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Existence and attractivity results on semi-infinite intervals for integrodifferential equations with non-instantaneous impulsions in Banach spaces

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Abstract

In this article, we study the existence of mild solutions of a non-instantaneous integrodifferential equations on unbounded domain via resolvent operators in Banach space. For our proofs, we employ the semigroups theory and Schauder's fixed point theorem. Moreover, we show that solutions of our problem are attractive. Finally, an example is given to validate the theory part.

1 Introduction

In domains such as population dynamics and optimal control, impulsive integral equations, impulsive integro-differential equations, and impulsive differential equations naturally occur (see the monographs [1, 10, 15, 36, 29, 31]). It appears that the earliest discussion of impulsive systems dates back to Krylov and Bogolyubov's work [33].

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In 1982, Grimmer started by utilizing resolvent operators to demonstrate the existence of integro-differential systems in [22, 27, 28]. The resolvent operator with fixed point technique is the most convenient and appropriate approach for solving integrodifferential equations. Readers may see [22, 28, 35] and the sources given therein for more information on resolvent operators. The existence of nonlocal analytic resolvent operator integro-differential equations in [34, 35] has been demonstrated using facts about resolvent operators and the regularity of evolution integro-differential systems. Because of its applicability in describing numerous issues in physics, fluid dynamics, biological models, and chemical kinetics, integral-differential equations over infinite intervals have sparked a lot of interest see [8, 9, 13, 30]. Many authors have examined qualitative properties such as existence, uniqueness, and stability for many integral, differential and integro-differential equations, (see [1, 18, 32, 14, 17, 30, 39, 3, 4, 5, 6, 7]), and with nonlocal condition in [19, 20, 21, 23].

In [16], Benchohra and Rezoug investigated the existence and local attractivity of the mild solution, defined on a semi-infinite positive real interval $J = [0, \infty)$, for non-autonomous semilinear second order evolution equation of mixed type in a real Banach space. They considered the following problem

$$y''(t) - A(t)y(t) = f\left(t, y(t), \int_0^t K(t,s,y(s))ds\right),\ t \in J,$$
$$y(0) = y_0,\ y'(0) = y_1,$$

where $\{A(t)\}_{0\leq t<+\infty}$ is a family of linear closed operators from E into E, $f : J\times\ E \times E \to E$ is a Carathéodory function, $K : \Delta \times E \to E$ is a continuous function, $\Delta := \{(t, s) \in J \times J : s \leq t\}, y_0, y_1 \in E$ and $(E, |\cdot|)$ is a real Banach space. The results are obtained by using the Mönch fixed point and the Kuratowski measure of noncompactness.

In [2], Abbas *et al.* discussed the existence of mild solutions for the following nonlocal problem of impulsive integrodifferential equations:

$$\begin{cases} u'(t) = Au(t) + \displaystyle\int_0^t \Upsilon(t-s)u(s)ds \\ \qquad\qquad +f(t,u(t)); \quad t \in I_k, k = 0, \ldots, m, \\ u\left(t_k^+\right) = u\left(t_k^-\right) + L_k\left(u\left(t_k^-\right)\right); \quad k = 1, \ldots, m, \\ u(0) + g(u) = u_0 \in E, \end{cases}$$

where $I_0 = [0, t_1], I_k := (t_k, t_{k+1}]; k = 1, \ldots, m, 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T, f : I_k \times E \longrightarrow E; k = 1, \ldots, m, L_k : E \longrightarrow E; k = 1, \ldots, m, g :$

$PC \longrightarrow E$ are given functions, the set PC is given later, E is a real (or complex) Banach space with the norm $\|\cdot\|, u'(t) := du/dt, A : D(A) \subset E \longrightarrow E$ generates a C_0-semigroup on the Banach space E, and $\Upsilon(t)$ is a closed linear operator on E with $D(A) \subset D(\Upsilon)$.

Motivated by the above mentioned works, we investigate the existence and attractivity of mild solutions to the following impulsive integrodifferential equations using resolvent operators:

$$\begin{cases} \chi'(\vartheta) = \Psi\chi(\vartheta) + f(\vartheta, \chi(\vartheta), (H\chi)(\vartheta)) \\ \qquad\qquad + \int_0^\vartheta \Phi(\vartheta-\varrho)\chi(\varrho)d\varrho;\ \text{if } \vartheta \in \Theta_k; k = 0, 1, \ldots, \\ \chi(\vartheta) = g_k\left(\vartheta, \chi\left(\vartheta_k^-\right)\right); \text{if } \vartheta \in \tilde{\Theta}_k; k = 1, 2, \ldots, \\ \chi(0) = \chi_0, \end{cases} \tag{1}$$

where $\Theta_0 = [0,\ \vartheta_1], \Theta_k := (\varrho_k, \vartheta_{k+1}]$ and $\tilde{\Theta}_k = (\vartheta_k, \varrho_k]$ with $0 = \varrho_0 < \vartheta_1 \leq \varrho_1 \leq \vartheta_2 < \ldots < \varrho_{m-1} \leq \vartheta_m \leq \varrho_m \leq \vartheta_{m+1} \leq \ldots \leq +\infty$, $\Psi : D(\Psi) \subset \Xi \to \Xi$ is the infinitesimal generator of a strongly continuous semigroup $\{T(\vartheta)\}_{\vartheta\geq 0}$, $\Phi(\vartheta)$ is a closed linear operator with domain $D(\Psi) \subset D(\Phi(\vartheta))$, the operator H is defined by

$$(H\chi)(\vartheta) = \int_0^a h(\vartheta, \varrho, \chi(\varrho))d\varrho,$$

for $a > 0$, $D_h = \{(\vartheta, \varrho) \in \mathbb{R}^2\ ;\ 0 \leq \varrho \leq \vartheta \leq a\}$, and $h : D_h \times \Xi \to \Xi$. The nonlinear term $f : \Theta_k \times \Xi \times \Xi \to \Xi;\ k = 0, 1, \ldots,\ g_k : \tilde{\Theta}_k \times \Xi \to \Xi;\ k = 1, 2, \ldots$, are a given functions, and $(\Xi, \|\cdot\|)$ is a Banach space. They base their arguments on the fixed point theory and the concept of measure of noncompactness with the help of the resolvent operator.

