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On an eigenvalue problem associated with the $(p, q)-$ Laplacian

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Dedicated to Professor Adrian Petruşel on the occasion of his 60th anniversary

Abstract

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$. Consider the following generalized Robin-Steklov eigenvalue problem associated with the operator $\mathcal{A}u = -\Delta_p u - \Delta_q u$

$$\begin{cases} \mathcal{A}u + \rho_1(x) \mid u \mid^{p-2} u + \rho_2(x) \mid u \mid^{q-2} u = \lambda\alpha(x) \mid u \mid^{r-2} u, \ x \in \Omega, \\ \frac{\partial u}{\partial \nu_A} + \gamma_1(x) \mid u \mid^{p-2} u + \gamma_2(x) \mid u \mid^{q-2} u = \lambda\beta(x) \mid u \mid^{r-2} u, \ x \in \partial\Omega, \end{cases}$$

where $p,\ q,\ r \in (1, \infty)$, $p < q$; $\alpha,\ \rho_i \in L^\infty(\Omega)$ and $\beta,\ \gamma_i \in L^\infty(\partial\Omega)$ are nonnegative functions satisfying $\int_\Omega \alpha\ dx + \int_{\partial\Omega} \beta\ d\sigma > 0$ and $\int_\Omega \rho_i\ dx + \int_{\partial\Omega} \gamma_i\ d\sigma > 0$, $i = 1, 2$.

We show that, if either $\big(r < p\big)$ or $\big(r > q$ with $r < q(N-1)/(N-q)$ in case $q < N\big)$, then the eigenvalue set (spectrum) of the above problem is precisely $(0, \infty)$. If $r \in \{p, q\}$ then the corresponding spectrum is a smaller interval (d, ∞), $d > 0$. On the other hand, if $\big(r \in (p, q)$ with $r < p(N-1)/(N-p)$ in case $p < N\big)$, then we are able to identify an interval of eigenvalues $[\lambda^*, \infty)$, where λ^* is a positive number depending on r.

Obviously, the spectrum of the above problem coincides with the spectra of the Neumann-like, Robin-like, and Steklov-like eigenvalue problems corresponding to the cases when some of the functions α, β, γ_i vanish.

Key Words: Eigenvalues, $(p, q)-$Laplacian, parametric boundary condition, Nehari manifold, variational methods.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$. Consider the eigenvalue problem associated with the operator $\mathcal{A}u = -\Delta_p u - \Delta_q u$

$$\begin{cases} \mathcal{A}u + \rho_1(x) \mid u \mid^{p-2} u + \rho_2(x) \mid u \mid^{q-2} u = \lambda\alpha(x) \mid u \mid^{r-2} u, x \in \Omega, \\ \frac{\partial u}{\partial \nu_{\mathcal{A}}} + \gamma_1(x) \mid u \mid^{p-2} u + \gamma_2(x) \mid u \mid^{q-2} u = \lambda\beta(x) \mid u \mid^{r-2} u, x \in \partial\Omega, \end{cases} \tag{1}$$

under the following hypotheses
(h_{pqr}) $p,\ q,\ r \in (1, \infty),\ p < q$;

$(h_{\alpha\beta})$ $\alpha \in L^\infty(\Omega)$ and $\beta \in L^\infty(\partial\Omega)$ are nonnegative functions satisfying

$$\int\limits_{\Omega} \alpha\ dx + \int\limits_{\partial\Omega} \beta\ d\sigma > 0; \tag{2}$$

$(h_{\rho_i\gamma_i})$ $\rho_i \in L^\infty(\Omega),\ i = 1, 2,$ and $\gamma_i \in L^\infty(\partial\Omega),\ i = 1, 2,$ are nonnegative functions such that

$$\int\limits_{\Omega} \rho_i\ dx + \int\limits_{\partial\Omega} \gamma_i\ d\sigma > 0,\ i = 1, 2. \tag{3}$$

Recall that, for $\theta \in (1, \infty)$, Δ_θ denotes the θ-Laplacian, $\Delta_\theta u = \operatorname{div}(|\nabla u|^{\theta-2}\nabla u)$. In the above boundary condition we have used the notation

$$\frac{\partial u}{\partial \nu_{\mathcal{A}}} := \left(\mid \nabla u \mid^{p-2} + \mid \nabla u \mid^{q-2} \right) \frac{\partial u}{\partial \nu},$$

where ν is the outward unit normal to $\partial\Omega$.

The operator $\left(\Delta_p + \Delta_q\right)$, called (p, q)-Laplacian, occurs in many applications that include models of elementary particles ([9], [14]), elasticity theory ([21]), reaction-diffusion equations ([12]).

The solution u of (1) is understood as an element of the Sobolev space $W := W^{1,q}(\Omega)$ satisfying equation $(1)_1$ in the sense of distributions and $(1)_2$ in the sense of traces.

Definition 1.1. *A scalar $\lambda \in \mathbb{R}$ is said to be an eigenvalue of the problem* (1) *if there exists* $u_\lambda \in W \setminus \{0\}$ *such that for all* $w \in W$

$$\begin{aligned}
&\int_\Omega \big(\mid \nabla u_\lambda \mid^{p-2} + \mid \nabla u_\lambda \mid^{q-2}\big)\nabla u_\lambda \cdot \nabla w \; dx \\
&\quad + \int_\Omega \big(\rho_1 \mid u_\lambda \mid^{p-2} + \rho_2 \mid u_\lambda \mid^{q-2}\big) u_\lambda w \; dx \\
&\quad + \int_{\partial\Omega} \big(\gamma_1 \mid u_\lambda \mid^{p-2} + \gamma_2 \mid u_\lambda \mid^{q-2}\big) u_\lambda w \; d\sigma \\
&\quad = \lambda \left(\int_\Omega \alpha \mid u_\lambda \mid^{r-2} u_\lambda w \; dx + \int_{\partial\Omega} \beta \mid u_\lambda \mid^{r-2} u_\lambda w \; d\sigma \right).
\end{aligned} \tag{4}$$

This u_λ *is called an eigenfunction of the problem* (1) *(corresponding to the eigenvalue* λ*).*

According to a Green type formula (see [11], p. 71), $u \in W \setminus \{0\}$ is a solution of (1) if and only if it satisfies (4).

Remark 1.2. *Choosing* $w = u_\lambda$ *in* (4) *shows that the eigenvalues of problem* (1) *cannot be negative. It is also obvious, taking into account the assumptions* $(h_{\rho_i \gamma_i})$, *that* 0 *can not be an eigenvalue of problem* (1).
Now, let us introduce the notations

$$\begin{aligned}
K_p(u) &:= \int_\Omega \big(\mid \nabla u \mid^p + \rho_1 \mid u \mid^p\big) dx + \int_{\partial\Omega} \gamma_1 \mid u \mid^p \, d\sigma, \\
K_q(u) &:= \int_\Omega \big(\mid \nabla u \mid^q + \rho_2 \mid u \mid^q\big) dx + \int_{\partial\Omega} \gamma_2 \mid u \mid^q \, d\sigma, \\
k_r(u) &:= \int_\Omega \alpha \mid u \mid^r \, dx + \int_{\partial\Omega} \beta \mid u \mid^r \, d\sigma \;\; \forall \; u \in W.
\end{aligned} \tag{5}$$

Note that any eigenfunction u_λ *corresponding to an eigenvalue* $\lambda > 0$ *satisfies* $k_r(u_\lambda) > 0$, *hence all eigenfunctions corresponding to positive eigenvalues necessarily belong to* $W \setminus \mathcal{Z}$, $\mathcal{Z} := \{v \in W; \; k_r(v) = 0\}$.

