viii) In any bounded hoop we get $p \leq p^{\sim\sim}$, $p \odot p^{\sim} = 0$ and $p^{\sim\sim\sim} = p^{\sim}$;
(ix) In any $\vee$-hoop, for any $n \in \mathbb{N}$, we get $(p \vee q)^n \to r = \bigwedge\{(x_1 \odot x_2 \odot \cdots \odot x_n) \to r \mid x_i \in \{p, q\}\}$;
(x) In any $\vee$-hoop, we obtain $p \odot (q \vee r) = (p \odot q) \vee (p \odot r)$.
(xi) $p \to (q \wedge r) = (p \to q) \wedge (p \to r)$.

Consider $\emptyset \neq \mathbb{F} \subseteq \mathcal{A}$ is said to be a ***filter*** of $\mathcal{A}$ if for any $p, q \in \mathcal{A}$,
$(F1)$ $p, q \in \mathbb{F}$ implies $p \odot q \in \mathbb{F}$,
$(F2)$ $p \leq q$ and $p \in \mathbb{F}$ imply $q \in \mathbb{F}$.
The set $\mathcal{F}(\mathcal{A})$ contains all filters of $\mathcal{A}$. Clearly, $1 \in \mathbb{F}$, for each $\mathbb{F} \in \mathcal{F}(\mathcal{A})$. $\mathbb{F} \in \mathcal{F}(\mathcal{A})$ is ***proper*** if $\mathbb{F} \neq \mathcal{A}$. Obviously, if $\mathbb{F}$ is a proper filter of $\mathcal{A}$, then $0 \notin \mathbb{F}$ where $\mathcal{A}$ is bounded.

Let $\mathbb{F} \in \mathcal{F}(\mathcal{A})$. Then for any $p, q \in \mathcal{A}$, define the relation $\sim_{\mathbb{F}}$ by

$$p \sim_{\mathbb{F}} q \;\Leftrightarrow\; p \to q \in \mathbb{F} \text{ and } q \to p \in \mathbb{F}.$$

Then we can see that the relation $\sim_{\mathbb{F}}$ is a congruence relation on $\mathcal{A}$ and the algebraic structure $(\frac{\mathcal{A}}{\mathbb{F}}, \otimes, \rightsquigarrow, \mathbb{F})$ is a hoop where

$$[p] \otimes [q] = [p \odot q], \quad [p] \rightsquigarrow [q] = [p \to q],$$

for any $[p], [q] \in \frac{\mathcal{A}}{\mathbb{F}}$.

Consider $\mathcal{A}$ and $\mathbb{M}$ are two hoops. A map $\mathfrak{h} : \mathcal{A} \to \mathbb{M}$ is a ***hoop-homomorphism*** if it satisfies in the following conditions:

$(\mathfrak{h}_1)$ $\mathfrak{h}(p \to q) = \mathfrak{h}(p) \to \mathfrak{h}(q)$,
$(\mathfrak{h}_2)$ $\mathfrak{h}(p \odot q) = \mathfrak{h}(p) \odot \mathfrak{h}(q)$.
A hoop-homomorphism $\mathfrak{h}$ is called a ***hoop-isomorphism*** if $\mathfrak{h}$ is bijective.

Definition 2.2. [1] Consider $\emptyset \neq \mathbb{I} \subseteq \mathcal{A}$. Then $\mathbb{I}$ is an ***ideal of*** $\mathcal{A}$ if
(I_1) $0 \in \mathbb{I}$,
(I_2) for each $p, q \in \mathbb{I}$, $p \oplus q \in \mathbb{I}$, where $p \oplus q = p^{\sim} \to q$.
(I_3) for each $p, q \in \mathcal{A}$, if $p \leq q$ and $q \in \mathbb{I}$, then $p \in \mathbb{I}$.
Obviously, $\mathcal{A}$ and $\{0\}$ are trivial ideals. All ideals of $\mathcal{A}$ is set by $\mathcal{I}d(\mathcal{A})$. $\mathbb{I} \in \mathcal{I}d(\mathcal{A})$ is a ***proper ideal*** if $\mathbb{I} \neq \mathcal{A}$. Clearly, $\mathbb{I} \in \mathcal{I}d(\mathcal{A})$ is proper iff $1 \notin \mathbb{I}$.

Proposition 2.3. [1] *Suppose $\emptyset \neq \mathbb{I} \subseteq \mathcal{A}$. Then, for any $p, q \in \mathcal{A}$, the next equivalent properties hold:*
(i) $\mathbb{I} \in \mathcal{I}d(\mathcal{A})$,
(ii) $0 \in \mathbb{I}$, *for any* $p, q \in \mathbb{I}$, $p \oplus q \in \mathbb{I}$ *and if* $p^{\sim} \odot q \in \mathbb{I}$ *and* $p \in \mathbb{I}$, *then* $q \in \mathbb{I}$.
(iii) $0 \in \mathbb{I}$, *for any* $p, q \in \mathbb{I}$, $p \oplus q \in \mathbb{I}$ *and if* $(p^{\sim} \to q^{\sim})^{\sim} \in \mathbb{I}$ *and* $p \in \mathbb{I}$, *then* $q \in \mathbb{I}$.

Definition 2.4. [1] Let $\emptyset \neq \mathbb{X} \subseteq \mathcal{A}$. The smallest ideal containing $\mathbb{X}$ in $\mathcal{A}$ is said ***the generated ideal by*** $\mathbb{X}$ in $\mathcal{A}$ and set it by $(\mathbb{X}]$.

Theorem 2.5. [1] *Consider $\emptyset \neq \mathbb{X} \subseteq \mathcal{A}$. Then*

$$(\mathbb{X}] = \{x \in \mathcal{A} \mid \exists n \in \mathbb{N} \text{ s.t. for } p_1, p_2, \cdots, p_n \in \mathbb{X},\ x \leq p_1 \oplus (p_2 \oplus \cdots \oplus (p_{n-1} \oplus p_n) \cdots)\}.$$

Proposition 2.6. [1] *Assume $\mathbb{I} \in \mathcal{I}d(\mathcal{A})$ and $x \in \mathcal{A}$. Then the following statements hold,*
(i) $(x] = \{p \in \mathcal{A} \mid \exists n \in \mathbb{N} \text{ s.t. } p \leq nx\}$, *where* $nx = x \oplus (x \oplus \cdots \oplus (x \oplus x) \cdots)$;
(ii) if $\mathcal{A}$ has (DNP), then $(\mathbb{I} \cup \{x\}] = \{p \in \mathcal{A} \mid \exists\, n \in \mathbb{N} \text{ s.t } p \odot (nx)^{\sim} \in \mathbb{I}\}$;
(iii) if $\vee$-hoop $\mathcal{A}$ has (DNP), then $(\mathbb{I} \cup \{p\}] \cap (\mathbb{I} \cup \{q\}] = (\mathbb{I} \cup \{p \wedge q\}]$.

Proper ideal $\mathbb{P}$ is a ***prime ideal*** if $p \wedge q \in \mathbb{P}$ implies $p \in \mathbb{P}$ or $q \in \mathbb{P}$, for any $p, q \in \mathcal{A}$. All prime ideals of $\mathcal{A}$ set by Spec$(\mathcal{A})$. A proper ideal $\mathbb{M}$ is a ***maximal ideal*** of $\mathcal{A}$ if there is $\mathbb{Q} \in \mathcal{I}d(\mathcal{A})$ that $\mathbb{M} \subseteq \mathbb{Q} \subseteq \mathcal{A}$, then $\mathbb{M} = \mathbb{Q}$ or $\mathbb{Q} = \mathcal{A}$. All maximal ideals of $\mathcal{A}$ set by Max$(\mathcal{A})$ (see [1]).

Proposition 2.7. [5] *Suppose $\mathcal{A}$ is a $\vee$-hoop with (DNP). Then*
(i) $\bigcap\{\mathbb{P} \mid \mathbb{P} \in Spec(\mathcal{A})\} = \{0\}$.
(ii) If $0 \neq p \in \mathcal{A}$, *then* $\exists \mathbb{P} \in Spec(\mathcal{A})$, *where* $p \notin \mathbb{P}$.

Proposition 2.8. [5] *Let $\mathcal{A}$ be a $\vee$-hoop with (DNP). Then every maximal ideal of $\mathcal{A}$ is a prime one.*

Proposition 2.9. [6] *Let $\mathbb{M} \in Max(\mathcal{A})$ and $\mathbb{I}, \mathbb{Q} \in \mathcal{I}d(\mathcal{A})$, where hoop has (DNP). Then $\mathbb{I} \odot \mathbb{Q} \subseteq \mathbb{M}$ if and only if $\mathbb{I} \subseteq \mathbb{M}$ or $\mathbb{Q} \subseteq \mathbb{M}$.*

3 Closure operators on elements of hoops

In this section, we define the notion of closure operator on elements of hoops and investigate some properties of it.

Note. In this section we let $(\mathcal{A}, \odot, \to, 1)$ be a hoop.

Definition 3.1. A map $c : \mathcal{A} \to \mathcal{A}$ is a ***closure operator*** if for any $p, q \in \mathcal{A}$ we have
(c_1) $p \leq c(p)$,
(c_2) If $p \leq q$, then $c(p) \leq c(q)$,
(c_3) $c(c(p)) = c(p)$.