We would like to point out that our work may be viewed as a continuation of the papers [2, 16]. Indeed, unlike the problem in [16], we have added non-instantaneous impulses to our problem, and we investigate our problem on an unbounded domain, as opposed to the problem in [2]. Finally, we investigate our problem's attractivity in a Banach space. The utilization of the assumed hypotheses and the fixed point theorem are novel in the framework of the investigated problem.

The following is how this work is organized. Section 2 presents preliminary results on the resolvent operators in the sense of Grimmer, as well as the noncompactness measure. Section 3 shows the existence and attractivity of system (1) in Banach space. We will also provide an example to demonstrate the abstract consequence of our effort.

2 Preliminaries

In this section, we will go over some of the notations, definitions, and theorems that will be used throughout the work. We refer to [37, 41] for more details on the notions used in this section.

Let $BC(\tilde{\Theta}, \Xi)$ be the Banach space of all bounded and continuous functions χ mapping $\tilde{\Theta} := [0, +\infty)$ into Ξ, with the usual supremum norm

$$\|\chi\|_{\infty} = \sup_{\vartheta \in \tilde{\Theta}} \|\chi(\vartheta)\|.$$

Theorem 2.1 ([37, 41])**.** *A strongly measurable function $f : \mathbb{R} \to \Xi$ be Bochner integrable if and only if $|f|$ is measurable.*

Let $L^1(\tilde{\Theta}, \Xi)$ be the Banach space of measurable functions $f : \tilde{\Theta} \to \Xi$ which are Bochner integrable, with the norm

$$\|f\|_{L^1} = \int_0^{+\infty} \|f(\vartheta)\| d\vartheta,$$

We consider the following Cauchy problem

$$\begin{cases} \chi'(\vartheta) = \Psi\chi(\vartheta) + \displaystyle\int_0^{\vartheta} \Phi(\vartheta - \varrho)\chi(\varrho)d\varrho; \quad \text{for } \vartheta \geq 0, \\ \chi(0) = \chi_0 \in \Xi. \end{cases} \tag{2}$$

The existence and properties of a resolvent operator has been discussed in [27]. In what follows, we suppose the following assumptions:

($R1$) Ψ is the infinitesimal generator of a uniformly continuous semigroup $\{T(\vartheta)\}_{\vartheta>0}$,

($R2$) For all $\vartheta \geq 0, \Phi(\vartheta)$ is closed linear operator from $D(\Psi)$ to Ξ and $\Phi(\vartheta) \in \Phi(D(\Psi), \Xi)$. For any $\chi \in D(\Psi)$, the map $\vartheta \to \Phi(\vartheta)\chi$ is bounded, differentiable and the derivative $\vartheta \to \Phi'(\vartheta)\chi$ is bounded uniformly continuous on $\mathbb{R}^+$.

Theorem 2.2 ([27])**.** *Assume that $(R1)-(R2)$ hold, then there exists a unique resolvent operator for the Cauchy problem (2).*

Consider the space

$$PC(\mathbb{R}^+,\Xi)=\Big\{\chi:\mathbb{R}^+\to\Xi\ :\ \ \chi|_{\tilde{\Theta}_k}=g_k;\ k=1,2,\dots,\chi|_{\Theta_k};\ k=0,1,\dots,\ \text{are continuous},\ \chi\left(\varrho_k^-\right),\ \chi\left(\varrho_k^+\right),\ \chi\left(\vartheta_k^-\right)\ \textit{and}\ \chi\left(\vartheta_k^+\right)\ \text{exist with}\ \chi\left(\vartheta_k^-\right)=\chi\left(\vartheta_k\right)\Big\},$$

and

$$X=BPC(\mathbb{R}^+,\Xi)=\left\{\chi\in PC(\mathbb{R}^+,\Xi):\chi\text{ is bounded on }\mathbb{R}^+\right\},$$

with respect to the norm

$$\|\chi\|_{BPC}=\sup_{\vartheta\in\mathbb{R}^+}\left\{\|\chi(\vartheta)\|\right\}.$$

Now, we define the Kuratowski measure of noncompactness.

Definition 2.3 ([11])**.** *Let Ξ be a Banach space and Ω_Ξ the bounded subsets of Ξ. The Kuratowski measure of noncompactness is the map $\mu:\Omega_\Xi\to[0,\infty]$ defined by*

$$\mu(\Phi)=\inf\{\epsilon>0:\Phi\subseteq\cup_{i=1}^n\Phi_i\ \textit{and}\ diam(\Phi_i)\le\epsilon\};\ \textit{here}\ \Phi\in\Omega_\Xi,$$

where

$$diam(\Phi_i)=sup\{\|u-v\|_E:u,v\in\Phi_i\}.$$

For more information about measure of noncompactness, see [40, 11, 12, 25].

Lemma 2.4 ([11])**.** *If $\{D\}_{n=0}^{+\infty}$ is sequence of nonempty, bounded and closed subsets of Ξ such that $D_{n+1}\subset D_n$; $(n=1,2,3\dots)$ and if $\lim_{n\to\infty}\mu(D_n)=0$, then the intersection*

$$D_\infty=\bigcap_{n=0}^{+\infty}D_n,$$

is nonempty and compact.

Theorem 2.5 ([26])**.** *Let X be a Banach space, Ω compact convex subset of X and $N:\Omega\to\Omega$ is a continuous map. Then N has at least one fixed point in Ω.*

3 The main result

In this section we discuss the existence of mild solution for the problem (1).