In order to state our main results, let us define

$$\widehat{\lambda}_q := \inf_{w \in W \setminus \mathcal{Z}} \frac{K_q(w)}{k_q(w)}, \; \widehat{\lambda}_p := \inf_{w \in W \setminus \mathcal{Z}} \frac{K_p(w)}{k_p(w)}, \tag{6}$$

$$\begin{aligned} &\lambda_* := \inf_{w\in W\setminus \mathcal{Z}} \Gamma \frac{K_p(w)^\omega K_q(w)^{1-\omega}}{k_r(w)},\ \lambda^* = \frac{r}{p^\omega q^{1-\omega}}\lambda_*,\\ &\omega := \frac{q-r}{q-p},\quad \Gamma := \frac{q-p}{(r-p)^{1-\omega}(q-r)^\omega}. \end{aligned} \tag{7}$$

Let us now state the main results of this paper.

Theorem 1.3. *Assume that (h_{pqr}), $(h_{\alpha\beta})$, $(h_{\rho_i\gamma_i})$ are fulfilled.*
(a) If $r = p$, then $\widehat{\lambda}_p > 0$ and the set of eigenvalues of problem (1) is precisely $(\widehat{\lambda}_p, \infty)$;
(b) If $r = q$, then $\widehat{\lambda}_q > 0$ and the set of eigenvalues of problem (1) is precisely $(\widehat{\lambda}_q, \infty)$.

Theorem 1.4. *Assume that (h_{pqr}), $(h_{\alpha\beta})$, $(h_{\rho_i\gamma_i})$ are fulfilled.*
(a) If either $\big(r < p\big)$ or $\big(r > q$ with $r < q(N-1)/(N-q)$ in case $q < N\big)$, then the set of eigenvalues of problem (1) equals $(0,\infty)$;
(b) If $p < r < q$ with $r < p(N-1)/(N-p)$ in case $p < N$, then $0 < \lambda_* < \lambda^*$ and every $\lambda \in [\lambda^*, \infty)$ is an eigenvalue of problem (1); for any $\lambda \in (-\infty, \lambda_*)$ problem (1) has only the trivial solution.

Moreover, the constants λ_*, λ^* can be expressed as follows

$$\lambda_* = \inf_{v\in W\setminus \mathcal{Z}} \frac{K_p(v) + K_q(v)}{k_r(v)},\quad \lambda^* = \inf_{v\in W\setminus \mathcal{Z}} \frac{\frac{1}{p}K_p(v) + \frac{1}{q}K_q(v)}{\frac{1}{r}k_r(v)}. \tag{8}$$

Remark 1.5. *Regarding the assumptions $\big(r < q(N-1)/(N-q)$ if $q < N\big)$, and $\big(r < p(N-1)/(N-p)$ if $1 < p < N\big)$ in Theorem 1.4 (a) and (b), respectively, we point out that these are directly related to the well-known compact embedding $W^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$ which holds when $1 < r < \theta^*$, where $\theta^* = \theta N/(N-\theta)$ if $1 < \theta < N$, and the trace compact embedding $W^{1,\theta}(\Omega) \hookrightarrow L^r(\partial\Omega)$ if $1 < r < \widetilde{\theta}$, where $\widetilde{\theta} = \theta(N-1)/(N-\theta)$ if $\theta < N$ (see [1], [10, Section 9.3]).*

If $\gamma_1 = \gamma_2 = \beta \equiv 0$ (i.e., the boundary condition is of Neumann type), Theorem 1.4 still holds if in the cases $q < N$ and $p < N$ the conditions $r < q(N-1)/(N-q)$ and $r < p(N-1)/(N-p)$ are replaced by the weaker conditions $r < qN/(N-q)$ and $r < qN/(N-q)$, respectively, since in this case we need only the compact embedding $W^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$, $\theta \in \{p,\ q\}$.

Eigenvalue problems for the $(p,q)-$Laplacian have been extensively investigated in recent years. For the case of the Dirichlet boundary condition we refer to Bobkov and Tanaka [8], Cherfils and Il'yasov [12], Faria, Miyagaki and Motreanu [15] and the references therein.

The spectrum of problem (1), which we describe herein, coincides with the Neumann-like, Robin-like, and Steklov-like eigenvalue problems corresponding to the cases when some of the functions α, β, γ_i vanish.

The generalized Steklov spectrum in the case $\rho_i \equiv 0,\ \gamma_i \equiv 0,\ i = 1, 2,$ was investigated by the authors in [3, 4, 6]. Note also that the particular case $\alpha \equiv 1, \beta \equiv 0$ and $\rho_i \equiv 0,\ i = 1, 2,\ \gamma_1 \equiv 0, \gamma_2 \equiv$ const. $> 0,\ r \in \{p, q\}$, i.e., the case of the (p, q)-Laplacian with a Robin boundary condition, was investigated by Gyulov and Moroşanu in [17]. Let us also mention the paper by Papageorgiou, Vetro and Vetro [19] concerning the case $\rho_1 \equiv 0,\ \gamma_1 \equiv 0,\ \gamma_2 \equiv$ const. $> 0,\ r = q$, with the potential function ρ_2 being sign changing. Also, the problem (1) in the case $r = q,\ a \equiv 0,\ b \equiv 1$ was studied by Barbu and Moroşanu in [5].

Notice that the arguments we shall use in the proof of Theorem 1.3 are essentially known from [2, 4, 5], but here those arguments are adapted to the present context and presented for the convenience of the reader.

While in the previous papers [17], [19] only subsets of the corresponding spectra were determined, in this paper the presence of the potential functions $\rho_i,\ \gamma_i$ satisfying assumptions $(h_{\rho_i\gamma_i})$ allows the full description of the spectrum in four cases out of five.

2 Preliminary results

In this section we state some auxiliary results which will be used in the proofs of our main results.