Note. By (c_1), clearly $c(1) = 1$.

Example 3.2. (i) Clearly, identity map $id_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ is a closure operator.
(ii) The constant map that sends every element to 1 is a closure operator.
(iii) Assume $\mathcal{A} = \{0, x, y, 1\}$ is a chain, where $0 \leq x \leq y \leq 1$. Define the operations $\odot$ and $\to$ on $\mathcal{A}$ as follows:

$\odot$	0	x	y	1
0	0	0	0	0
x	0	x	x	x
y	0	x	x	y
1	0	x	y	1

$\to$	0	x	y	1
0	1	1	1	1
x	0	1	1	1
y	0	y	1	1
1	0	x	y	1

Then $(\mathcal{A}, \odot, \to, 0, 1)$ is a bounded hoop. Define $c : \mathcal{A} \to \mathcal{A}$ as $c(0) = 0$, $c(1) = 1$ and $c(x) = c(y) = y$. Obviously, c is a closure operator on $\mathcal{A}$.

Definition 3.3. A map $c : \mathcal{A} \to \mathcal{A}$ is a ***semi-closure operator*** if for any $p, q \in \mathcal{A}$ it satisfies in (c_1)-(c_3), and we have
(c_4) $c(p) \odot c(q) \leq c(p \odot q)$.

Remark 3.4. Clearly, every semi-closure operator is a closure operator. But the converse is not true.

Example 3.5. (i) According to Example 3.2(iii), clearly c is a semi-closure operator.
(ii) Consider $\mathcal{A} = \{0, x, y, 1\}$ is a chain, where $0 \leq x \leq y \leq 1$. Define the operations $\odot$ and $\to$ on $\mathcal{A}$ as follows:

$\odot$	0	x	y	1
0	0	0	0	0
x	0	0	x	x
y	0	x	y	y
1	0	x	y	1

$\to$	0	x	y	1
0	1	1	1	1
x	x	1	1	1
y	0	x	1	1
1	0	x	y	1

Then $(\mathcal{A}, \odot, \to, 0, 1)$ is a bounded hoop. Define $c : \mathcal{A} \to \mathcal{A}$ as $c(0) = 0$, $c(1) = 1$ and $c(x) = c(y) = y$. Obviously, c is a closure operator on $\mathcal{A}$ but it is not a semi-closure operator on $\mathcal{A}$. Because

$$c(x) \odot c(x) = y \odot y = y \not\leq 0 = c(0) = c(x \odot x).$$

Theorem 3.6. *If $c : \mathcal{A} \to \mathcal{A}$ is a semi-closure operator on $\mathcal{A}$, then $D = \{p \in \mathcal{A} \mid c(p) = 1\}$ is a filter of $\mathcal{A}$.*

Proof. Since $c(1) = 1$, we get $1 \in D$ and so $D \neq \emptyset$. Assume $p, q \in D$. Then $c(p) = c(q) = 1$. Since c is a semi-closure operator on $\mathcal{A}$, by (c_4) we have

$$1 = 1 \odot 1 = c(p) \odot c(q) \leq c(p \odot q).$$

Thus $c(p \odot q) = 1$, and so $p \odot q \in D$. Suppose $p \in D$ and $q \in \mathcal{A}$ such that $p \leq q$. By (c_2), we have $1 = c(p) \leq c(q)$. Hence, $c(q) = 1$ and so $q \in D$. Therefore, D is a filter of $\mathcal{A}$. □

The following example shows that the condition semi-closure operator is necessary in Theorem 3.6.

Example 3.7. In Example 3.5(ii), define $c : \mathcal{A} \to \mathcal{A}$ by $c(0) = 0$ and $c(x) = c(y) = c(1) = 1$. Clearly c is a closure operator but it is not a semi-closure operator, because

$$c(x) \odot c(x) = 1 \odot 1 = 1 \not\leq 0 = c(0) = c(x \odot x).$$

Also, $D = \{x, y, 1\}$ is not a filter of $\mathcal{A}$, since $x \odot x = 0 \notin D$.

Theorem 3.8. *Let $c : \mathcal{A} \to \mathcal{A}$ be a semi-closure operator on $\mathcal{A}$ and $c(\mathcal{A}) = \{p \in \mathcal{A} \mid c(p) = p\}$. Then we have the following statements:*
(i) $c(\mathcal{A}) = \{c(p) \mid p \in \mathcal{A}\}$ and $c(\mathcal{A})$ is closed under $\wedge$ and $\to$.
(ii) If for any $p \in \mathcal{A}$, $p^2 = p$, then $c(p \odot q) = c(p) \odot c(q)$. In addition, in this case, $c(\mathcal{A})$ is $\odot$-closed.
(iii) If $\mathcal{A}$ is idempotent, then $(c(\mathcal{A}), \odot_c, \to_c, c(0), 1)$ is a hoop, and for any $p, q \in c(\mathcal{A})$ we have

$$p \odot_c q = c(p \odot q), \quad \textit{and} \quad p \to_c q = c(p \to q).$$

Proof. (*i*) Let $B = \{c(p) \mid p \in \mathcal{A}\}$. Assume $p \in B$. Then there exists $x \in \mathcal{A}$ such that $c(x) = p$. Since c is a closure operator, by (c_3) we have

$$c(p) = c(c(x)) = c(x) = p.$$

Thus $c(p) = p$ and so $p \in c(\mathcal{A})$. Conversely, suppose $p \in c(\mathcal{A})$. Then $c(p) = p$ and so $p \in B$. Hence, $c(\mathcal{A}) = \{c(p) \mid p \in \mathcal{A}\}$.

Now, suppose that c is a semi-closure operator on $\mathcal{A}$ and $p, q \in c(\mathcal{A})$. Then $c(p) = p$ and $c(q) = q$. By (c_2) and since $p \wedge q \leq p, q$ we have $c(p \wedge q) \leq c(p) \wedge c(q)$. Also, by (c_1) we obtain

$$p \wedge q \leq c(p \wedge q) \leq c(p) \wedge c(q) = p \wedge q. \tag{3.1}$$

Hence $c(p \wedge q) = p \wedge q$ and so $c(\mathcal{A})$ is $\wedge$-closed. In addition, by (c_1) we have $p \to q \leq c(p \to q)$. Since c is a semi-closure operator, by (c_4) and (3.1) we have

$$c(p) \odot c(p \to q) \leq c(p \odot (p \to q)) = c(p \wedge q) = c(p) \wedge c(q) \leq c(q).$$

By Proposition 2.1(ii), we get $c(p \to q) \leq c(p) \to c(q) = p \to q$. Hence, $c(p \to q) = p \to q$. Therefore, $c(\mathcal{A})$ is $\to$-closed, too.
(ii) Since c is a semi-closure operator, clearly, for any $p \in \mathcal{A}$, $c(p) \in \mathcal{A}$ and by hypothesis, $c^2(p) = c(p)$. By (c_4), we know $c(p) \odot c(q) \leq c(p \odot q)$, for $p, q \in \mathcal{A}$. By Proposition 2.1(iii), $p \odot q \leq p, q$ and by (c_2) we get $c(p \odot q) \leq c(p), c(q)$. Then $c^2(p \odot q) \leq c(p) \odot c(q)$. By assumption, we obtain $c(p \odot q) \leq c(p) \odot c(q)$. Hence $c(p \odot q) = c(p) \odot c(q)$.
(iii) Consider $p, q, r \in c(\mathcal{A})$. Then by (i) we have

$$p \to_c p = c(p \to p) = c(1) = 1.$$

Thus $p \to_c p = 1$. By *(i)* and *(ii)* we get $c(\mathcal{A})$ is $\wedge$-closed and $\odot$-closed, then

$$\begin{aligned} p \odot_c (p \to_c q) &= p \odot_c c(p \to q) = c(p \odot c(p \to q)) \\ &= c(p \odot (p \to q)) = p \odot (p \to q) = q \odot (q \to p) \\ &= c(q \odot (q \to p)) = c(q \odot c(q \to p)) = q \odot_c c(q \to p) \\ &= q \odot_c (q \to_c p). \end{aligned}$$

Also,

$$\begin{aligned} p \to_c (q \to_c r) &= p \to_c c(q \to r) = c(p \to c(q \to r)) \\ &= c(p \to (q \to r)) = p \to (q \to r) \\ &= q \to (p \to r) = q \to c(p \to r) \\ &= c(q \to c(p \to r)) = c(q \to (p \to_c r)) \\ &= q \to_c (p \to_c r). \end{aligned}$$

Now, it is enough to prove that $(c(\mathcal{A}), \odot_c)$ is a commutative monoid. For this, by (ii) we have

$$p \odot_c q = c(p \odot q) = c(q \odot p) = q \odot_c p, \quad \text{and} \quad p \odot_c 1 = c(p \odot 1) = p.$$

Suppose $(c(\mathcal{A}), \odot_c)$ is not associative. Assume $p \odot_c (q \odot_c r) \not\preceq (p \odot_c q) \odot_c r$. Then there exists $y \in c(\mathcal{A})$ such that $p \odot_c (q \odot_c r) \leq y \prec (p \odot_c q) \odot_c r$. Then by (c_2) we have

$$\begin{aligned}
p \odot_c (q \odot_c r) \leq y &\Leftrightarrow q \odot_c r \leq p \rightarrow_c y \\
&\Leftrightarrow q \leq r \rightarrow_c (p \rightarrow_c y) \\
&\Leftrightarrow q \leq r \rightarrow_c c(p \rightarrow y) \\
&\Leftrightarrow q \leq c(r \rightarrow c(p \rightarrow y)) \\
&\Leftrightarrow q \leq c(r \rightarrow (p \rightarrow y)) \\
&\Leftrightarrow q \leq r \rightarrow (p \rightarrow y) \\
&\Leftrightarrow q \leq c(r) \rightarrow (c(p) \rightarrow c(y)) \\
&\Leftrightarrow q \leq c(p) \rightarrow (c(r) \rightarrow c(y)) \\
&\Leftrightarrow q \leq p \rightarrow_c c(r \rightarrow y) \\
&\Leftrightarrow (p \odot_c q) \odot_c r \leq y
\end{aligned}$$

which is a contradiction. So, $p \odot_c (q \odot_c r) \leq (p \odot_c q) \odot_c r$ and similarly, $(p \odot_c q) \odot_c r \leq p \odot_c (q \odot_c r)$. Hence, $(c(\mathcal{A}), \odot_c)$ is a commutative monoid. Therefore, $(c(\mathcal{A}), \odot_c, \rightarrow_c, c(0), 1)$ is a hoop. □

The following example shows that all the conditions in Theorem 3.8 are necessary.