3.1 Existence of mild solutions

Let us recollect the following particular measure of noncompactness that derives from [12], and will be utilized in our main results in order to establish a measure of noncompactness in the space $BPC(\tilde{\Theta}, \Xi)$. Let us fix a nonempty bounded subset H in the space $BPC(\tilde{\Theta}, \Xi)$, for $v \in H$, $T > 0$, $\epsilon > 0$ *and* $\kappa, \tau \in [0, T]$, such that $|\kappa - \tau| \leq \epsilon$. We denote $\omega^T(v, \epsilon)$ the modulus of continuity of the function v on the interval $[0, T]$, namely,

$$\begin{array}{lcl} \omega^T(v,\epsilon) & = & \sup\{\|v(\kappa) - v(\tau)\| \ ; \ \kappa, \tau \in [0,T]\}, \\ \omega^T(H,\epsilon) & = & \sup\{\omega^T(v,\epsilon) \ ; \ v \in H\}, \\ \omega_0^T(H) & = & \lim_{\epsilon \to 0}\{\omega^T(H,\epsilon)\}, \\ \omega_0(H) & = & \lim_{T \to +\infty} \omega_0^T(H). \end{array}$$

If ϑ is fixed from $\tilde{\Theta}$, let us denote $H(\vartheta) = \{v(\vartheta) \in \Xi \ ; \ v \in H\}$ and

$$d^{\Delta}(H(\vartheta)) = \text{diam } (H(\vartheta)) = \sup\{\|u(\vartheta) - v(\vartheta)\| \ ; \ u, \ v \in H\}.$$

Finally, consider the function χ_{BPC} defined on the family of subset of $BPC(\tilde{\Theta}, \Xi)$ by the formula

$$\chi_{BPC}(H) = \omega_0(H) + \lim_{\vartheta \to \infty} \sup_{\vartheta \in \tilde{\Theta}} d^{\Delta}(H(\vartheta)).$$

It can be shown similar to [12] that the function χ_{BPC} is a sublinear measure of noncompactness on the space $BPC(\tilde{\Theta}, \Xi)$.

Definition 3.1. *A function $\chi \in BPC(\tilde{\Theta}, \Xi)$ is called a mild solution of problem (1) if it satisfies*

$$\chi(\vartheta) = \begin{cases} R(\vartheta)\chi_0 + \displaystyle\int_0^{\vartheta} R(\vartheta - \varrho) f(\varrho, \chi(\varrho), (H\chi)(\varrho)) d\varrho; \quad \textit{if } \vartheta \in \Theta_0, \\ \\ R(\vartheta - \varrho_k) \left[g_k(\varrho_k, \chi(\vartheta_k^-)) \right] \\ + \displaystyle\int_{\varrho_k}^{\vartheta} R(\vartheta - \varrho) f(\varrho, \chi(\varrho), (H\chi)(\varrho)) d\varrho; \ \vartheta \in \Theta_k, \\ \\ g_k(\vartheta, \chi(\vartheta_k^-)), \ \textit{if } \vartheta \in \tilde{\Theta}_k, \end{cases}$$

where $k = 1, 2, \ldots$

The hypotheses:

$(H1)$ $f:\tilde{\Theta}\times\Xi\times\Xi\to\Xi$ is a Carathéodory function and there exist a function $\zeta\in L^1(\tilde{\Theta},\mathbb{R}^+)$ and a continuous nondecreasing function $\varphi:\tilde{\Theta}\to(0,+\infty)$, such that :

$$\|f(\vartheta,u,\bar{u})\|\le\zeta(\vartheta)\varphi(\|u\|+\|\bar{u}\|),\qquad\text{for } u,\ \bar{u}\in\Xi.$$

$(H2)$ The function $h:D_h\times\Xi\to\Xi$ is continuous and there exists $c_h>0$, such that

$$\|h(\vartheta,\varrho,u)-h(\vartheta,\varrho,\bar{u})\|\le c_h\|u-\bar{u}\|,\ \text{ for each } (\vartheta,\varrho)\in D_h \textit{ and } u,\ \bar{u}\in\Xi,$$

with

$$h^*=\sup\{\|h(\vartheta,\varrho,0)\|\ ,\ (\vartheta,\varrho)\in D_h\}<\infty.$$

$(H3)$ $g_k:\tilde{\Theta}_k\times\Xi\to\Xi$ are continuous and there exists $\beta_{g_k}\in(0,1);\ k\in\mathbb{N}$, such that

$$\|g_k(\vartheta,u)-g_k(\vartheta,v)\|\le\beta_{g_k}\|u-v\|,\ \text{ for all } u,v\in\Xi,\ k=1,2,\ldots$$

and

$$\max_{k\in\mathbb{N}}\{\beta_{g_k}\}=\beta^*_{g_k}.$$

$(H4)$ Assume that there exist $M_R\ge1$ and $\beta\ge0$, such that

$$\|R(\vartheta)\|_{\Phi(\Xi)}\le M_Re^{-\beta\vartheta},$$

and

$$\lim_{\vartheta\to+\infty}\sup_{\vartheta\in\tilde{\Theta}}\int_0^{\vartheta}e^{-\beta(\vartheta-\varrho)}\zeta(\varrho)d\varrho=0.$$

Theorem 3.2. *Assume that the conditions $(H1)-(H4)$ are satisfied. If*

$$M_R\beta^*_{g_k}<1,$$

then the system (1) has at least one mild solution.