First of all, note that for $\theta \in (1, \infty)$ and $r < \widetilde{\theta}$ if $\theta < N$, $u \to \big(k_r(u)\big)^{\frac{1}{r}}$ is a seminorm on $W^{1,\theta}(\Omega)$ which satisfies
(i) $\exists d > 0$ such that $k_r(u)^{\frac{1}{r}} \le d \parallel u \parallel_{W^{1,\theta}(\Omega)} \quad \forall u \in W^{1,\theta}(\Omega)$, and
(ii) if $u \equiv$ const., then $k_r(u) = 0$ implies $u \equiv 0$.
Hence, from [13, Proposition 3.9.55] we obtain the following result

Lemma 2.1. *Assume that assumptions $(h_{\alpha\beta})$ are fulfilled, $\theta,\ r \in (1, \infty)$ and $r < \widetilde{\theta}$ if $\theta < N$. Then*

$$\parallel u \parallel_{\theta,r} := \parallel \nabla u \parallel_{L^\theta(\Omega)} + \big(k_r(u)\big)^{\frac{1}{r}} \ \forall\ u \in W^{1,\theta}(\Omega)$$

is a norm on $W^{1,\theta}(\Omega)$, equivalent to the standard one.

Remark 2.2. *Under assumptions $(h_{\rho_i\gamma_i})$, $K_p(\cdot)^{1/p}$ and $K_q(\cdot)^{1/q}$ are norms equivalent to the usual norms of the Sobolev spaces $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$, respectively.*

Next, for $\theta > 1$, we consider the eigenvalue problem

$$\begin{cases} -\Delta_\theta u + \rho(x) \mid u \mid^{\theta-2} u = \lambda\alpha(x) \mid u \mid^{\theta-2} u \ \text{ in } \Omega, \\ \mid \nabla u \mid^{\theta-2} \frac{\partial u}{\partial \nu} + \gamma(x) \mid u \mid^{\theta-2} u = \lambda\beta(x) \mid u \mid^{\theta-2} u \text{ on } \partial\Omega, \end{cases} \tag{9}$$

where $\rho \in L^\infty(\Omega)$ and $\gamma \in L^\infty(\partial\Omega)$ are given nonnegative functions satisfying

$$\int_\Omega \rho \; dx + \int_{\partial\Omega} \gamma \; d\sigma > 0. \tag{10}$$

As usual, the number $\lambda \in \mathbb{R}$ is said to be an eigenvalue of problem (9) if there exists a function $u_\lambda \in W^{1,\theta}(\Omega) \setminus \{0\}$ such that for all $w \in W^{1,\theta}(\Omega)$

$$\int_\Omega \mid \nabla u_\lambda \mid^{\theta-2} \nabla u_\lambda \cdot \nabla w \; dx + \int_\Omega \rho \mid u_\lambda \mid^{\theta-2} u_\lambda w \; dx + \int_{\partial\Omega} \gamma \mid u_\lambda \mid^{\theta-2} u_\lambda w \; d\sigma$$
$$= \lambda \left(\int_\Omega \alpha \mid u_\lambda \mid^{\theta-2} u_\lambda w \; dx + \int_{\partial\Omega} \beta \mid u_\lambda \mid^{\theta-2} u_\lambda w \; d\sigma \right).$$

By arguments similar to those used in the case of problem (1) we can show that the eigenvalues of problem (9) are positive and the corresponding eigenfunctions belong to $W^{1,\theta}(\Omega) \setminus \mathcal{Z}_\theta$, where

$$\mathcal{Z}_\theta := \left\{ w \in W^{1,\theta}(\Omega);\; k_\theta(w) := \int_\Omega \alpha \mid u \mid^\theta dx + \int_{\partial\Omega} \beta \mid u \mid^\theta d\sigma = 0 \right\},$$

Define the C^1 functional

$$\Theta_\theta : W^{1,\theta}(\Omega) \setminus \mathcal{Z}_\theta \to (0,\infty),\; \Theta_\theta(v) := \frac{K_\theta(v)}{k_\theta(v)} \;\; \forall \; v \in W^{1,\theta}(\Omega) \setminus \mathcal{Z}_\theta,$$

where $K_\theta(u) := \int_\Omega \big(\mid \nabla u \mid^p + \rho \mid u \mid^p \big) dx + \int_{\partial\Omega} \gamma \mid u \mid^p d\sigma$.

Lemma 2.3. *Assume that the assumption $(h_{\alpha\beta})$ is fulfilled and $\rho \in L^\infty(\Omega)$, $\gamma \in L^\infty(\partial\Omega)$ are given nonnegative functions satisfying* (10)*. Then there exists $u_* \in W^{1,\theta}(\Omega) \setminus \mathcal{Z}_\theta$ such that*

$$\Theta_\theta(u_*) = \lambda_\theta := \inf_{w \in W^{1,\theta}(\Omega) \setminus \mathcal{Z}_\theta} \Theta_\theta(w) > 0.$$

In addition, λ_θ is the smallest eigenvalue of the problem (9) *and u_* is an eigenfunction corresponding to λ_θ.*

The proof of this result is based on arguments similar to those used in the proof of Lemma 2 in [5] (see also [18, Proposition 3.1]), so we omit it.

Now, for $\lambda > 0$ define the C^1 energy functional for problem (1)

$$\mathcal{J}_\lambda : W \to \mathbb{R},\; \mathcal{J}_\lambda(u) = \frac{1}{p} K_p(u) + \frac{1}{q} K_q(u) - \frac{\lambda}{r} k_r(u) \;\forall\; u \in W. \tag{11}$$

Obviously, according to Definition 1.1, λ is an eigenvalue of problem (1) with corresponding eigenfunction $u_\lambda \in W \setminus \{0\}$ if and only if u_λ is a critical point of $\mathfrak{J}_\lambda$, i.e. $\mathfrak{J}'_\lambda(u_\lambda) = 0$.

The following lemma will be an important ingredient in the proof of our main results.

Lemma 2.4. *Assume that* (h_{pqr}), $(h_{\alpha\beta})$, $(h_{\rho_i\gamma_i})$ *are fulfilled and* $r \in (1,\ q)$. *Then the functional* $\mathfrak{J}_\lambda$ *is coercive on* W, *i.e.,* $\lim\limits_{\|u\|_W \to \infty} \mathfrak{J}_\lambda(u) = \infty$.

Proof. Assume by way of contradiction that functional $\mathfrak{J}_\lambda$ is not coercive. So, there exist a positive constant C and a sequence $\left(u_n\right)_n \subset W$ such that $\| u_n \|_W \to \infty$ as $n \to \infty$ and $\mathfrak{J}_\lambda(u_n) \leq C\ \forall\ n \geq 1$. Therefore

$$\frac{1}{p} K_p(u_n) + \frac{1}{q} K_q(u_n) - \frac{\lambda}{r} k_r(u_n) \leq C \quad \forall\ n \geq 1. \tag{12}$$

In particular,

$$0 \leq \frac{1}{q} K_q(u_n) \leq \frac{\lambda}{r} k_r(u_n) + C\ \forall\ n \geq 1. \tag{13}$$

It follows from estimate (13) and Remark 2.2 that $k_r(u_n) \to \infty$ as $n \to \infty$.