Example 3.9. (i) According to Example 3.5(ii), c is a closure operator and

$$c(x \rightarrow 0) = c(x) = y \neq 0 = y \rightarrow 0 = c(x) \rightarrow c(0).$$

Hence the condition semi-closure operator in Theorem 3.8(i) is necessary.
(ii) According to Example 3.2(iii), clearly c is a semi-closure operator and $\mathcal{A}$ is not idempotent since $y \odot y = x \neq y$. Obviously, $c(p \odot q) \neq c(p) \odot c(q)$, for any $p, q \in \mathcal{A}$, because

$$c(y) \odot c(y) = y \odot y = x \neq y = c(x) = c(y \odot y).$$

Thus the condition idempotent in Theorem 3.8(ii) is necessary.

Example 3.10. (i) Consider $\mathbb{I} = [0, 1]$ and define two operations $\odot$ and $\rightarrow$ on $\mathbb{I}$ as follows:

$$p \odot q = \min\{p, q\}, \quad p \rightarrow q = \begin{cases} 1, & p \leq q \\ q & \text{otherwise} \end{cases}$$

Then $(\mathbb{I}, \odot, \to, 1)$ is a hoop, where for any $p \in \mathbb{I}$, $p \odot p = p$. Define $c : \mathbb{I} \to \mathbb{I}$ as follows:

$$c(p) = \begin{cases} \frac{1}{3}, & 0 \leq p \leq \frac{1}{3} \\ \frac{2}{3}, & \frac{1}{3} < p \leq \frac{2}{3} \\ 1, & \frac{2}{3} < p \leq 1 \end{cases} \tag{3.2}$$

Obviously, c is a semi-closure operator and by Theorem 3.8, $\left\langle \mathbb{I}, \odot_c, \to_c, \frac{1}{3}, 1 \right\rangle$ is a hoop.

(*ii*) Consider $\mathbb{I} = [0, 1]$ and define two operations $\odot$ and $\to$ on $\mathbb{I}$ as follows:

$$p \odot q = \max\{0, p + q - 1\}, \quad \text{and} \quad p \to q = \min\{1, 1 - p + q\}.$$

Then $(\mathbb{I}, \odot, \to, 1)$ is a hoop. Consider the map defined in (3.2). Clearly, $c(p) \odot c(q) \not\leq c(p \odot q)$, for any $p, q \in \mathbb{I}$, because

$$\frac{2}{3} = \frac{2}{3} \odot 1 = c(\frac{1}{2}) \odot c(\frac{5}{6}) \not\leq c(\frac{1}{2} \odot \frac{5}{6}) = c(\frac{1}{3}) = \frac{1}{3}.$$

Hence, c is not a semi-closure operator and so $\left\langle \mathbb{I}, \odot_c, \to_c, \frac{1}{3}, 1 \right\rangle$ is not a hoop, since

$$1 = c(\frac{1}{2} \to \frac{1}{3}) \neq c(\frac{1}{2}) \to c(\frac{1}{3}) = \frac{2}{3} \to \frac{1}{3} = \frac{2}{3}.$$

Theorem 3.11. *Consider* $(\mathcal{A}, \odot, \to, 1)$ *and* $(c(\mathcal{A}), \odot_c, \to_c, 1)$ *are two idempotent hoops, where* c *is a semi-closure operator on* $\mathcal{A}$*. Then*

(*i*) $c : \mathcal{A} \to c(\mathcal{A})$ *is a hoop homomorphism.*

(*ii*) *The map* $\bar{c} : \dfrac{\mathcal{A}}{D} \to c(\mathcal{A})$ *defined by* $\bar{c}([p]) = c(p)$ *is a hoop isomorphism, where* $D = \{p \in \mathcal{A} \mid c(p) = 1\}$.

Proof. (*i*) Define $c(p * q) = p *_c q$, where $* \in \{\odot, \to\}$. Then by Theorem 3.8(iii), the proof is clear.

(*ii*) By Theorem 3.6, D is a filter of $\mathcal{A}$ and so $\dfrac{\mathcal{A}}{D}$ is well-defined. We show that $\bar{c}$ is well-defined. Suppose $[p] = [q]$. Then $(p \to q) \odot (q \to p) \in D$, and so $c\left((p \to q) \odot (q \to p)\right) = 1$. Since by Proposition 2.1(iii), $(p \to q) \odot (q \to p) \leq p \to q,\ q \to p$, by (c_2) we get

$$1 = c\left((p \to q) \odot (q \to p)\right) \leq c(p \to q),\ c(q \to p),$$

and so $c(p \to q) = c(q \to p) = 1$. By (i), since c is a hoop homomorphism we have $c(p) \to c(q) = 1$ and $c(q) \to c(p) = 1$. Hence, $c(p) = c(q)$, and so $\overline{c}([p]) = \overline{c}([q])$. Therefore, $\overline{c}$ is well-defined. Now, we prove $\overline{c}$ is a hoop homomorphism. By (i) we have

$$\overline{c}([p] \to [q]) = \overline{c}([p \to q]) = c(p \to q) = c(p) \to c(q) = \overline{c}([p]) \to \overline{c}([q]).$$

Also,

$$\overline{c}([p] \odot [q]) = \overline{c}([p \odot q]) = c(p \odot q) = c(p) \odot c(q) = \overline{c}([p]) \odot \overline{c}([q]).$$

Hence $\overline{c}$ is a hoop homomorphism and clearly, $\overline{c}$ is surjective. Suppose for $p, q \in \mathcal{A}$, $c(p) = c(q)$. Then $c(p) \to c(q) = c(q) \to c(p) = 1$. Thus since c is a hoop homomorphism we have

$$c\left((p \to q) \odot (q \to p)\right) = (c(p) \to c(q)) \odot (c(q) \to c(p)) = 1.$$

So, $(p \to q) \odot (q \to p) \in D$. Hence, $[p] = [q]$. Therefore, the map $\overline{c}$ is a hoop isomorphism. □

Theorem 3.12. *Let $\mathcal{A}$ and $\mathbb{X}$ be two hoops and $h : \mathcal{A} \to \mathbb{X}$ be a hoop homomorphism. Suppose $c_1 : \mathcal{A} \to \mathcal{A}$ and $c_2 : \mathbb{X} \to \mathbb{X}$ are two semi-closure operators. Assume $D_1 = \{p \in \mathcal{A} \mid c_1(p) = 1\}$ and $D_2 = \{q \in \mathbb{X} \mid c_2(q) = 1\}$. Then the following statements hold:*
(i) If for any $p \in \mathcal{A}$, $h(c_1(p)) \leq c_2(h(p))$, then the map $\overline{h} : \dfrac{\mathcal{A}}{D_1} \to \dfrac{\mathbb{X}}{D_2}$ where $\overline{h}([p]) = [h(p)]$ is a hoop homomorphism.
(ii) If h is surjective such that for any $p \in \mathcal{A}$, $h(c_1(p)) = c_2(h(p))$, and $h(c_1(p)) = 1$ implies $c_1(p) = 1$, then a map $\overline{h} : \dfrac{\mathcal{A}}{D_1} \to \dfrac{\mathbb{X}}{D_2}$ is a hoop isomorphism.

Proof. (i) Suppose $[p] = [q]$. Then $(p \to q) \odot (q \to p) \in D_1$, and so $c_1\left((p \to q) \odot (q \to p)\right) = 1$. Since h is a hoop homomorphism, we have

$$h\left(c_1\left((p \to q) \odot (q \to p)\right)\right) = 1.$$

Then by assumption, we get

$$\begin{aligned} 1 &= h\left(c_1\left((p \to q) \odot (q \to p)\right)\right) \\ &\leq c_2\left(h\left((p \to q) \odot (q \to p)\right)\right) \\ &= c_2\left((h(p) \to h(q)) \odot (h(q) \to h(p))\right). \end{aligned}$$

Thus $c_2\left((h(p) \to h(q)) \odot (h(q) \to h(p))\right) = 1$, and so $(h(p) \to h(q)) \odot (h(q) \to h(p)) \in D_2$. Hence, $[h(p)] = [h(q)]$, and so $\overline{h}$ is well-defined. The proof of homomorphism is clear.