Proof. Transform the problem (1) into a fixed point problem. Consider the operator $\aleph:BPC(\tilde{\Theta},\Xi)\to BPC(\tilde{\Theta},\Xi)$ defined by:

$$\aleph\chi(\vartheta)=\begin{cases}R(\vartheta)\chi_0+\displaystyle\int_0^{\vartheta}R(\vartheta-\varrho)f(\varrho,\chi(\varrho),(H\chi)(\varrho))d\varrho; & \text{if } \vartheta\in\Theta_0,\\[2ex] R(\vartheta-\varrho_k)\left[g_k(\varrho_k,\chi(\vartheta_k^-))\right] & \\ +\displaystyle\int_{\varrho_k}^{\vartheta}R(\vartheta-\varrho)f(\varrho,\chi(\varrho),(H\chi)(\varrho))d\varrho;\ \vartheta\in\Theta_k, & \\[2ex] g_k(\vartheta,\chi(\vartheta_k^-)),\ \vartheta\in\tilde{\Theta}_k, & \end{cases}$$

where $k = 1, 2, \ldots$. Obviously, the fixed points of the operator $\aleph$ are mild solutions of the problem (1). Let $D_\rho = \{\chi \in BPC(\tilde{\Theta}, \Xi) : \|\chi\| \leq \rho\}$, with

$$\max\left\{M_R(\|\chi_0\| + \varphi(K_\rho^*)\|\zeta\|_{L^1}), \frac{M_R(g_0 + \varphi(K_\rho^*)\|\zeta\|_{L^1})}{1 - M_R\beta_{g_k}^*}\right\} \leq \rho,$$

where

$$K_\rho^* = \big((c_h + 1)\rho + ah^*\big),$$

the set D_ρ is bounded, closed and convex.

Step 1 : $\aleph(D_\rho) \subset D_\rho$.

- *Case* 1 : for $\vartheta \in \Theta_0$.
  For each $\chi \in D_\rho$ and by $(H1)$, we have

$$\begin{aligned}\|\aleph\chi(\vartheta)\| &\leq M_R\|\chi_0\| + M_R\int_0^\vartheta \varphi(\|\chi\| + \|H\chi\|)\zeta(\varrho)d\varrho\\ &\leq M_R\|\chi_0\| + M_R\varphi((c_h+1)\rho + ah^*)\|\zeta\|_{L^1},\end{aligned}$$

  then

$$\|\aleph\chi\|_X \leq M_R\Big[\|\chi_0\| + \varphi((c_h+1)\rho + ah^*)\|\zeta\|_{L^1}\Big].$$

- *Case* 2 : for $\vartheta \in \Theta_k$.
  For each $\chi \in D_\rho$ by $(H1), (H2)$ and $(H3)$, we have

$$\|g_k(., u(.))\| \leq \beta_{g_k}^*\|u(\cdot)\| + g_0,$$

  thus

$$\begin{aligned}\|\aleph\chi\|_X &\leq M_R\big[\beta_{g_k}^*\rho + g_0 + \varphi((c_h+1)\rho + ah^*)\|\zeta\|_{L^1}\big],\\ &\leq \rho.\end{aligned}$$

- *Case* 3 : for $\vartheta \in \tilde{\Theta}_k$.
  For each $\chi \in D_\rho$, we have

$$\|\aleph\chi(\vartheta)\| \leq \beta_{g_k}^*\rho + g_0.$$

Hence,

$$\|\aleph\chi\|_X \leq \rho.$$

Step 2: $\aleph$ is continuous.

Let $(\chi_n)_{n\in\mathbb{N}}$ be a sequence such that $\chi_n \to \chi_*$ in Ξ, then

- *Case* 1 : for $\vartheta \in \Theta_0$. We have

$$\begin{aligned}&\|(\aleph\chi_n)(\vartheta) - (\aleph\chi_*)(\vartheta)\| \\ &\quad \leq M_R \int_0^{\vartheta} \|f(\varrho, \chi_n(\varrho), H\chi_n(\varrho)) - f(\varrho, \chi_*(\varrho), H\chi_*(\varrho))\| d\varrho.\end{aligned}$$

By the continuity of h and f, we have

$$h(\vartheta, \varrho, \chi_n(\varrho)) \to h(\vartheta, \varrho, \chi_*(\varrho)) \quad as \quad n \to +\infty,$$

and

$$\|h(\vartheta, \varrho, \chi_n(\varrho)) - h(\vartheta, \varrho, \chi_*(\varrho))\| \leq c_h \|\chi_n - \chi_*\|.$$

By Lebesgue dominated convergence theorem

$$\int_0^{\vartheta} h(\vartheta, \varrho, \chi_n(\varrho)) d\varrho \to \int_0^{\vartheta} h(\vartheta, \varrho, \chi_*(\varrho)) d\varrho, \quad as \quad n \to +\infty,$$

then by $(H1)$, we get

$$f(\varrho, \chi_n(\varrho), H\chi_n(\varrho)) \to f(\varrho, \chi_*(\varrho), H\chi_*(\varrho)), \quad as \quad n \to +\infty,$$

consequently

$$\|\aleph\chi_n - \aleph\chi_*\|_X \to 0, \quad as \quad n \to +\infty.$$

- *Case* 2 : for $\vartheta \in \Theta_k$. We have

$$\begin{aligned}&\|(\aleph\chi_n)(\vartheta) - (\aleph\chi_*)(\vartheta)\| \\ &\quad \leq M_R \|g_k(\varrho_k, \chi_n(\vartheta_k^-)) - g_k((\varrho_k, \chi_*(\vartheta_k^-)))\| \\ &\qquad + M_R \int_{\varrho_k}^{\vartheta} \|f(\varrho, \chi_n(\varrho), H\chi_n(\varrho)) - f(\varrho, \chi_*(\varrho), H\chi_*(\varrho))\| d\varrho.\end{aligned}$$

Similar to Case 1, by the continuity of h, f *and* g_k, we get

$$\|\aleph\chi_n - \aleph\chi_*\|_X \to 0, \quad as \quad n \to +\infty.$$

- *Case* 3 : for $\vartheta \in \tilde{\Theta}_k$. We have

$$\|(\aleph\chi_n)(\vartheta) - (\aleph\chi_*)(\vartheta)\| \quad \leq \quad \|g_k(\vartheta, \chi_n(\vartheta_k^-)) - g_k((\vartheta, \chi_*(\vartheta_k^-)))\|.$$