Define $v_n := u_n / \left(k_r(u_n)^{1/r}\right)\ \ \forall\ n \geq 1$ and divide inequality (13) by $k_r(u_n)^{q/r}$. As $r < q$, we obtain that $K_q(v_n) \to 0$ as $n \to \infty$. Hence, $v_n \to 0$ in W (see Remark 2.2) as well as in $L^r(\Omega)$ and in $L^r(\partial\Omega)$. In particular, $k_r(v_n) \to 0$, as $n \to \infty$, but this contradicts the fact that $k_r(v_n) = 1$ for all $n \geq 1$. So, $\mathfrak{J}_\lambda$ is coercive on W. $\square$

Remark 2.5. *Let* $\lambda > 0$ *be fixed. Under the assumptions of Lemma 2.4, if there exists* $u_{0\lambda} \in W \setminus \{0\}$ *such that* $\mathfrak{J}_\lambda(u_{0\lambda}) < 0$, *then* λ *is an eigenvalue of problem* (1)*. Indeed, taking into account Lemma 2.4, the functional* $\mathfrak{J}_\lambda$ *is coercive on* W*. Obviously,* $\mathfrak{J}_\lambda$ *is also weakly lower semicontinuous on* W*. So, there exists a global minimizer* $u_* \in W$ *for* $\mathfrak{J}_\lambda$, *i.e.,* $\mathfrak{J}_\lambda(u_*) = \min_W \mathfrak{J}_\lambda$ *(see, e.g., [20, Theorem 1.2]). We notice that* $\mathfrak{J}_\lambda(u_*) \leq \mathfrak{J}_\lambda(u_{0\lambda}) < 0$, *which implies* $u_* \neq 0$*. In addition,* $\mathfrak{J}'_\lambda(u_*) = 0$ *and so* u_* *is an eigenfunction of problem* (1) *corresponding to the eigenvalue* λ.

3 Proof of Theorem 1.3

Throughout this section we assume that (h_{pqr}), $(h_{\alpha\beta})$ and $(h_{\rho_i\gamma_i})$ are fulfilled and will be used without mentioning them in the statements below.

3.1 Proof of Theorem 1.3 (a)

In this subsection we will address the case $r = p$. The proof of Theorem 1.3 (a) is based on the following two lemmas.

Lemma 3.1. *If $r = p$, then $\widehat{\lambda}_p > 0$ and there is no eigenvalue of problem* (1) *in $(-\infty, \widehat{\lambda}_p]$. Moreover, we have the equality*

$$\widetilde{\lambda}_p := \inf_{w \in W \setminus \mathcal{Z}} \frac{\frac{1}{q} K_q(w) + \frac{1}{p} K_p(w)}{\frac{1}{q} k_p(w)} = \widehat{\lambda}_p. \tag{14}$$

Proof. First, we deduce from Lemma 2.3 with $\theta = p$ that $\widehat{\lambda}_p > \lambda_p = \Theta_p(u^*) > 0$. As we pointed out in Remark 1.2, all the eigenvalues of problem (1) must be positive.

Now, let us check that there is no eigenvalue of problem (1) in $(-\infty, \widehat{\lambda}_p]$. Assume the contrary, that there is an eigenpair $(\lambda, u_\lambda) \in (-\infty, \widehat{\lambda}_p] \times (W \setminus \mathcal{Z})$. Then (4) with $w = u_\lambda$ will imply

$$\lambda = \frac{K_q(u_\lambda) + K_p(u_\lambda)}{k_p(u_\lambda)} \leq \widehat{\lambda}_p. \tag{15}$$

On one hand, if $\lambda < \widehat{\lambda}_p$, we have a contradiction with the definition of $\widehat{\lambda}_p$. On the other hand, if $\lambda = \widehat{\lambda}_p$ we have $K_q(u_\lambda) = 0$ which implies $u_\lambda \equiv 0$ (see Remark 2.2). This is impossible since u_λ was assumed to be an eigenfunction.

Finally, let us check the equality (14). The estimate $\widehat{\lambda}_p \leq \widetilde{\lambda}_p$ is obvious. On the other hand, for each $v \in W \setminus \mathcal{Z}$ and $t > 0$, we have

$$\widetilde{\lambda}_p = \inf_{w \in W \setminus \mathcal{Z}} \frac{\frac{p}{q} K_q(w) + K_p(w)}{k_p(w)} \leq \frac{K_p(v)}{k_p(v)} + t^{q-p} \frac{p K_q(v)}{q k_p(v)}.$$

Now letting $t \to 0$, then passing to infimum over all $v \in W \setminus \mathcal{Z}$, we get the claimed inequality. □

Lemma 3.2. *If $r = p$, then every $\lambda > \widehat{\lambda}_p$ is an eigenvalue of problem* (1).

Proof. Let $\lambda > \widehat{\lambda}_p$ be fixed. From Lemma 2.4 with $r = p$, the functional $\mathcal{J}_\lambda$ is coercive on W.

On the other hand, from Lemma 3.1 we get $\widehat{\lambda}_p = \widetilde{\lambda}_p$ hence, as $\lambda > \widehat{\lambda}_p$, there is some $u_{0\lambda} \in W \setminus \mathcal{Z}$ satisfying $\mathcal{J}_\lambda(u_{0\lambda}) < 0$. Consequently, according to Remark 2.5, λ is an eigenvalue of problem (1). □

Finally, the conclusions of Theorem 1.3 (a) follow from Lemmas 3.1 and 3.2.

3.2 Proof of Theorem 1.3 (b)

If $r = q$ we cannot expect to have coercivity on W of the functional $\mathcal{J}_\lambda$. So, we need to use another approach. Consider the Nehari type manifold defined by

$$\mathcal{N}_\lambda = \{v \in W \setminus \{0\}; \langle \mathcal{J}'_\lambda(v), v\rangle = K_p(v) + K_q(v) - \lambda k_q(v) = 0\}.$$

We shall consider the restriction of $\mathcal{J}_\lambda$ to $\mathcal{N}_\lambda$ since any possible eigenfunction corresponding to λ belongs to $\mathcal{N}_\lambda$. Note that on $\mathcal{N}_\lambda$ functional $\mathcal{J}_\lambda$ has the form

$$\mathcal{J}_\lambda(u) = \frac{q-p}{qp} K_p(u) > 0 \ \forall\ u \in \mathcal{N}_\lambda. \tag{16}$$

Remark 3.3. *Taking into account assumptions* $(h_{\rho_i \gamma_i})$, *it is obvious that* $\mathcal{N}_\lambda \subset W \setminus \mathcal{Z}$.

As in the preceding case, we have

Lemma 3.4. *If* $r = q$, *then* $\widehat{\lambda}_q > 0$ *and there is no eigenvalue of problem* (1) *in* $(-\infty, \widehat{\lambda}_q]$. *Moreover, we have the equality*

$$\widetilde{\lambda}_q := \inf_{w \in W \setminus \mathcal{Z}} \frac{\frac{q}{p} K_p(w) + K_q(w)}{k_q(w)} = \widehat{\lambda}_q. \tag{17}$$

The proof is similar to the proof of Lemma 3.1, so we omit it.
In what follows, until further notice, $\lambda > \widehat{\lambda}_q$ will be a fixed real number.