(*ii*) By (i), $\overline{h}$ is a hoop homomorphism. Since h is surjective, obviously $\overline{h}$ is surjective, too. Suppose $\overline{h}[p] = \overline{h}[q]$. Then $[h(p)] = [h(q)]$ and so $(h(p) \to h(q)) \odot (h(q) \to h(p)) \in D_2$. Thus

$$c_2\left((h(p) \to h(q)) \odot (h(q) \to h(p))\right) = 1.$$

Since h is a hoop homomorphism we have $c_2\left(h((p \to q) \odot (q \to p))\right) = 1$. By assumption, $h(c_1((p \to q) \odot (q \to p))) = 1$, and so $c_1((p \to q) \odot (q \to p)) = 1$. Hence $(p \to q) \odot (q \to p) \in D_1$. Thus $[p] = [q]$. Therefore, $\overline{h}$ is a hoop isomorphism. □

Assume $\mathbb{X}$ is a set. Define two operations $\odot$ and $\to$ on $P(\mathbb{X})$ as follows:

$$\mathbb{I} \odot \mathbb{Q} = \mathbb{I} \cap \mathbb{Q}, \quad \text{and} \quad \mathbb{I} \to \mathbb{Q} = \mathbb{I}^c \cup \mathbb{Q}, \tag{3.3}$$

where $\mathbb{I}, \mathbb{Q} \subseteq \mathbb{X}$ and $\mathbb{I}^c = \mathbb{X} \setminus \mathbb{I}$. Then $P(\mathbb{X})$ (the power set of $\mathbb{X}$) with above operation is a hoop.

Example 3.13. Let $\mathcal{A} = \{x, y, z\}$ and $\mathbb{M} = \{w, u\}$ be two sets. Clearly, $(P(\mathcal{A}), \odot, \to, \mathcal{A})$ and $(P(\mathbb{M}), \odot, \to, \mathbb{M})$ with operations defined in (3.3) are hoops. Define a map $\mathfrak{h} : P(\mathcal{A}) \to \mathbb{M}$ as follows:

$$\begin{gathered}
\mathfrak{h}(\mathcal{A}) = \mathbb{M}, \quad \mathfrak{h}(\emptyset) = \emptyset \\
\mathfrak{h}(\{x\}) = \{w\}, \quad \mathfrak{h}(\{y\}) = \{u\}, \quad \mathfrak{h}(\{z\}) = \emptyset, \\
\mathfrak{h}(\{x, y\}) = \{w, u\}, \quad \mathfrak{h}(\{x, z\}) = \{w\}, \quad \mathfrak{h}(\{y, z\}) = \{u\},
\end{gathered}$$

Obviously, $\mathfrak{h}$ is a hoop-homomorphism. Now, consider two maps c_1 and c_2 on $P(\mathcal{A})$ and $P(\mathbb{M})$, respectively, as follows:

$$c_1(\mathbb{I}) = \begin{cases} \{x, z\}, & \mathbb{I} \subset \{x, z\} \\ \mathcal{A}, & \text{otherwise} \end{cases} \qquad \text{and} \quad c_2(\mathbb{Q}) = \begin{cases} \{w\}, & \mathbb{Q} \subset \{w\} \\ \mathbb{M}, & \text{otherwise} \end{cases}$$

Clearly, c_1 and c_2 are two semi-closure operators. In addition, we have

$$\mathfrak{h}(c_1(\mathbb{I})) = \begin{cases} \mathfrak{h}\left(\{x, z\}\right) = \{w\} = c_2\left(\mathfrak{h}(\mathbb{I})\right), & \mathbb{I} \subset \{x, z\} \\ \mathfrak{h}(\mathcal{A}) = \mathbb{M} = c_2\left(\mathfrak{h}(\mathbb{I})\right), & \text{otherwise} \end{cases}$$

Moreover, $D_1 = \{\{y\}, \{x, y\}, \{y, z\}, \mathcal{A}\}$, $D_2 = \{\{u\}, \mathbb{M}\}$. It is routine that

$$\overline{\mathfrak{h}} : \frac{P(\mathcal{A})}{D_1} \to \frac{P(\mathbb{M})}{D_2}, \text{ where } \overline{\mathfrak{h}}\left([\{x, z\}]\right) = [\{w\}], \text{ and } \overline{\mathfrak{h}}\left([\mathcal{A}]\right) = [\mathbb{M}].$$

Since $\mathfrak{h}(c_1(\mathbb{I})) = c_2(\mathfrak{h}(\mathbb{I}))$, for any $\mathbb{I} \subset \mathcal{A}$ and $\mathfrak{h}(c_1(\mathcal{A})) = \mathbb{M}$ implies $c_1(\mathbb{I}) = \mathcal{A}$. Hence, by Theorem 3.12, $\overline{\mathfrak{h}}$ is a hoop-isomorphism.

4 Closure operator on ideals of hoops

In this section, we define the notion of closure operator on the set of all ideals of hoops and investigate some properties of it.

Definition 4.1. Define the map $C : \mathcal{I}d(\mathcal{A}) \to \mathcal{I}d(\mathcal{A})$ is called a ***closure operator on ideals of*** $\mathcal{A}$ if C has the following conditions:
(C_1) $\mathbb{I} \subseteq \mathbb{I}^c$,
(C_2) if $\mathbb{I} \subseteq \mathbb{Q}$, then $\mathbb{I}^c \subseteq \mathbb{Q}^c$,
(C_3) $\mathbb{I}^{cc} = \mathbb{I}^c$,
where $\mathbb{I}^c = C(\mathbb{I})$. An ideal $\mathbb{I}$ is called ***C-closed*** if $\mathbb{I}^c = \mathbb{I}$.

Example 4.2. (i) Clearly, $id : \mathcal{I}d(\mathcal{A}) \to \mathcal{I}d(\mathcal{A})$ is a closure operator on $\mathcal{I}d(\mathcal{A})$.
(ii) Obviously, $\mathbb{I}^c$ is a C-closed ideal of $\mathcal{A}$.
(iii) Let $\mathcal{A}$ be a hoop and $\mathbb{I} \in \mathcal{I}d(\mathcal{A})$. Consider

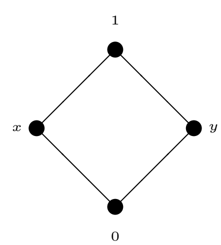
$$\mathbb{I}^c = \bigcap \{\mathbb{Q} \in \mathcal{I}d(\mathcal{A}) \mid \mathbb{I} \subseteq \mathbb{Q}\}.$$

Then C is a closure operator on ideal of $\mathcal{A}$. Clearly, $\mathbb{I} \subseteq \bigcap \{\mathbb{Q} \in \mathcal{I}d(\mathcal{A}) \mid \mathbb{I} \subseteq \mathbb{Q}\} = \mathbb{I}^c$. Assume $\mathbb{I} \subseteq \mathbb{Q}$. Since $\mathbb{I}^c = \bigcap \{\mathbb{Q} \in \mathcal{I}d(\mathcal{A}) \mid \mathbb{I} \subseteq \mathbb{Q}\} \subseteq \mathbb{Q}$ and $\mathbb{Q} \subseteq \mathbb{Q}^c$, we get $\mathbb{I}^c \subseteq \mathbb{Q}^c$.

In addition, $\mathbb{I}^c \subseteq \mathbb{I}^c$, then $\mathbb{I}^{cc} \subseteq \mathbb{I}^c$. Also, by (C_1), $\mathbb{I}^c \subseteq \mathbb{I}^{cc}$. Hence, $\mathbb{I}^c = \mathbb{I}^{cc}$. Therefore, $\mathbb{I}^c$ is a closure operator on ideal.

For every $\mathbb{I} \in \mathcal{I}d(\mathcal{A})$, $\mathbb{M}_{\mathbb{I}}$ is the set of all maximal ideals of $\mathcal{A}$ containing $\mathbb{I}$ and $\text{Rad}(\mathbb{I}) = \bigcap \mathbb{M}_{\mathbb{I}}$. Also, $\mathbb{I} \in \mathcal{I}d(\mathcal{A})$ is called a ***radical ideal*** of $\mathcal{A}$ if $\text{Rad}(\mathbb{I}) = \mathbb{I}$.

Example 4.3. Assume $\mathcal{A} = \{0, x, y, 1\}$ is a poset where $0 \leq x, y \leq 1$. Define $\odot$ and $\to$ on $\mathcal{A}$ by:



$\odot$	0	x	y	1
0	0	0	0	0
x	0	x	0	x
y	0	0	y	y
1	0	x	y	1

$\to$	0	x	y	1
0	1	1	1	1
x	y	1	y	1
y	x	x	1	1
1	0	x	y	1

Then $(\mathcal{A}, \odot, \to, 0, 1)$ is a bounded hoop, $\mathcal{I}d(\mathcal{A}) = \{\{0\}, \{0, x\}, \{0, y\}, \mathcal{A}\}$ and for $\mathbb{I} = \{0\}$ we have $\mathbb{M}_{\mathbb{I}} = \{\{0, x\}, \{0, y\}\}$. Thus

$$\text{Rad}(\mathbb{I}) = \bigcap \mathbb{M}_{\mathbb{I}} = \{0, x\} \cap \{0, y\} = \{0\} = \mathbb{I}.$$

Hence, $\{0\}$ is a radical ideal of $\mathcal{A}$.