By the continuity of g_k, we obtain

$$\|\aleph\chi_n - \aleph\chi_*\|_X \to 0, \quad as \quad n \to +\infty.$$

Thus, $\aleph$ is continuous.
Step 3: In the sequel, we consider the sequence of sets $\{\Omega_n\}_{n=0}^{+\infty}$ defined by induction as follows :

$$\Omega_0 = D_\rho, \quad \Omega_{n+1} = conv\big(\aleph(\Omega_n)\big); \text{ for } n = 0,1,2\ldots, \quad \Omega_\infty = \bigcap_{n=0}^{+\infty} \Omega_n,$$

this sequence is nondecreasing, i.e. $\Omega_{n+1} \subset \Omega_n$ for each $n \in \mathbb{N}$.
Now we will prove that $\lim_{n\to+\infty} \chi_{BPC}(\Omega_n)$ =0, so for $T > 0$ *and* $k_0 \in \mathbb{N}$, with $T \geq \vartheta_{k_0}$ and $\chi \in \Pi$, we have

- *Case* 1 : for $\kappa, \tau \in \Theta_0$.

$$\begin{aligned}\|\aleph\chi(\kappa) - \aleph\chi(\tau)\| &\leq \|R(\kappa) - R(\tau)\|\|\chi_0\| \\ &\quad + \int_0^\kappa \|R(\kappa-\varrho) - R(\tau-\varrho)\|\zeta(\varrho)d\varrho \\ &\quad \times \varphi(\|\chi\|_{BPC} + \|H\chi\|_{BPC}) \\ &\quad + \int_\kappa^\tau \|R(\tau-\varrho)\|\zeta(\varrho)\varphi(K_\rho^*)d\varrho, \\ &\leq \|R(\kappa) - R(\tau)\|\|\chi_0\| \\ &\quad + \varphi(K_\rho^*)\int_0^\kappa \|R(\kappa-\varrho) - R(\tau-\varrho)\|\zeta(\varrho)d\varrho \\ &\quad + M_R\varphi(K_\rho^*)\int_\kappa^\tau \zeta(\varrho)d\varrho.\end{aligned}$$

  By the strong continuity of $R(\vartheta)$ and $(H1)$, we get

$$\|\aleph\chi(\kappa) - \aleph\chi(\tau)\| \to 0, \ as \ \kappa \to \tau.$$

- *Case* 2 : for $\kappa, \tau \in \Theta_k$.

$$\begin{aligned}\|\aleph\chi(\kappa) - \aleph\chi(\tau)\| &\leq \|R(\kappa-\varrho_k) - R(\tau-\varrho_k)\|\|g(\varrho_k, \chi(\vartheta_k^-))\| \\ &\quad + \int_{\varrho_k}^\kappa \|R(\kappa-\varrho) - R(\tau-\varrho)\|\zeta(\varrho)\varphi(K_\rho^*)d\varrho \\ &\quad + \int_\kappa^\tau \|R(\tau-\varrho)\|\zeta(\varrho)\varphi(K_\rho^*)d\varrho \\ &\leq \|R(\kappa-\varrho_k) - R(\tau-\varrho_k)\|(\beta_{g_k}^*\rho + g_0) \\ &\quad + \varphi(K_\rho^*)\int_{\varrho_k}^\kappa \|R(\kappa-\varrho) - R(\tau-\varrho)\|\zeta(\varrho)d\varrho \\ &\quad + M_R\varphi(K_\rho^*)\int_\kappa^\tau \zeta(\varrho)d\varrho.\end{aligned}$$

Since $R(\vartheta)$ is norm continuous and by $(H1)$, we obtain

$$\|\aleph\chi(\kappa) - \aleph\chi(\tau)\| \to 0, \ as \ \kappa \to \tau.$$

- $Case\ 3$: for $\kappa, \tau \in \tilde{\Theta}_k$. We have

$$\|\aleph\chi(\kappa) - \aleph\chi(\tau)\| = \|g_k(\kappa, \chi(\vartheta_k^-)) - g_k(\tau, \chi(\vartheta_k^-))\|.$$

From $(H3)$, the set $\left\{g_k(\vartheta, \chi(\vartheta_k^-)\right\}_{k=1}^{k_0}$ is equicontinuous, then

$$\|\aleph\chi(\kappa) - \aleph\chi(\tau)\| \to 0, \ as \ \kappa \to \tau.$$

Finally, the set $\aleph(\Omega_{n+1})$ is equicontinuous, then $\omega_0(\aleph(\Omega_{n+1})) = 0$.
Now for $u, \chi \in \Omega_n$ and $\vartheta \in [0, T]$, we have three cases:

- $Case\ 1$: for $\vartheta \in \Theta_0$.

$$\begin{aligned}
&\|(\aleph\chi)(\vartheta) - (\aleph u)(\vartheta)\| \\
&\quad \leq M_R \int_0^{\vartheta} e^{-\beta(\vartheta-\varrho)} \|f(\varrho, \chi(\varrho), H\chi(\varrho)) - f(\varrho, u(\varrho), Hu(\varrho))\| d\varrho \\
&\quad \leq 2M_R\varphi(K_\rho^*) \int_0^{\vartheta} e^{-\beta(\vartheta-\varrho)} \zeta(\varrho) d\varrho.
\end{aligned}$$