Lemma 3.5. *If* $r = q$, *then there exists a point* $u_* \in \mathcal{N}_\lambda$ *where* $\mathcal{J}_\lambda$ *attains its minimal value over* $\mathcal{N}_\lambda$, $m_\lambda := \inf_{w \in \mathcal{N}_\lambda} \mathcal{J}_\lambda(w) > 0$.

Proof. We shall follow an argument similar to that used in [2, Case 2, Steps 1-4]. So, we split the proof into four steps.

Step 1. $\mathcal{N}_\lambda \neq \emptyset$.

In fact, from $\lambda > \widehat{\lambda}_q$ and the definition of $\widehat{\lambda}_q$ (see (6)) there exists $v_0 \in W \setminus \mathcal{Z}$ such that $K_q(v_0) < \lambda k_q(v_0)$. In addition, taking into account assumptions $(h_{\rho_i \gamma_i})$, we have $K_p(v_0) > 0$.

We claim that for a convenient $\tau > 0$, $\tau v_0 \in \mathcal{N}_\lambda$. Indeed, the condition $\tau v_0 \in \mathcal{N}_\lambda$, $\tau > 0$, reads $\tau^p K_p(v_0) + \tau^q K_q(v_0) = \lambda \tau^q k_q(v_0)$. This equation can be solved for τ, and hence, for this τ we have $\tau v_0 \in \mathcal{N}_\lambda$.

Step 2. Every minimizing sequence $\big(u_n\big)_n \subset \mathcal{N}_\lambda$ for $\mathcal{J}_\lambda$ restricted to $\mathcal{N}_\lambda$ is bounded in W.

Let $\big(u_n\big)_n \subset \mathcal{N}_\lambda$ be such a minimizing sequence for $\mathcal{J}_\lambda$. Assume by contradiction that $\big(u_n\big)_n$ is unbounded in W hence, on a subsequence, again denoted $\big(u_n\big)_n$, we have $\|u_n\| \to \infty$. Since $\big(u_n\big)_n \subset \mathcal{N}_\lambda$, we have (see equality (16))

$$\mathcal{J}_\lambda(u_n) = \frac{q-p}{qp} K_p(u_n) \to m_\lambda \geq 0 \text{ as } n \to \infty, \tag{18}$$

and

$$0 < \lambda k_q(u_n) = K_p(u_n) + K_q(u_n) \;\forall\; n \geq 1. \tag{19}$$

It follows from (19) and Remark 2.2 that

$$k_q(u_n) \to \infty \quad \text{as} \quad n \to \infty. \tag{20}$$

Set $v_n = u_n/(k_q(u_n)^{1/q}),\ n \geq 1$. Obviously, $k_q(v_n) = 1 \;\forall\; n \geq 1$. Now, from (19) it follows $K_q(v_n) \leq \lambda$ for all $n \geq 1$, so $\big(v_n\big)_n$ is bounded in W. Therefore, there exists $v_0 \in W$ such that $v_n \rightharpoonup v_0$ in W (hence also in $W^{1,p}(\Omega)$ to the same v_0) and $v_n \to v_0$ in $L^q(\Omega)$ as well as in $L^q(\partial\Omega)$. In addition, we also have $k_q(v_0) = 1$. Now, dividing (18) by $k_q(u_n)^{p/q}$ and making use of (20), we see that $K_p(v_n) \to 0$, and so $v_0 \equiv 0$ which contradicts the fact that $k_q(v_0) = 1$. Therefore, $\big(u_n\big)_n$ is bounded in W.

Step 3. $m_\lambda := \inf\limits_{w \in \mathcal{N}_\lambda} \mathcal{J}_\lambda(w) > 0$.

Otherwise, suppose $m_\lambda = 0$ and let $\big(u_n\big)_n \subset \mathcal{N}_\lambda$ be a minimizing sequence for $\mathcal{J}_\lambda$. By Step 2, $\big(u_n\big)_n$ is bounded in W, so for some $u_0 \in W$, $u_n \rightharpoonup u_0$ (on a subsequence) in W (and also weakly in $W^{1,p}(\Omega)$ to the same u_0), and $u_n \to u_0$ in $L^q(\Omega)$ as well as in $L^q(\partial\Omega)$. We have (see (18)) $K_p(u_n) \to 0$, hence $u_0 \equiv 0$ (see Remark 2.2).

Note that $k_q(u_n) > 0$ for all $n \geq 1$ (see Remark 3.3) and $k_q(u_n) \to 0$ as $n \to \infty$. Now, from (19) we obtain that $K_q(v_n) \leq \lambda$ for all $n \geq 1$, so the sequence $\big(v_n\big)_n$ is bounded in W (see Step 2 for the definition of $\big(v_n\big)_n$). Hence, on a subsequence, $v_n \rightharpoonup v_0$ in W and $v_n \to v_0$ in $L^q(\Omega)$ as well as in $L^q(\partial\Omega)$, for some $v_0 \in W$. Now, we divide (19) by $k_q(u_n)^{p/q}$ to obtain

$$K_p(v_n) = k_q(u_n)^{(q-p)/q}\Big[\lambda - K_q(v_n) - k_q(v_n)\Big] \to 0.$$

This implies $v_n \rightharpoonup 0$ in $W^{1,p}(\Omega)$. In particular, $k_q(v_n) \to 0$, which is a contradiction. This contradiction shows that $m_\lambda > 0$.

Step 4. There exists $u_* \in \mathcal{N}_\lambda$ such that $\mathcal{J}_\lambda(u_*) = m_\lambda$.

Let $\big(u_n\big)_n \subset \mathcal{N}_\lambda$ be a minimizing sequence, i.e., $\mathcal{J}_\lambda(u_n) \to m_\lambda$. In particular, the sequence $\big(u_n\big)_n$ satisfies (19) and is bounded in W by Step 3, thus on a subsequence $u_n \rightharpoonup u_* \in W$ and strongly in both $L^q(\Omega)$ and $L^q(\partial\Omega)$ (to the same u_*). In addition, $u_* \not\equiv 0$. Otherwise, if $u_* \equiv 0$, we infer by (19) that $\big(K_p(u_n)\big)_n$ converges to 0. Then (18) will give $m_\lambda = 0$ thus contradicting the statement of Step 3. By passing to limit as $n \to \infty$ in (19), we find

$$K_p(u_*) + K_q(u_*) \leq \lambda k_q(u_*). \tag{21}$$

If we have equality in (21), then $u_* \in \mathcal{N}_\lambda$ and the proof is complete since in this case $\mathcal{J}_\lambda(u_*) = m_\lambda$. In what follows we show that the strict inequality

$$K_p(u_*) + K_q(u_*) < \lambda k_q(u_*) \tag{22}$$

is impossible. Let us assume by contradiction that (22) holds true. We check that there exists $\tau \in (0,1)$ such that $\tau u_* \in \mathcal{N}_\lambda$. For this purpose, we consider the function

$$f : (0,\infty) \to \mathbb{R},\ f(t) := t^{p-q} K_p(u_*) + K_q(u_*) - \lambda k_q(u_*).$$