Note. Similar to Example 4.2(iii), $\mathrm{Rad}(\mathbb{I})$ is a closure operator on ideal of $\mathcal{A}$.

Proposition 4.4. *Suppose C is a closure operator on $\mathcal{I}d(\mathcal{A})$ and $\{\mathbb{I}_\alpha\}_{\alpha\in\Lambda}$ is a family of ideals of $\mathcal{A}$. Then*
(i) for any $\mathbb{I}\in\mathcal{I}d(\mathcal{A})$, $\mathbb{I}^c=\bigcap\{\mathbb{Q}\mid\mathbb{Q}\in\mathcal{I}d(\mathcal{A}),\ \mathbb{I}\subseteq\mathbb{Q},$ and $\mathbb{Q}$ is a C-closed ideal of $\mathcal{A}\}$.
(ii) If for any $\alpha\in\Lambda$, $\mathbb{I}_\alpha$ is C-closed, then $\bigcap\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha$ is C-closed.
(iii) $\langle\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}^c_\alpha\rangle^c=\langle\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha\rangle^c$.

Proof. (i) Let

$$\mathcal{C}=\bigcap\{\mathbb{Q}\mid\mathbb{Q}\in\mathcal{I}d(\mathcal{A}),\ \mathbb{I}\subseteq\mathbb{Q},\ \text{and }\mathbb{Q}\text{ is a }C\text{-closed ideal of }\mathcal{A}\}.$$

Suppose $\mathbb{Q}\in\mathcal{C}$. Then $\mathbb{Q}\in\mathcal{I}d(\mathcal{A})$, $\mathbb{I}\subseteq\mathbb{Q}$ and $\mathbb{Q}$ is C-closed. Then by (C_2), $\mathbb{I}^c\subseteq\mathbb{Q}^c$, and since $\mathbb{Q}$ is C-closed, we get $\mathbb{Q}^c=\mathbb{Q}$, and so $\mathbb{I}^c\subseteq\mathbb{Q}$. Hence, $\mathcal{C}\subseteq\mathbb{I}^c$.

Conversely, by (C_1), $\mathbb{I}\subseteq\mathbb{I}^c$ and by (C_3), $\mathbb{I}^c$ is C-closed. Thus $\mathbb{I}^c\in\mathcal{C}$, and so $\mathbb{I}^c\subseteq\mathcal{C}$. Therefore,

$$\mathbb{I}^c=\bigcap\{\mathbb{Q}\mid\mathbb{Q}\in\mathcal{I}d(\mathcal{A}),\ \mathbb{I}\subseteq\mathbb{Q},\ \text{and }\mathbb{Q}\text{ is a }C\text{-closed ideal of }\mathcal{A}\}.$$

(ii) Clearly, $\bigcap\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha\subseteq\mathbb{I}_\alpha$ and by (C_2), $\left(\bigcap\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha\right)^c\subseteq\mathbb{I}^c_\alpha$. Since $\mathbb{I}_\alpha$ is C-closed, we get $\left(\bigcap\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha\right)^c\subseteq\mathbb{I}_\alpha$, and so $\left(\bigcap\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha\right)^c\subseteq\bigcap\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha$. On the other side, since $\bigcap\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha\in\mathcal{I}d(\mathcal{A})$, by (C_1), $\bigcap\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha\subseteq\left(\bigcap\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha\right)^c$. Hence, $\bigcap\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha$ is C-closed.
(iii) By (C_1), for any $\alpha\in\Lambda$, $\mathbb{I}_\alpha\subseteq\mathbb{I}^c_\alpha$, thus $\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha\subseteq\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}^c_\alpha$, and so $\langle\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha\rangle\subseteq\langle\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}^c_\alpha\rangle$. Hence, by (C_1), we have $\langle\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha\rangle^c\subseteq\langle\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}^c_\alpha\rangle^c$.

Conversely, since $\mathbb{I}_\alpha\subseteq\langle\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha\rangle$, by (C_2) we get $\mathbb{I}^c_\alpha\subseteq\langle\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha\rangle^c$, and so $\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}^c_\alpha\subseteq\langle\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha\rangle^c$. Thus $\langle\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}^c_\alpha\rangle\subseteq\langle\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha\rangle^c$. Then $\langle\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}^c_\alpha\rangle^c\subseteq\langle\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha\rangle^{cc}$. By (C_3), $\langle\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}^c_\alpha\rangle^c\subseteq\langle\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha\rangle^c$. Therefore, $\langle\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}^c_\alpha\rangle^c=\langle\bigcup\limits_{\alpha\in\Lambda}\mathbb{I}_\alpha\rangle^c$. □

Corollary 4.5. *There is a one-to-one correspondence between the closure operators on $\mathcal{A}$ and the subsets of $\mathcal{I}d(\mathcal{A})$ that are arbitrary closed.*

Proof. Consider C is a closure operator on ideal of $\mathcal{I}d(\mathcal{A})$ and $\mathcal{I}_c$ is the set of all C-closed ideals of $\mathcal{A}$. For any subset $\mathcal{J}$ of $\mathcal{I}d(\mathcal{A})$ that is closed under arbitrary intersection, define $c(\mathcal{J}) : \mathcal{I}d(\mathcal{A}) \to \mathcal{I}d(\mathcal{A})$ where $\mathcal{I}^{c(\mathcal{J})} = \bigcap\{\mathbb{Q} \mid \mathbb{Q} \in \mathcal{J}$ and $\mathbb{I} \subseteq \mathbb{Q}\}$. Similar to Example 4.2(iii), we can see that $c(\mathcal{J})$ is a closure operator on ideal of $\mathcal{A}$. By Proposition 4.4(i), we get $c(\mathcal{J}(C)) = C$ and by Example 4.2(iii), $c(\mathcal{J})$-closed ideals are exactly $\mathcal{J}$. Hence, $\mathcal{J}(c(\mathcal{J})) = \mathcal{J}$. □

Proposition 4.6. *Consider $\mathcal{A}$ be a $\vee$-hoop with (DNP). Then $Rad(\mathbb{I} \cap \mathbb{Q}) = Rad(\mathbb{I}) \cap Rad(\mathbb{Q})$, for any $\mathbb{I}, \mathbb{Q} \in \mathcal{I}d(\mathcal{A})$.*

Proof. According to definition of Rad($\mathbb{I}$), we have to prove $\bigcap \mathbb{M}_{\mathbb{I}\cap\mathbb{Q}} = \bigcap \mathbb{M}_{\mathbb{I}} \cap \bigcap \mathbb{M}_{\mathbb{Q}}$. For this, let $\mathbb{M} \in \bigcap \mathbb{M}_{\mathbb{I}} \cap \bigcap \mathbb{M}_{\mathbb{Q}}$. Then $\mathbb{M} \in \bigcap \mathbb{M}_{\mathbb{I}}$ and $\mathbb{M} \in \bigcap \mathbb{M}_{\mathbb{Q}}$. Thus $\mathbb{I} \subseteq \mathbb{M}$ and $\mathbb{Q} \subseteq \mathbb{M}$, and so $\mathbb{I} \cap \mathbb{Q} \subseteq \mathbb{M}$. Hence, $\mathbb{M} \in \bigcap \mathbb{M}_{\mathbb{I}\cap\mathbb{Q}}$.

Conversely, assume $\mathbb{M} \in \bigcap \mathbb{M}_{\mathbb{I}\cap\mathbb{Q}}$. Then $\mathbb{I} \cap \mathbb{Q} \subseteq \mathbb{M}$. Suppose $\mathbb{M} \notin \bigcap \mathbb{M}_{\mathbb{I}} \cap \bigcap \mathbb{M}_{\mathbb{Q}}$. Thus, $\mathbb{M} \notin \bigcap \mathbb{M}_{\mathbb{I}}$ or $\mathbb{M} \notin \bigcap \mathbb{M}_{\mathbb{Q}}$, and so $\mathbb{I} \not\subseteq \mathbb{M}$ or $\mathbb{Q} \not\subseteq \mathbb{M}$. Then there exist $p \in \mathbb{I} \setminus \mathbb{M}$ or $q \in \mathbb{Q} \setminus \mathbb{M}$. Since $p \wedge q \leq p, q$, we get $p \wedge q \in \mathbb{I} \cap \mathbb{Q} \subseteq \mathbb{M}$. Hence, $p \wedge q \in \mathbb{M}$. By Proposition 2.8, $\mathbb{M}$ is a prime ideal of $\mathcal{A}$, and so $p \in \mathbb{M}$ or $q \in \mathbb{M}$, which is a contradiction. Thus, $\mathbb{M} \in \mathbb{M}_{\mathbb{I}}$ and $\mathbb{M} \in \mathbb{M}_{\mathbb{Q}}$. Hence, $\mathrm{Rad}(\mathbb{I}) \cap \mathrm{Rad}(\mathbb{Q}) \subseteq \mathrm{Rad}(\mathbb{I}) \subseteq \mathbb{M}$ or $\mathrm{Rad}(\mathbb{I}) \cap \mathrm{Rad}(\mathbb{Q}) \subseteq \mathrm{Rad}(\mathbb{Q}) \subseteq \mathbb{M}$, and so

$$\mathrm{Rad}(\mathbb{I} \cap \mathbb{Q}) \subseteq \mathrm{Rad}(\mathbb{I}) \cap \mathrm{Rad}(\mathbb{Q}).$$

Therefore, $\mathrm{Rad}(\mathbb{I} \cap \mathbb{Q}) = \mathrm{Rad}(\mathbb{I}) \cap \mathrm{Rad}(\mathbb{Q})$. □

Definition 4.7. The closure operator C is ***stable*** if for any $\mathbb{I}, \mathbb{Q} \in \mathcal{I}d(\mathcal{A})$, $(\mathbb{I} \cap \mathbb{Q})^c = \mathbb{I}^c \cap \mathbb{Q}^c$.