Then

$$\sup_{\vartheta \in \Theta_0} d^{\Delta}(\aleph\{\Omega_n(\vartheta)\}) \leq 2M_R\varphi(K_\rho^*) \sup_{\vartheta \in \tilde{\Theta}} \int_0^{\vartheta} e^{-\beta(\vartheta-\varrho)} \zeta(\varrho) d\varrho,$$

when $\vartheta \to +\infty$ and by $(H4)$, we get

$$\lim_{n \to +\infty} \chi_{BPC}(\Omega_n) = 0.$$

- $Case\ 2$: for $\vartheta \in \Theta_k$. We have

$$\begin{aligned}
&\|(\aleph\chi)(\vartheta) - (\aleph u)(\vartheta)\| \\
&\quad \leq M_R \|g_k(\varrho_k, u(\vartheta_k^-)) - g_k((\varrho_k, \chi(\vartheta_k^-)))\| \\
&\qquad + M_R \int_{\varrho_k}^{\vartheta} e^{-\beta(\vartheta-\varrho)} \|f(\varrho, u(\varrho), Hu(\varrho)) - f(\varrho, \chi(\varrho), H\chi(\varrho))\| d\varrho \\
&\quad \leq M_R\beta_{g_k}^* \|u(\varrho_k) - \chi(\varrho_k)\| + 2M_R\varphi(K_\rho^*) \int_{\varrho_k}^{\vartheta} e^{-\beta(\vartheta-\varrho)} \zeta(\varrho) d\varrho,
\end{aligned}$$

when $\vartheta \to +\infty$ and by $(H4)$, we get

$$\chi_{BPC}(\aleph(\Omega_n)) \leq (M_R\beta^*_{g_k})\chi_{BPC}(\Omega_n),$$

so

$$\chi_{BPC}(\Omega_{n+1}) \leq (M_R\beta^*_{g_k})\chi_{BPC}(\Omega_n).$$

- *Case* 3 : for $\vartheta \in \tilde{\Theta}_k$. We have

$$\begin{aligned}\|(\aleph\chi)(\vartheta) - (\aleph u)(\vartheta)\| &\leq \|g_k(\vartheta, u(\vartheta_k^-)) - g_k((\vartheta, \chi(\vartheta_k^-)))\| \\ &\leq \beta^*_{g_k}\|u(\vartheta) - \chi(\vartheta)\|.\end{aligned}$$

then

$$\chi_{BPC}(\aleph(\Omega_n)) \leq \beta^*_{g_k}\ \chi_{BPC}(\Omega_n),$$

Therefore

$$\chi_{BPC}(\Omega_{n+1}) \leq \beta^*_{g_k}\chi_{BPC}(\Omega_n).$$

By the method of mathematical induction, we can get

$$\begin{aligned}\chi_{BPC}(\Omega_{n+1}) &\leq (M_R\beta^*_{g_k})^{n+1}\chi_{BPC}(\Omega_0), \qquad \text{for all } \vartheta \in \Theta_k,\ k = 1, 2, \ldots \\ \chi_{BPC}(\Omega_{n+1}) &\leq (\beta^*_{g_k})^{n+1}\chi_{BPC}(\Omega_0), \qquad \text{for all } \vartheta \in \tilde{\Theta}_k,\ k = 0, 1, \ldots\end{aligned}$$

Then, we obtain

$$\lim_{n\to+\infty} \chi_{BPC}(\Omega_n) = 0.$$

Taking into account Lemma 2.4, we infer that $\Omega_\infty = \cap_{n=0}^{+\infty}\Omega_n$ is nonempty, convex and compact. As a consequence of these three steps together with Theorem 2.5, we can conclude that $\aleph : \Omega_\infty \to \Omega_\infty$, has at least one fixed point, which is a mild solution of problem (1). □

3.2 Attractivity of solutions

In this section we study the local attractivity of solutions for the problem (1). Firstly, we introduce the following concept of attractivity of solutions.

Definition 3.3 ([24]). *We say that the solutions of (1) are locally attractive if there exists a closed ball $\Phi(z^*, \gamma)$ in the space X for some $z^* \in X$ such that for arbitrary solutions z and $\widetilde{z}$ of (1) belonging to $\Phi(z^*, \gamma)$, we have that*

$$\lim_{\vartheta\to+\infty}(z(\vartheta) - \widetilde{z}(\vartheta)) = 0.$$

When the last limit is uniform with respect to $\Phi(z^*, \gamma)$, solutions of problem (1) are said to be uniformly locally attractive (or equivalently that solutions of (1) are locally asymptotically stable).

Lets z^* be a solution of (1), $\Phi_\gamma = \Phi(z^*, \gamma)$ the closed ball in X and a constant
$K^*_\gamma = ((c_h + 1)\gamma + ah^*)$, depends on a positive constant γ.

Theorem 3.4. *Suppose that hypotheses $(H1)-(H4)$ and (H^*) hold, with*

$$\max\left\{2M_R(\|\chi_0\| + \varphi(K^*_\gamma)\|\zeta\|_{L^1}),\ 2M_R(\beta^*_{g_k}\gamma + g_0 + \varphi(K^*_\gamma)\|\zeta\|_{L^1})\right\} \leq \gamma,$$

and

$$M_R \max\{\beta^*_{g_k}\} < \frac{1}{2},$$

then, the solutions of problem (1) are uniformly locally attractive.