As $K_p(u_*) > 0$, we have $f(t) \to \infty$ as $t \to 0_+$. Since $f(1) < 0$ (see (22)), there exists $\tau \in (0,1)$ such that $f(\tau) = 0$ which implies $\tau u_* \in \mathcal{N}_\lambda$. But then,

$$0 < m_\lambda \le \mathcal{J}_\lambda(\tau u_*) = \tau^p \frac{q-p}{qp} K_p(u_*) \le \tau^p \lim_{n\to\infty} \mathcal{J}_\lambda(u_n) = \tau^p m_\lambda < m_\lambda,$$

which is impossible. □

Lemma 3.6. *If $r = q$, the minimizer $u_* \in \mathcal{N}_\lambda$ from Lemma 3.5 is an eigenfunction of problem* (1) *with corresponding eigenvalue λ.*

Proof. First, note that u_* is a solution of the constraint minimization problem.

$$\min_{v \in W\setminus\{0\}} \mathcal{J}_\lambda(v), \quad g_q(v) := K_p(v) + K_q(v) - \lambda k_q(v) = 0.$$

Next, we are going to check that $\mathcal{R}(g_q'(u_*)) = \mathbb{R}$, i.e., for all $\xi \in \mathbb{R}$ there exists a $w \in W \setminus \{0\}$ such that $\langle g_q'(u_*), w\rangle = \xi$ (here $\mathcal{R}(g_q'(u_*))$ stands for the range of $g_q'(u_*)$). Indeed, if we choose in the above equations w of the form $w = \chi u_*,\ \chi \in \mathbb{R}$ making use of $u_* \in \mathcal{N}_\lambda$, we obtain

$$\chi\Big(pK_p(u_*) + q\big(K_q(u_*) - \lambda k_q(u_*)\big)\Big) = \xi \ \Leftrightarrow\ \chi K_p(u_*)(p-q) = \xi$$

which has a unique solution χ (by Remark 2.2). Applying the Lagrange multiplier rule [16, Theorem 3.29, p. 496], we can find $\mu \in \mathbb{R}$ such that

$$\langle \mathcal{J}_\lambda'(u_*), v\rangle + \mu\langle g_q'(u_*), v\rangle = 0, \quad \forall\ v \in W.$$

Testing with $v = u_*$ and using the fact that $u_* \in \mathcal{N}_\lambda$, we derive that

$$\mu(p-q)K_p(u_*) = 0,$$

which implies $\mu = 0$. Therefore $\langle \mathcal{J}_\lambda'(u_*), v\rangle = 0$, $\forall\ v \in W$, i.e., λ is an eigenvalue of problem (1). □

Finally, we see that Theorem 1.3 (b) follows from Lemmas 3.4, 3.5 and 3.6.

4 Proof of Theorem 1.4

We shall prove Theorem 1.4 through a series of lemmas based on the assumptions (h_{pqr}), $(h_{\alpha\beta})$ and $(h_{\rho_i\gamma_i})$ which will not be mentioned explicitly in the statements.

4.1 Proof of Theorem 1.4 (a)

The proof of Theorem 1.4 (a) is based on some intermediate results, as follows.

Lemma 4.1. *If $r < p$, then every $\lambda > 0$ is an eigenvalue of problem* (1).

Proof. Let $\lambda > 0$ be fixed. According to Lemma 2.4, the functional $\mathcal{J}_\lambda$ is coercive.
Now, for $t > 0$, we have

$$t \to \frac{\mathcal{J}_\lambda}{t^r}(t) = \frac{t^{p-r}}{p}k_1 + \frac{t^{q-r}}{q}k_2 - \frac{\lambda}{r}k_r(1) \to -\frac{\lambda}{r}k_r(1) < 0 \text{ as } t \to 0_+,$$

where $k_i := \int\limits_\Omega \rho_i \, dx + \int\limits_{\partial\Omega} \gamma_i \, dx > 0,\ i = 1, 2$. Hence $\mathcal{J}_\lambda(u_*) < 0$ and, according to Remark 2.2, the proof is complete. □

In the rest of this subsection we suppose that $q < r$, and $r < \widetilde{q}$ if $q < N$. Let $\lambda > 0$ be a fixed number. As in Subsection 3.2, under these assumptions we cannot expect to have coercivity of the functional $\mathcal{J}_\lambda$ on W. So, we need to consider another approach involving the Nehari manifold

$$\mathcal{N}_\lambda = \{v \in W \setminus \{0\}; \langle \mathcal{J}_\lambda'(w), w\rangle = K_p(v) + K_q(v) - \lambda k_r(v) = 0\}. \tag{23}$$

Notice that on $\mathcal{N}_\lambda$ the functional $\mathcal{J}_\lambda$ is given by

$$\mathcal{J}_\lambda(u) = \frac{r-p}{pr}K_p(u) + \frac{r-q}{qr}K_q(u) > 0. \tag{24}$$

Also, we claim that $\mathcal{N}_\lambda \neq \emptyset$. In this respect, we define

$$h(t) := t^p K_p(1) + t^q K_q(1) - t^r \lambda k_r(1) \ \forall\ t > 0.$$

Observing that the function $t \mapsto h(t)$ is continuous on $(0, \infty)$ and

$$t^{-p}h(t) \to K_p(1) > 0 \ \text{ as } t \to 0_+,\ t^{-r}h(t) \to -\lambda k_r(1) < 0 \ \text{ as } t \to \infty,$$

we infer that there exists $\tau \in (0, \infty)$ such that $h(\tau) = 0$, so $w \equiv \tau \in \mathcal{N}_\lambda$.

The proofs of the next two results can be achieved by using arguments similar to those from the proofs of Lemmas 3.5 and 3.6 above, so we omit them.

Lemma 4.2. *Assume that $q < r$, and $r < \widetilde{q}$ if $q < N$. Then there exists a point $u_* \in \mathcal{N}_\lambda$ where $\mathcal{J}_\lambda$ attains its minimal value over $\mathcal{N}_\lambda$, $m_\lambda := \inf\limits_{w \in \mathcal{N}_\lambda} \mathcal{J}_\lambda(w) > 0$.*

Lemma 4.3. *Assume that $q < r$, and $r < \widetilde{q}$ if $q < N$. Then the minimizer $u_* \in \mathcal{N}_\lambda$ from Lemma 4.2 is an eigenfunction of problem* (1) *corresponding to the eigenvalue λ.*

Summing up, we see that Lemmas 4.1, 4.2, and 4.3 fully complete the proof of Theorem 1.4 (a).

4.2 Proof of Theorem 1.4 (b)

Lemma 4.4. *Assume that $p < r < q$, and $r < \widetilde{p}$ if $p < N$. Then $0 < \lambda_* < \lambda^*$, where λ_* and λ^* are the constants defined in* (7).