Example 4.8. (i) Obviously, the identity map and the constant map, where $\mathbb{I}^c = \mathcal{A}$, are stable closure operators.
(ii) Consider $\mathcal{A}$ be a $\vee$-hoop with (DNP). Since Rad($\mathbb{I}$) is a closure operator on ideal of $\mathbb{I}$, by Proposition 4.6, we have $\mathrm{Rad}(\mathbb{I} \cap \mathbb{Q}) = \mathrm{Rad}(\mathbb{I}) \cap \mathrm{Rad}(\mathbb{Q})$, and so Rad($\mathbb{I}$) is a stable closure operator on ideals of $\mathcal{A}$.

Let $\mathcal{CO}(\mathcal{A})$ be the set of all closure operators on $\mathcal{A}$. Then define an order on $\mathcal{CO}(\mathcal{A})$ such that
(1) $(\mathcal{CO}(\mathcal{A}), \preccurlyeq)$ where $C \preccurlyeq C'$ if and only if $\mathbb{I}^c \subseteq \mathbb{I}^{c'}$, for any $\mathbb{I} \in \mathcal{I}d(\mathcal{A})$.
(2) $(\mathcal{CO}(\mathcal{A}), \circ)$ is closed and make a monoid, where $\circ$ is the composition operation.
(3) $(\mathcal{CO}(\mathcal{A}), \preccurlyeq)$ is a distributive lattice. According to definition of $(\mathcal{CO}(\mathcal{A}), \preccurlyeq)$, we consequence that $(\mathcal{CO}(\mathcal{A}), \preccurlyeq)$ is equivalent to $(\mathcal{I}d(\mathcal{A}), \subseteq)$. Since $(\mathcal{I}d(\mathcal{A}), \subseteq)$ is a bounded complete lattice, then $(\mathcal{CO}(\mathcal{A}), \preccurlyeq)$ is a bounded complete lattice, where the least element is id and the greatest element is $C : \mathcal{I}d(\mathcal{A}) \to \mathcal{I}d(\mathcal{A})$, where $C(\mathbb{I}) = \mathcal{A}$.

Example 4.9. If $(\mathcal{I}d(\mathcal{A}), \subseteq)$ is a chain, then every ideal of $\mathcal{A}$ is stable. Suppose $\mathbb{I}, \mathbb{Q} \in \mathcal{I}d(\mathcal{A})$. Then $\mathbb{I} \subseteq \mathbb{Q}$ or $\mathbb{Q} \subseteq \mathbb{I}$. If $\mathbb{I} \subseteq \mathbb{Q}$, then $\mathbb{I}^c \subseteq \mathbb{Q}^c$. Thus $\mathbb{I}^c \cap \mathbb{Q}^c = \mathbb{I}^c = (\mathbb{I} \cap \mathbb{Q})^c$. The proof of other case is similar.

Let $\mathcal{A}$ be a $\vee$-hoop. Then $\mathcal{A}$ is called an ***H-hoop*** if for any $p \in \mathcal{A}$, there exists $n \geq 1$ such that $p \vee (p^n)^{\sim} = 1$, where $p^n = p \odot p^{n-1}$ and $p^0 = 1$.

Example 4.10. (i) Consider $\mathcal{A}$ is the hoop as in Example 4.3. Clearly, $\mathcal{A}$ is an H-hoop.
(ii) Assume $\mathcal{A}$ is the hoop as in Example 3.2(iii). Then $\mathcal{A}$ is not an H-hoop, since for any $n \in \mathbb{N}$ we have

$$x \vee (x^n)^{\sim} = x \vee x^{\sim} = x \vee 0 = x \neq 1.$$

Theorem 4.11. *In any H-hoop, every prime ideal of $\mathcal{A}$ is contained in a maximal ideal of $\mathcal{A}$.*

Proof. Let $\mathbb{P} \in \mathrm{Spec}(\mathcal{A})$ and $\mathbb{M} \in \mathcal{I}d(\mathcal{A})$ where $\mathbb{P} \subset \mathbb{M} \subset \mathcal{A}$. We show that $\mathbb{M} \in \mathrm{Max}(\mathcal{A})$. For this, by assumption, $\mathcal{A}$ is an H-hoop, then for any $p \in \mathbb{P}$, there exists $n \in \mathbb{N}$ such that $p \vee (p^n)^{\sim} = 1$. Thus for any $p \in \mathbb{P}$, by Proposition 2.1(ix), $p^{\sim} \wedge (p^n)^{\sim\sim} = 0$ and by Proposition 2.1(viii) we have $p^{\sim} \wedge p^n = 0$. Then $p^{\sim} \wedge p^n \in \mathbb{P}$. Since $\mathbb{P} \in \mathrm{Spec}(\mathcal{A})$, we get $p^{\sim} \in \mathbb{P}$ or $p^n \in \mathbb{P}$. If $p^{\sim} \in \mathbb{P}$, then $p \oplus p^{\sim} = 1 \in \mathbb{P}$, which is a contradiction. Thus $p^{\sim} \notin \mathbb{P}$ and so $p^{\sim} \notin \mathbb{M}$, since $\mathbb{M}$ is a proper ideal of $\mathcal{A}$. Then $\mathbb{M} \subset (\mathbb{M} \cup \{p^{\sim}\}]$. On the other side, $p \in \mathbb{P} \subset \mathbb{M}$, thus $p \in \mathbb{M}$, and so we get $1 \in \langle \mathbb{M} \cup \{p^{\sim}\} \rangle$. Thus, $(\mathbb{M} \cup \{p^{\sim}\}] = \mathcal{A}$. Hence, $\mathbb{M}$ is a maximal ideal of $\mathcal{A}$ that contain $\mathbb{P}$. $\square$

A hoop $\mathcal{A}$ is called a ***hyper-archimedean hoop*** if every ideal of $\mathcal{A}$ is a radical ideal of $\mathcal{A}$.

Example 4.12. According to Example 4.3, clearly $\mathcal{A}$ is a hyper-archimedean hoop.

Example 4.13. Assume $\mathcal{A} = \{0, x, y, z, 1\}$ where $0 \leq x \leq z \leq 1$ and $0 \leq y \leq z \leq 1$. Define the operations $\rightarrow$ and $\odot$ on $\mathcal{A}$ by:

$\rightarrow$	0	x	y		1
0	1	1	1	1	1
x	y	1	y	1	1
y	x	x	1	1	1
	0	x	y	1	1
1	0	x	y	z	1

$\odot$	0	x	y		1
0	0	0	0	0	0
x	0	x	0	x	x
y	0	0	y	y	y
	0	x	y	z	z
1	0	x	y	z	1

Then $(\mathcal{A}, \rightarrow, \odot, 0, 1)$ is a hyper-archimedean hoop which is not an H-hoop, since for any $n \in \mathbb{N}$, $z \vee (z^n)^{\sim} = z \vee 0 = z \neq 1$.

Proposition 4.14. *Consider $\mathcal{A}$ had (DNP). Let $\mathbb{I}, \mathbb{Q}$ be two radical ideals of $\mathcal{A}$ such that $\mathbb{I} \odot \mathbb{Q}$ be a radical ideal. Then $\mathbb{I} \odot \mathbb{Q} = \mathbb{I} \cap \mathbb{Q}$.*

Proof. Let $\mathbb{I}, \mathbb{Q}$ be two radical ideals of $\mathcal{A}$ such that $\mathbb{I} \odot \mathbb{Q}$ be a radical ideal. Then $\mathbb{I} = \mathrm{Rad}(\mathbb{I}) = \bigcap \mathbb{M}_{\mathbb{I}}$ and $\mathbb{Q} = \mathrm{Rad}(\mathbb{Q}) = \bigcap \mathbb{M}_{\mathbb{Q}}$. Since $\mathbb{I} \odot \mathbb{Q}$ is a radical ideal, we get $\mathbb{I} \odot \mathbb{Q} = \mathrm{Rad}(\mathbb{I} \odot \mathbb{Q}) = \bigcap \mathbb{M}_{\mathbb{I} \odot \mathbb{Q}}$. Thus

$$\mathbb{I} \cap \mathbb{Q} = \left(\bigcap \mathbb{M}_{\mathbb{I}}\right) \cap \left(\bigcap \mathbb{M}_{\mathbb{Q}}\right) = \bigcap (\mathbb{M}_{\mathbb{I}} \cap \mathbb{M}_{\mathbb{Q}}).$$

So, it is enough to prove that $\mathbb{M}_{\mathbb{I}} \cap \mathbb{M}_{\mathbb{Q}} = \mathbb{M}_{\mathbb{I} \odot \mathbb{Q}}$. For this, suppose $p \in \mathbb{I} \odot \mathbb{Q}$. Then there exist $x \in \mathbb{I}$ and $y \in \mathbb{Q}$ such that $p = x \odot y$. Since $x \odot y \leq x, y$ and $\mathbb{I}, \mathbb{Q} \in \mathcal{I}d(\mathcal{A})$, we get $p = x \odot y \in \mathbb{I}, \mathbb{Q}$. Hence, it is clear that $\mathbb{M}_{\mathbb{I}} \cap \mathbb{M}_{\mathbb{Q}} \subseteq \mathbb{M}_{\mathbb{I} \odot \mathbb{Q}}$.