Proof. For $z \in \Phi(z^*, \gamma)$ by $(H1)$ and $(H3)$, we get

- *Case* 1 : for $\vartheta \in \Theta_0$. We have

$$\begin{aligned}\|(\aleph z)(\vartheta) - z^*(\vartheta)\| &= \|(\aleph z)(\vartheta) - (\aleph z^*)(\vartheta)\| \\ &\leq M_R \int_0^\vartheta \|f(\varrho, z(\varrho), Hz(\varrho)) - f(\varrho, z^*(\varrho), Hz^*(\varrho))\| d\varrho \\ &\leq 2M_R(\|\chi_0\| + \varphi(K^*_\gamma)\|\zeta\|_{L^1}) \\ &\leq \gamma.\end{aligned}$$

This proves that $\aleph(\Phi_\gamma) \subset \Phi_\gamma$.
So, for each $z, \widetilde{z} \in \Phi(z^*, \gamma)$ solutions of problem (1) and $\vartheta \in \Theta_0$, we have

$$\begin{aligned}\|z(\vartheta) - \widetilde{z}(\vartheta)\| &= \|(\aleph z)(\vartheta) - (\aleph \widetilde{z})(\vartheta)\| \\ &\leq 2M_R\varphi(K^*_\gamma) \sup_{\vartheta\in\Theta_0} \int_0^\vartheta e^{-\beta(\vartheta-\varrho)}\zeta(\varrho)d\varrho \\ &\leq 2M_R\varphi(K^*_\gamma) \sup_{\vartheta\in\tilde{\Theta}} \int_0^\vartheta e^{-\beta(\vartheta-\varrho)}\zeta(\varrho)d\varrho,\end{aligned}$$

by $(H4)$, we conclude that

$$\|z(\vartheta) - \widetilde{z}(\vartheta)\| \to 0, \quad as\ \vartheta \to +\infty.$$

- *Case* 2 : for $\vartheta \in \Theta_k$. we have

$$\begin{aligned}\|(\aleph z)(\vartheta) - z^*(\vartheta)\| &\leq M_R\|g_k(\varrho_k, z(\vartheta_k^-)) - g_k((\varrho_k, z^*(\vartheta_k^-)))\| \\ &+ M_R \int_{\varrho_k}^\vartheta \|f(\varrho, z(\varrho), Hz(\varrho)) - f(\varrho, z^*(\varrho), Hz^*(\varrho))\| d\varrho \\ &\leq 2M_R(\beta^*_{g_k}\gamma + g_0 + \varphi(K^*_\gamma)\|\zeta\|_{L^1}) \\ &\leq \gamma.\end{aligned}$$

Therefore, $\aleph(\Phi_\gamma) \subset \Phi_\gamma$.
So, for each $z, \widetilde{z} \in \Phi(z^*, \gamma)$ solutions of problem (1) and $\vartheta \in \Theta_k$, we have

$$\begin{aligned}\|z(\vartheta) - \widetilde{z}(\vartheta)\| &= \|(\aleph z)(\vartheta) - (\aleph \widetilde{z})(\vartheta)\| \\ &\leq M_R e^{-\beta(\vartheta-\varrho_k)} \|g_k(\varrho_k, z(\vartheta_k^-)) - g_k((\varrho_k, \widetilde{z}(\vartheta_k^-)))\| \\ &\quad + 2M_R \varphi(K_\gamma^*) \sup_{\vartheta \in \Theta_k} \int_{\varrho_k}^{\vartheta} e^{-\beta(\vartheta-\varrho)} \zeta(\varrho) d\varrho \\ &\leq M_R e^{-\beta(\vartheta-\varrho_k)} \beta_{g_k}^* \|z(\varrho_k) - \widetilde{z}(\varrho_k)\| \\ &\quad + 2M_R \varphi(K_\gamma^*) \sup_{\vartheta \in \tilde{\Theta}} \int_0^{\vartheta} e^{-\beta(\vartheta-\varrho)} \zeta(\varrho) d\varrho,\end{aligned}$$

Then, using $(H4)$, we obtain

$$\|z(\vartheta) - \widetilde{z}(\vartheta)\| \to 0, \quad as\ \vartheta \to +\infty.$$

- *Case* 3 : for $\vartheta \in \tilde{\Theta}_k$. we have

$$\begin{aligned}\|(\aleph z)(\vartheta) - z^*(\vartheta)\| &= \|(\aleph z)(\vartheta) - (\aleph z^*)(\vartheta)\| \\ &\leq \|g(\vartheta, z(\vartheta_k^-)) - g((\vartheta, z^*(\vartheta_k^-)))\| \\ &\leq 2\beta_{g_k}^* \gamma \\ &\leq \gamma.\end{aligned}$$

Thus, $\aleph(\Phi_\gamma) \subset \Phi_\gamma$.
So, for each $z, \widetilde{z} \in \Phi(z^*, \gamma)$ solutions of problem (1) and $\vartheta \in \Theta_k$, we have

$$\begin{aligned}\|z(\vartheta) - \widetilde{z}(\vartheta)\| &= \|(\aleph z)(\vartheta) - (\aleph \widetilde{z})(\vartheta)\| \\ &\leq \|g(\vartheta, z(\vartheta_k^-)) - g((\vartheta, \widetilde{z}(\vartheta_k^-)))\| \\ &\leq \beta_{g_k}^* \|z(\vartheta) - \widetilde{z}(\vartheta)\|,\end{aligned}$$

then

$$(1 - \beta_{g_k}^*)\|z(\vartheta) - \widetilde{z}(\vartheta)\| \leq 0,$$

hence

$$\|z(\vartheta) - \widetilde{z}(\vartheta)\| = 0.$$

As a result, the solutions of the problem (1) are uniformly locally attractive.
□

4 An example

Consider the following impulsive integro-differential equation :

$$\begin{cases} \frac{\partial}{\partial\vartheta}\xi(\vartheta,x) = \varepsilon^2 \frac{\partial^2\xi(\vartheta,x)}{\partial x^2} - \int_0^\vartheta \Lambda(\vartheta-\varrho)\frac{\partial^2\xi(\varrho,x)}{\partial x^2}d\varrho \\ \qquad + \frac{\mu e^{-(1-b^{-1})\vartheta}}{1+e^\vartheta} \arctan\left(\int_0^a e^{-\varrho\vartheta-\varrho}|\xi(\varrho,x)|d\varrho\right), \\ \qquad - \frac{\mu\|\xi(\vartheta,x)\|_{L^2}}{e^{(1-b^{-1})\vartheta}(1+\sqrt{\vartheta})}, \quad \text{if } \vartheta\in\Theta_k,\ x\in(0,\pi), \\ \xi(\vartheta,x) = \frac{\xi(k^- - 1,x)}{33(1+|\xi(k^- - 1,x)|)}, \text{ if } \vartheta\in\tilde{\Theta}_k,\ x\in(0,\pi), \\ \xi(\vartheta,0) = \xi(\vartheta,\pi) = 0, \quad \vartheta\in\mathbb{R}^+, \\ \xi(0,x) = \xi_0(x), \quad x\in(0,\pi), \end{cases} \tag{3}$$