Proof. Taking into account Lemma 2.1 with $\theta = p$, Remark 2.2, and the following continuous embeddings

$$W^{1,q}(\Omega) \hookrightarrow W^{1,p}(\Omega) \hookrightarrow L^r(\Omega),\ W^{1,q}(\Omega) \hookrightarrow W^{1,p}(\Omega) \hookrightarrow L^r(\partial\Omega),$$

we see that

$$K_p(v) \geq M_1 k_r(v)^{\frac{p}{r}}, \quad K_q(v) \geq M_2 k_r(v)^{\frac{q}{r}}\ \forall\ v \in W \setminus \mathcal{Z}, \tag{25}$$

where M_1, M_2 are two positive constants (independent of v). Next, from (25), since $p\omega + q(1-\omega) = r$, we obtain that there exists a positive constant M (independent of v) such that

$$\Gamma \frac{K_p(v)^\omega K_q(v)^{1-\omega}}{k_r(v)} \geq M\ \forall\ v \in W \setminus \mathcal{Z}.$$

Finally, taking the infimum over all $v \in W \setminus \mathcal{Z}$ in the above inequality, we infer that $M \leq \lambda_*$.

To complete the proof we need to show that $\lambda_* < \lambda^*$. This inequality is equivalent to (see (7))

$$\frac{r}{p^\omega q^{1-\omega}} > 1 \quad \Leftrightarrow \quad r^{q-p} > p^{q-r} q^{r-p},$$

which can be rewritten as

$$\left(1 + \frac{q-p}{p}\right)^{\frac{p}{q-p}} < \left(1 + \frac{r-p}{p}\right)^{\frac{p}{r-p}}.$$

So, the desired inequality follows, since the function $x \to (1+x)^{\frac{1}{x}}$ is decreasing on $(0, \infty)$ and $q - p > r - p$. $\square$

Lemma 4.5. *Assume that $p < r < q$, and $r < \widetilde{p}$ if $p < N$. Then the constants λ_* and λ^* defined in* (7) *can be equivalently expressed by* (8).

Proof. Let $v \in W \setminus \mathcal{Z}$ be fixed. Define

$$T_*(v) := \frac{K_p(v) + K_q(v)}{k_r(v)}, \quad T^*(v) := \frac{\frac{r}{p} K_p(v) + \frac{r}{q} K_q(v)}{k_r(v)}, \tag{26}$$

$$
\begin{aligned}
g_v(t) &:= T_*(tv) = \frac{t^{p-r}K_p(v) + t^{q-r}K_q(v)}{k_r(v)}, \\
h_v(t) &:= T^*(tv) = \frac{\frac{r}{p}t^{p-r}K_p(v) + \frac{r}{q}t^{q-r}K_q(v)}{k_r(v)} \quad \forall\, t > 0.
\end{aligned} \tag{27}
$$

It is easy to see that the function g_v achieves its minimal value $\lambda(v) > 0$ on $(0,\infty)$ for $t = t(v) > 0$, where

$$
\lambda(v) = \Gamma \frac{K_p(v)^{\omega} K_q(v)^{1-\omega}}{k_r(v)}, \; t(v) = \left[\frac{(r-p)K_p(v)}{(q-r)K_q(v)}\right]^{\frac{1}{q-p}}. \tag{28}
$$

It follows that

$$
\lambda(v) = T_*\big(t(v)v\big) = \inf_{t>0} \frac{K_p(tv) + K_q(tv)}{k_r(tv)}.
$$

Now, taking the infimum over all $v \in W \setminus \mathcal{Z}$, we obtain that

$$
\lambda_* = \inf_{v \in W\setminus\mathcal{Z}} \left(\inf_{t>0} \frac{K_p(tv) + K_q(tv)}{k_r(tv)} \right). \tag{29}
$$

Put $\xi_* := \inf\limits_{v\in W\setminus\mathcal{Z}} T_*(v)$, $\quad \xi^* := \inf\limits_{v\in W\setminus\mathcal{Z}} T^*(v)$. The estimate $\xi_* \geq \lambda_*$ is obvious. On the other hand, for each $v \in W \setminus \mathcal{Z}$ we have $t(v)v \in W \setminus \mathcal{Z}$, so $\lambda(v) \geq \xi_*$, and taking the infimum over all $v \in W \setminus \mathcal{Z}$, we infer that $\xi_* \leq \lambda_*$. Hence, $\xi_* = \lambda_*$.

By a similar reasoning, as the function h_v achieves its minimal value $\widehat{\lambda}(v) = \frac{r}{p^{\omega}q^{1-\omega}}\lambda(v) > 0$ on $(0,\infty)$ for $t = \widehat{t}(v) = \left(\frac{q}{p}\right)^{1/(q-p)} t(v) > 0$, we first get the equality

$$
\lambda^* = \inf_{v \in W\setminus\mathcal{Z}} \left(\inf_{t>0} \frac{\frac{r}{p}K_p(tv) + \frac{r}{q}K_q(tv)}{k_r(tv)} \right), \tag{30}
$$

and finally $\xi^* = \lambda^*$. $\square$

Next, let us show that there exists $u_* \in W \setminus \mathcal{Z}$ such that $\lambda_* = \lambda(u_*)$. To this purpose, we define the functional $\Phi : W \setminus \mathcal{Z} \to (0,\infty)$, $v \to \lambda(v)$, that is

$$
\Phi(v) := \Gamma \frac{K_p(v)^{\omega} K_q(v)^{1-\omega}}{k_r(v)} \quad \forall\, v \in W \setminus \mathcal{Z}.
$$

Lemma 4.6. *Assume that $p < r < q$, and $r < \widetilde{p}$ if $p < N$. Then there exists $u_* \in W \setminus \mathcal{Z}$ such that $\lambda_* = \Phi(u_*) = \inf\limits_{w\in W\setminus\mathcal{Z}} \Phi(w)$.*

Proof. First of all, taking into account the equality $p\omega + q(1-\omega) = r$, we see that functional Φ is positively homogeneous of zero degree, that is

$$\Phi(tv) = \Phi(v) \ \forall\ t > 0,\ v \in W \setminus \mathcal{Z}.$$

Hence, we can find a minimizing sequence $\big(u_n\big)_n$ for λ_* such that $\big(u_n\big)_n \subset W \setminus \mathcal{Z}$ and $k_r(u_n) = 1 \ \ \forall\ n \in \mathbb{N}$, i.e.,

$$\Phi(u_n) = \Gamma K_p(u_n)^{\omega} K_q(u_n)^{1-\omega} \to \inf_{w \in W \setminus \mathcal{Z}} \Phi(w) = \lambda_*. \tag{31}$$

Let us prove that $\big(u_n\big)_n$ is bounded in W. Assume the contrary, that there exists a subsequence of $\big(u_n\big)_n$, again denoted $\big(u_n\big)_n$, such that $\|u_n\|_W \to \infty$. From Lemma 2.1 we infer that $K_q(u_n) \to \infty$ as $n \to \infty$ and so, using the inequality (25) and $k_r(u_n) = 1$ for all $n \geq 1$, we get $K_q(u_n) \geq M_2 > 0$ for all $n \geq 1$. Therefore, since $\omega > 0$, we have $\Gamma K_p(u_n)^{\omega} K_q(u_n)^{1-\omega} \to \infty$, which contradicts (31). Thus, the sequence $\big(u_n\big)_n$ is bounded in W so there exist $u_* \in W$ and a subsequence of $\big(u_n\big)_n$, again denoted $\big(u_n\big)_n$, such that $u_n \rightharpoonup u_*$ in W, (also in $W^{1,p}(\Omega)$) and $u_n \to u_*$ in $L^r(\Omega)$ as well as in $L^r(\partial\Omega)$. In particular, $k_r(u_*) = 1$, thus $u_* \notin \mathcal{Z}$.