Conversely, suppose $\mathbb{M} \in \mathbb{M}_{\mathbb{I} \odot \mathbb{Q}}$. Then $\mathbb{I} \odot \mathbb{Q} \subseteq \mathbb{M}$. By Proposition 2.9, we get $\mathbb{I} \subseteq \mathbb{M}$ or $\mathbb{Q} \subseteq \mathbb{M}$. Consider $\mathbb{I} \subseteq \mathbb{M}$. Since $\mathbb{I} \cap \mathbb{Q} \subseteq \mathbb{I}$ we have $\mathbb{I} \cap \mathbb{Q} \subseteq \mathbb{M}$, and so $\mathbb{M} \in \mathbb{M}_{\mathbb{I} \cap \mathbb{Q}} = \mathbb{M}_{\mathbb{I}} \cap \mathbb{M}_{\mathbb{Q}}$, by Proposition 4.6. Therefore, $\mathbb{I} \odot \mathbb{Q} = \mathbb{I} \cap \mathbb{Q}$. □

Definition 4.15. The operator $C : \mathcal{I}d(\mathcal{A}) \to \mathcal{I}d(\mathcal{A})$ is called a ***semiprime operator*** if for any $\mathbb{I}, \mathbb{Q} \in \mathcal{I}d(\mathcal{A})$, $\mathbb{I}^c \odot \mathbb{Q}^c \subseteq (\mathbb{I} \odot \mathbb{Q})^c$.

Example 4.16. (i) Obviously, the identity map and the constant map, where $\mathbb{I}^c = \mathcal{A}$, are semiprime operators.
(ii) According to Example 4.13 we have $\mathcal{I}d(\mathcal{A}) = \{\{0\}, \{0, x\}, \{0, y\}, \mathcal{A}\}$. Define $C : \mathcal{I}d(\mathcal{A}) \to \mathcal{I}d(\mathcal{A})$ such that $C(\{0\}) = \{0\}$ and $C(\{0, x\}) = C(\{0, y\}) = C(\mathcal{A}) = \mathcal{A}$. Then C is a semiprime operator.
(iii) According to Example 4.3, define $C : \mathcal{I}d(\mathcal{A}) \to \mathcal{I}d(\mathcal{A})$ such that $C(\{0\}) = \{0\}$ and $C(\{0, x\}) = C(\{0, y\}) = C(\mathcal{A}) = \mathcal{A}$. Then C is not a semiprime operator, because

$$C(\{0, x\}) \odot C(\{0, y\}) = \mathcal{A} \odot \mathcal{A} = \mathcal{A} \nsubseteq \{0\} = C(\{0\}) = C(\{0, x\} \odot \{0, y\}).$$

Proposition 4.17. *In any hyper-archimedean hoop, a stable closure operator and semiprime are coincide.*

Proof. Since $\mathcal{A}$ is a hyper-archimedean hoop, by Proposition 4.14, for any $\mathbb{I}, \mathbb{Q} \in \mathcal{I}d(\mathcal{A})$, $\mathbb{I} \cap \mathbb{Q} = \mathbb{I} \odot \mathbb{Q}$. Suppose $C : \mathcal{I}d(\mathcal{A}) \to \mathcal{I}d(\mathcal{A})$ is a stable closure operator on $\mathcal{I}d(\mathcal{A})$. Then for any $\mathbb{I}, \mathbb{Q} \in \mathcal{I}d(\mathcal{A})$, $\mathbb{I}^c \cap \mathbb{Q}^c \subseteq (\mathbb{I} \cap \mathbb{Q})^c$. By assumption, $\mathbb{I}^c \cap \mathbb{Q}^c = \mathbb{I}^c \odot \mathbb{Q}^c$ and $\mathbb{I} \cap \mathbb{Q} = \mathbb{I} \odot \mathbb{Q}$. Hence, $\mathbb{I}^c \odot \mathbb{Q}^c = \mathbb{I}^c \cap \mathbb{Q}^c \subseteq (\mathbb{I} \cap \mathbb{Q})^c = (\mathbb{I} \odot \mathbb{Q})^c$. Thus C is a semiprime closure operator on $\mathcal{I}d(\mathcal{A})$.

Conversely, suppose $C : \mathcal{I}d(\mathcal{A}) \to \mathcal{I}d(\mathcal{A})$ is a semi-prime operator. Then for any $\mathbb{I}, \mathbb{Q} \in \mathcal{I}d(\mathcal{A})$, $\mathbb{I}^c \odot \mathbb{Q}^c \subseteq (\mathbb{I} \odot \mathbb{Q})^c$. Since $\mathcal{A}$ is hyper-archimedean, we have

$$\mathbb{I}^c \cap \mathbb{Q}^c = \mathbb{I}^c \odot \mathbb{Q}^c \subseteq (\mathbb{I} \odot \mathbb{Q})^c = (\mathbb{I} \cap \mathbb{Q})^c.$$

Clearly, $(\mathbb{I} \cap \mathbb{Q})^c \subseteq \mathbb{I}^c \cap \mathbb{Q}^c$, since C is a closure operator. Therefore, in any hyper-archimedean hoop, a stable closure operator and semiprime are coincide. □

Note. Let C be a closure operator on $\mathcal{I}d(\mathcal{A})$ such that $\mathbb{I}^c \neq \mathcal{A}$, for any proper $\mathbb{I} \in \mathcal{I}d(\mathcal{A})$. Then C is called a ***proper closure operator***.

Example 4.18. Obviously, the identity map and Rad, where $\mathbb{I}^c = \mathcal{A}$, are proper closure operators.

Remark 4.19. If C is a proper closure operator, then every maximal ideal of $\mathcal{A}$ is C-closed. Since C is proper, for any $\mathbb{M} \in \mathrm{Max}(\mathcal{A})$, $\mathbb{M} \subseteq \mathbb{M}^c \neq \mathcal{A}$. From $\mathbb{M} \in \mathrm{Max}(\mathcal{A})$, we get $\mathbb{M} = \mathbb{M}^c$, and so $\mathbb{M}$ is C-closed.

Example 4.20. Obviously, the identity map and the $\mathrm{Rad}(\mathcal{A})$ map, where $id : \mathrm{Spec}(\mathcal{A}) \to \mathrm{Spec}(\mathcal{A})$ and $\mathrm{Rad} : \mathrm{Max}(\mathcal{A}) \to \mathrm{Max}(\mathcal{A})$, respectively, are C-closed operators.

Note. Denote $\mathrm{Spec}_c(\mathcal{A}) = \mathrm{Spec}(\mathcal{A}) \cap \mathcal{I}(c)$, where $\mathcal{I}(c)$ is the set of all C-closed ideals of $\mathcal{A}$.

Proposition 4.21. *Let $\mathcal{A}$ be a $\vee$-hoop with (DNP). Then the following equivalent statements hold:*
(i) $\mathcal{A}$ is an H-hoop,
(ii) for any $C \in \mathcal{CO}(\mathcal{A})$, we have $Spec(\mathcal{A}) \cap \mathbb{I}^c = Spec_c(\mathcal{A}) = Max(\mathcal{A})$.