where $\Theta_k = (k, k+1], k = 0, 1, \dots,\ \tilde{\Theta}_k = (k-1, k], k = 1, 2, \dots, \varepsilon > 0,\ b > 1,$ $\mu \in \left(0, \frac{5}{12}\right)$ and $\Lambda : \mathbb{R}^+ \to \mathbb{R}$ is a C^1-function. Now, we define

$$\xi(\vartheta)(x) = \xi(\vartheta, x),$$

$$f(\vartheta,\xi(\vartheta),H\xi(\vartheta))(x) = -\frac{\mu\|\xi(\vartheta,x)\|_{L^2}}{e^{(1-b^{-1})\vartheta}(1+\sqrt{\vartheta})} + \frac{\mu e^{-(1-b^{-1})\vartheta}}{(1+e^\vartheta)} \arctan\left(\int_0^a e^{-\varrho\vartheta-\varrho}|\xi(\varrho,x)|d\varrho\right),$$

$$H\xi(\vartheta)(x) = \int_0^a e^{-\varrho\vartheta-\varrho}|\xi(\varrho,x)|d\varrho,$$

$$g_k(\vartheta,\xi(\vartheta_{k^-},x)) = \frac{\xi(k^- - 1,x)}{33(1+|\xi(k^- - 1,x)|)},$$

$$\Phi(\vartheta) = \Gamma(\vartheta)\Psi.$$

To rewrite system (3) in the abstract form, we introduce the space $X = L^2(0,\pi)$ and let Ψ be defined by

$$\begin{cases} D(\Psi) = \{\varphi\in L^2(0,\pi)\ /\ \varphi,\ \varphi''\in L^2(0,\pi)\ ,\ \varphi(0)=\varphi(\pi)=0\}, \\ (\Psi\varphi)(x) = \frac{\partial^2\varphi(\vartheta,x)}{\partial x^2}, \end{cases}$$

It is well known that Ψ generates a strongly continuous semigroup $(S(\vartheta))_{\vartheta\geq 0}$, which is dissipative and compact with $\|S(\vartheta)\| \leq e^{-\varepsilon^2\vartheta}$, and for some $b > \frac{1}{\varepsilon^2}$,

we assume that $\|\Gamma(\vartheta)\| \leq \frac{e^{-\varepsilon^2\vartheta}}{b}$, and $\|\Gamma'(\vartheta)\| \leq \frac{e^{-\varepsilon^2\vartheta}}{b^2}$. It follows from [27], that $\|R(\vartheta)\| \leq e^{-\varkappa\vartheta}$, where $\varkappa = 1 - b^{-1}$.
More appropriate conditions on operator Φ, $(H4)$ hold with $M_R = 1$ and $\beta = 1 - b^{-1}$.
Then, system (3) takes the following abstract form

$$\begin{cases} \xi'(\vartheta) = \Psi\xi(\vartheta) + f\left(\vartheta, \xi(\vartheta), (H\xi)(\vartheta)\right) + \int_0^\vartheta \Phi(\vartheta - \varrho)\xi(\varrho)d\varrho, \text{ if } \vartheta \in \Theta_k, \\ \xi(\vartheta) = g_k\left(\vartheta, \xi\left(\vartheta_k^-\right)\right), \text{if } \vartheta \in \tilde{\Theta}_k, \\ \xi(0) = \xi_0, \end{cases} \tag{4}$$

where $Y = BPC(\mathbb{R}^+, X)$. With the help of simple computation, we find that

$$\zeta(\vartheta) = \frac{\mu e^{-(1-b^{-1})\vartheta}}{1+e^\vartheta},\ \varphi(x) = \varepsilon + x,\ \beta^*_{g_k} = \frac{1}{33},$$

$$L_\wp = \frac{\pi^2}{24},\ c_h = a = \frac{12 - \pi^2 - 24\|\zeta\|_{L^1}}{24}.$$

Then

$$|f(\vartheta, \xi_1(\vartheta), \xi_2(\vartheta))| \leq \frac{\mu e^{-(1-b^{-1})\vartheta}}{1+e^\vartheta}\bigg(|\xi_1(\vartheta)| + |\xi_2(\vartheta)| + \varepsilon\bigg),\ for\ \varepsilon > 0.$$

The function $\varphi(\vartheta) = \varepsilon + x$ is continuous nondecreasing from $\tilde{\Theta}$ to $[\varepsilon, +\infty)$, we have

$$\zeta \in \quad L^1(\tilde{\Theta}, \mathbb{R}^+),$$

$$\lim_{\vartheta\to+\infty} \sup_{\vartheta\in\tilde{\Theta}} \int_0^\vartheta e^{-\beta(\vartheta-\varrho)}\zeta(\varrho)d\varrho = \lim_{\vartheta\to+\infty} \sup_{\vartheta\in\tilde{\Theta}} \int_0^\vartheta \frac{\mu e^{-(1-b^{-1})\vartheta}}{1+e^\varrho}d\varrho = 0,$$

and

$$M_R \max\{\beta^*_{g_k}\} = 0,4112\ldots < \frac{1}{2}.$$

Also, we can choose the values of ρ and γ so that Theorem 3.2 and Theorem 3.4 are applicable. Consequently, the problem (3) has at least one mild solution defined on $\mathbb{R}^+$, which is uniformly locally attractive.

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