Also, the functionals K_p and K_q are weakly lower semicontinuous on $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$, respectively. Therefore we have

$$K_p(u_*) \leq \liminf_{n\to\infty} K_p(u_n) := \underline{K}_p,\ K_q(u_*) \leq \liminf_{n\to\infty} K_q(u_n) := \underline{K}_q.$$

Consequently, since $k_r(u_*) = k_r(u_n) = 1 \ \forall\ n \in \mathbb{N}$, we have

$$\begin{aligned}\Phi(u_*) &= \Gamma K_p(u_*)^{\omega} K_q(u_*)^{1-\omega} \leq \Gamma \underline{K}_p^{\omega} \underline{K}_q^{1-\omega} \\ &\leq \liminf_{n\to\infty} \Gamma K_p(u_n)^{\omega} K_q(u_n)^{1-\omega} = \lambda_*,\end{aligned} \tag{32}$$

and so $\Phi(u_*) = \lambda_*$. $\square$

As a consequence of Lemma 4.6 we obtain that λ^* is an eigenvalue of problem (1).

Lemma 4.7. *Assume that $p < r < q$, and $r < \widetilde{p}$ if $p < N$. If $u_* \in W \setminus \mathcal{Z}$ is the minimizer determined in Lemma 4.6, then*

$$u^* = \left(\frac{q}{p}\right)^{\frac{1}{q-p}} t(u_*) u_* \in W \setminus \mathcal{Z}, \tag{33}$$

with $t(u_*)$ defined in $(28)_2$, is an eigenfunction of problem (1) corresponding to the eigenvalue λ^*. In addition, $\mathcal{J}_{\lambda^*}(u^*) = 0$.

Proof. From Lemma 4.6, since functional Φ is a C^1 functional on $W \setminus \mathcal{Z}$, we have $\Phi'(u_*) = 0$, that is

$$\begin{aligned}
\langle \Phi'(u_*), w\rangle = \frac{1}{k_r(u_*)}\Bigg[&\Gamma\omega p\Big(\frac{K_p(u_*)}{K_q(u_*)}\Big)^{\omega-1}\Big(\int\limits_{\Omega} \mid \nabla u_* \mid^{p-2} \nabla u_* \cdot \nabla w \; dx \\
&+ \int\limits_{\Omega} \rho_1 \mid u_* \mid^{p-2} u_* w \; dx + \int\limits_{\partial\Omega} \gamma_1 \mid u_* \mid^{p-2} u_* w \; d\sigma\Big) \\
&+ \Gamma(1-\omega)q\Big(\frac{K_p(u_*)}{K_q(u_*)}\Big)^{\omega}\Big(\int\limits_{\Omega} \mid \nabla u_* \mid^{q-2} \nabla u_* \cdot \nabla w \; dx \\
&+ \int\limits_{\Omega} \rho_2 \mid u_* \mid^{q-2} u_* w \; dx + \int\limits_{\partial\Omega} \gamma_2 \mid u_* \mid^{q-2} u_* w \; d\sigma\Big) \\
&- \Phi(u_*) r\Bigg(\int\limits_{\Omega} \alpha \mid u_* \mid^{r-2} u_* w \; dx + \int\limits_{\partial\Omega} \beta \mid u_* \mid^{r-2} u_* w \; d\sigma\Bigg)\Bigg] = 0,
\end{aligned}$$

for every $w \in W$. Multiplying the above equality by $1/(p^\omega q^{1-\omega})$ and taking into account (33) and $\Phi(u_*) = p^\omega q^{1-\omega}\lambda^*/r$, we obtain

$$\begin{aligned}
&\int\limits_{\Omega} \Big(\mid \nabla u^* \mid^{p-2} + \mid \nabla u^* \mid^{q-2} \Big)\nabla u^* \cdot \nabla w \; dx \\
&\quad + \int\limits_{\Omega} \Big(\rho_1 \mid u_* \mid^{p-2} + \rho_2 \mid u_* \mid^{q-2} \Big) u_* w \; dx \\
&\quad + \int\limits_{\partial\Omega} \Big(\gamma_1 \mid u_* \mid^{p-2} + \gamma_2 \mid u_* \mid^{q-2} \Big) u_* w \; d\sigma \\
&\quad = \lambda^* \Big(\int\limits_{\Omega} a \mid u^* \mid^{r-2} u^* w \; dx + \int\limits_{\partial\Omega} b \mid u^* \mid^{r-2} u^* w \; d\sigma\Big) \;\forall\; w \in W.
\end{aligned}$$

According to Definition 1.1, u^* is an eigenfunction of problem (1) corresponding to the eigenvalue λ^*.

Finally, a simple computation shows that $\mathcal{J}_{\lambda^*}(u^*) = 0$. □

Lemma 4.8. *Assume that $p < r < q$, and $r < \widetilde{p}$ if $p < N$. Then every number $\lambda \in (\lambda^*, \infty)$ is an eigenvalue of problem (1), and for any $\lambda \in (-\infty, \lambda_*) \setminus \{0\}$ problem (1) has only the trivial solution.*

Proof. Let $\lambda > \lambda^*$ be fixed. Note that the eigenfunction u_* from Lemma 4.7

satisfies $k_r(u^*) \neq 0$ and $\mathcal{J}_{\lambda^*}(u^*) = 0$, so we have

$$\begin{aligned} 0 =& \mathcal{J}_{\lambda^*}(u^*) = \frac{1}{p}K_p(u^*) + \frac{1}{q}K_q(u) - \frac{\lambda^*}{r}k(u^*) \\ &> \frac{1}{p}K_p(u^*) + \frac{1}{q}K_q(u^*) - \frac{\lambda}{r}k_r(u^*) = \mathcal{J}_\lambda(u^*). \end{aligned}$$

Finally, making use of Remark 2.5, λ is an eigenvalue of problem (1).

The second statement is a simple consequence of Lemma 4.5. Assume, by way of contradiction, that there exists a $\lambda \in (0, \lambda_*)$ and $u_\lambda \in W \setminus \mathcal{Z}$ which satisfy the relation (4). Choosing here $w = u_\lambda$ yields

$$\lambda = \frac{K_p(u_\lambda) + K_q(u_\lambda)}{k_r(u_\lambda)},$$

which, by virtue of the equivalent definition of λ_* in (7), implies that $\lambda \geq \lambda_*$. This contradicts the choice of λ. $\square$

Summarizing, we see that Lemmas 4.4, 4.5, 4.7, and 4.8 fully complete the proof of Theorem 1.4 (b).

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