Proof. $(i \Rightarrow ii)$ By Proposition 2.8, $\mathrm{Max}(\mathcal{A}) \subseteq \mathrm{Spec}(\mathcal{A})$. Also, by Remark 4.19, every maximal ideal of $\mathcal{A}$ is C-closed, and so $\mathrm{Max}(\mathcal{A}) \subseteq \mathcal{I}(c)$. Thus, $\mathrm{Max}(\mathcal{A}) \subseteq \mathcal{I}(c) \cap \mathrm{Spec}(\mathcal{A}) \subseteq \mathrm{Spec}(\mathcal{A})$. Hence, $\mathrm{Max}(\mathcal{A}) \subseteq \mathrm{Spec}_c(\mathcal{A}) \subseteq \mathrm{Spec}(\mathcal{A})$. By (i), since $\mathcal{A}$ is an H-hoop, by Theorem 4.11, $\mathrm{Spec}(\mathcal{A}) \subseteq \mathrm{Max}(\mathcal{A})$. Therefore, $\mathrm{Spec}_c(\mathcal{A}) = \mathrm{Max}(\mathcal{A})$.
$(ii \Rightarrow i)$ Suppose $\mathrm{Spec}_c(\mathcal{A}) = \mathrm{Max}(\mathcal{A})$, for any proper closure operator on $\mathcal{I}d(\mathcal{A})$. Let $C = id_{\mathcal{I}d(\mathcal{A})}$. Clearly, $id_{\mathcal{I}d(\mathcal{A})}$ is proper and C-closed. Thus, by assumption $\mathrm{Spec}_{id_{\mathcal{I}d(\mathcal{A})}}(\mathcal{A}) = \mathrm{Max}(\mathcal{A}) = \mathrm{Spec}(\mathcal{A})$. Hence, $\mathcal{A}$ is an H-hoop. Since if $\mathcal{A}$ is not an H-hoop, by Theorem 4.11, there is a prime ideal which is not maximal, and so $\mathrm{Max}(\mathcal{A}) \neq \mathrm{Spec}(\mathcal{A})$, which is a contradiction. □

Proposition 4.22. *If the only proper closure operator on $\mathcal{A}$ is identity, then $\mathcal{A}$ is an H-hoop.*

Proof. Suppose $\mathcal{A}$ is not an H-hoop. Then by Theorem 4.11, $\mathcal{A}$ has a prime ideal such as $\mathbb{P}$ which is not maximal, and so $\mathrm{Rad}(\mathcal{A}) \neq \mathbb{P}$. On the other side, clearly $\mathrm{Rad}(\mathcal{A})$ is a proper closure operator on $\mathcal{A}$ and $\mathrm{Rad} \neq id$, which is a contradiction. Therefore, $\mathcal{A}$ is an H-hoop. □

Proposition 4.23. *Every closure operator C on $\mathcal{A}$ induces a closure operator $\overline{C}$ on $\dfrac{\mathcal{A}}{\mathbb{I}}$. If C is stable (semi-prime), then $\overline{C}$ is stable (semi-prime), too.*

Proof. If $\mathbb{I} \subseteq \mathbb{Q}$, for any $\mathbb{I}, \mathbb{Q} \in \mathcal{I}d(\mathcal{A})$, since $\mathbb{Q} \subseteq \mathbb{Q}^c$, then $\mathbb{I} \subseteq \mathbb{Q}^c$, and so $\frac{\mathbb{Q}^c}{\mathbb{I}}$ is well-defined. Define $\overline{C} : \mathcal{I}d\left(\frac{\mathcal{A}}{\mathbb{I}}\right) \to \mathcal{I}d\left(\frac{\mathcal{A}}{\mathbb{I}}\right)$, where $\left(\frac{\mathbb{Q}}{\mathbb{I}}\right)^{\overline{C}} = \frac{\mathbb{Q}^c}{\mathbb{I}}$. Clearly, $\overline{C}$ is a closure operator on $\dfrac{\mathcal{A}}{\mathbb{I}}$ and if c is stable, then

$$\overline{C}\left(\frac{\mathbb{Q}}{\mathbb{I}} \bigcap \frac{\mathbb{M}}{\mathbb{I}}\right) = \overline{C}\left(\frac{\mathbb{Q} \cap \mathbb{M}}{\mathbb{I}}\right) = \frac{(\mathbb{Q} \cap \mathbb{M})^c}{\mathbb{I}} = \frac{\mathbb{Q}^c \cap \mathbb{M}^c}{\mathbb{I}} = \frac{\mathbb{Q}^c}{\mathbb{I}} \bigcap \frac{\mathbb{M}^c}{\mathbb{I}}$$
$$= \overline{C}\left(\frac{\mathbb{Q}}{\mathbb{I}}\right) \bigcap \overline{C}\left(\frac{\mathbb{M}}{\mathbb{I}}\right).$$

The proof of other case is similar. □

Let $\mathrm{Prc}(\mathcal{A})$ be the set of all proper closure operators on $\mathcal{A}$. Consider $C \in \mathrm{Prc}(\mathcal{A})$. For any $\mathbb{I} \in \mathcal{I}d(\mathcal{A})$ and $\mathbb{M} \in \mathrm{Max}(\mathcal{A})$, where $\mathbb{I} \subseteq \mathbb{M}$, by $(C3)$, $\mathbb{I}^c \subseteq \mathbb{M}^c$. Since $C \in \mathrm{Prc}(\mathcal{A})$, by Remark 4.19, $\mathbb{M}^c = \mathbb{M}$, and so $\mathbb{I}^c \subseteq \mathbb{M}$. Thus, $\mathbb{I}^c \subseteq \mathrm{Rad}(\mathbb{I})$, for any $\mathbb{I} \in \mathcal{I}d(\mathcal{A})$. Hence, by Example 4.18, $C \leq \mathrm{Rad}$. On the other side, we know that $id_{\mathcal{I}d(\mathcal{A})}, \mathrm{Rad} \in \mathrm{Prc}(\mathcal{A})$, which are minimum and maximum proper closure operators on $\mathcal{I}d(\mathcal{A})$. Define two operators $\sqcap$ and $\sqcup$ on $\mathrm{Prc}(\mathcal{A})$ as follows:

$$(c \sqcap d)(\mathbb{I}) = c(\mathbb{I}) \cap d(\mathbb{I}), \quad \text{and} \quad (c \sqcup d)(\mathbb{I}) = (c(\mathbb{I}) \cup d(\mathbb{I})],$$

for any $c, d \in \mathrm{Prc}(\mathcal{A})$. If $\mathrm{Prc}(\mathcal{A})$ is $\sqcup$-closed, then $\langle \mathrm{Prc}(\mathcal{A}), \sqcap, \sqcup, id_{\mathcal{I}d(\mathcal{A})}, \mathrm{Rad}\rangle$ is a bounded lattice.

Notice that the condition $\vee$-closed is necessary, since according to Example 4.3, $\mathbb{I}_1 = \{0, x\}$ and $\mathbb{I}_2 = \{0, y\}$ are proper ideals of $\mathcal{A}$, but $\mathbb{I}_1 \cup \mathbb{I}_2$ is not an ideal of $\mathcal{A}$ since $x \oplus y = 1 \notin \mathbb{I}_1 \cup \mathbb{I}_2$ and so $(\mathbb{I}_1 \cup \mathbb{I}_2] = \mathcal{A}$ which is not proper.

5 Conclusions and future works

The notion of closure operators are defined on elements and ideals of hoops. The properties of closure operators and hoop-homomorphism on hoops are investigated. It is shown that the image of a closure operator on a hoop is isomorphic to a quotient hoop. In addition, by using the notion of closure operator on ideals of hoop, some related results are proved. The concept of proper closer operators on ideals of hoop is defined and it is proved that the set of all proper closure operators on hoops makes a bounded lattice by some operations.

6 Acknowledgements

The authors are very indebted to the editor and anonymous referees for their careful reading and valuable suggestions which helped to improve the readability of the paper.

References

[1] Aaly Kologani. M., Borzooei. R.A., (2019), **On ideal theory of hoop algebras**, *Mathematica Bohemica*, 145(2), pp. 1–22.

[2] Aglianó. P., Ferreirim. I.M.A., Montagna. F., (2000), **Basic hoops: An algebraic study of continuous t-norm**, draft.

[3] Borzooei. R.A., Aaly Kologani. M., (2015), **Local and perfect semi-hoops**, *Journal of Intelligent and Fuzzy Systems*, 29, pp. 223–234.

[4] Borzooei. R.A., Aaly Kologani. M., (2020), **Results on hoops**, *Journal of Algebraic Hyperstructures and Logical Algebras*, 1(1), pp. 61–77.

[5] Borzooei. R.A., Aaly Kologani. M., (2020), **Minimal prime ideals in hoops**, *Journal of Algebraic Hyperstructures and Logical Algebras*, 2(3), pp. 109-120.

[6] Borzooei, B.A., Aaly Kologani, M., **Product of ideals in hoops**, *Soft Computing* 27(17):1-11. DOI: 10.1007/s00500-023-08807-4.

[7] Borzooei. R.A., Aaly Kologani. M., Xin. X.L., Jun. Y.B., **On annihilators in hoops**, *Soft computing*, 26 (2022), 6969-6980.

[8] Bosbach. B., (1969), **Komplementäre halbgruppen. Axiomatik und arithmetik**, *Fundamenta Mathematicae*, 64, pp. 257–287.

[9] Bosbach. B., (1970), **Komplementäre halbgruppen. Kongruenzen and quotienten**, *Fundamenta Mathematicae*, 69, pp. 1–14.

[10] Georgescu. G., Leustean. L., Preoteasa. V., (2005), **Pseudo-hoops**, *Journal of Multiple-Valued logic and Soft Computing*, 11(1-2), pp. 153–184.

[11] Heubo-Kwegna, O.A., Nganou, J.B., (2020), **Closure operations on MV-algebras**, Fuzzy. Set. Sys., 418(6), DOI:10.1016/j.fss.2020.06.020.

[12] Ko, J.M., Kim, Y.C., (2004), **Closure operator on BL-algebras**, Commun. Korean Math. Soc., 19(2), pp. 219–232.

[13] Namdar. A., Borzooei. R.A., Borumand Saeid. A., Aaly Kologani. M., (2017), **Some results in hoop algebras**, *Journal of Intelligent and Fuzzy Systems*, 32, pp. 1805–1813.

[14] Xie. F., Liu. H., (2020), **Ideals in pseudo-hoop algebras**, *Journal of Algebraic Hyperstructures and Logical Algebras*, 1(4), 39–53.

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