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# $H_v$-module of functions over $H_v$-ring of arithmetics and it's fundamental module

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**Abstract**

After introducing the definition of hypergroups by Marty, the study of hyperstructures and its connections with other fields has been of great importance. In this paper, we continue the investigation between hyperstructure theory and number theory. More precisely, we define an $H_v$-module of complex valued functions over the $H_v$-ring of arithmetics, classify its complete parts, the strongly regular relations on it and identify its fundamental module.

## 1 Introduction

Algebraic hyperstructures represent a natural generalization of classical algebraic structures and they were introduced by F. Marty [13] in 1934 at the eighth Congress of Scandinavian Mathematicians. Where he generalized the notion of a group to that of a hypergroup. Marty defined a hypergroups as a non-empty set equipped with an associative and reproductive hyperoperation. Hypergroups are considered as natural generalizations of groups because in a group, the composition of two elements is an element whereas in a hypergroup, the composition of two elements is a non-empty set. Since then, many different kinds of hyperstructures (hyperring, hypermodule, hypervector space, ...) were widely studied from the theoretical point of view and for

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their applications to many subjects of pure and applied mathematics. For applications of hyperstructure theory, the reader may refer to [9]. A wider class of hyperstructures is obtained when some axioms concerning the above hyperstructures are replaced by their corresponding weak axioms, i.e., the equality sign is replaced by non-empty intersection. This generalization of the well-known algebraic hyperstructures (hypergroups, hyperrings, hypermodules) is known as $H_v$-structures ($H_v$-groups, $H_v$-rings, $H_v$-modules) and it has been introduced by T. Vougiouklis [16]. Many problems in life and in other sciences can be expressed by models using $H_v$-structures (see [10]).

A connection between hyperstructures and arithmetic functions has been established in 2010, by Asghari and Davvaz [3], where they defined a hypergroup on the set of arithmetic functions. Later, the authors [1] generalized the work in [3] by defining an $H_v$-ring of arithmetic functions and studying its properties. Then in [2], they studied strongly regular relations of the defined $H_v$-ring of arithmetic functions, characterized them and proved that it's fundamental ring is the ring of complex numbers under standard addition and multiplication. In this paper, we extend the work of [1] and [2] to $H_v$-modules and it is constructed as follows: After an Introduction, Section 2 presents definitions related to hyperstructure theory and fundamental relations. Section 3 presents the $H_v$-ring of arithmetics that is defined by the authors in [1] and defines an $H_v$-module of functions over it. Section 4 classifies the complete parts of the $H_v$-module of functions and finds it's fundamental module. Finally, Section 5 characterizes the strongly regular relations on the $H_v$-module of functions.

## 2 Preliminaries

In this section, we present some definitions related to hyperstructure theory and fundamental relations that are used throughout the paper. The reader may refer to [4, 6, 7, 8, 11, 12, 15] for more details.

Let $H$ be a non-empty set. Then, a mapping $\circ : H \times H \rightarrow \mathcal{P}^*(H)$ is called a *binary hyperoperation* on $H$, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of $H$. The couple $(H, \circ)$ is called a *hypergroupoid*. In this definition, if $A$ and $B$ are two non-empty subsets of $H$ and $x \in H$, then we define:

$$A \circ B = \bigcup_{\substack{a\in A\\ b\in B}} a \circ b,\ x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a:

1. *semihypergroup* if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$;

2. *quasi-hypergroup* if for every $x \in H$, $x \circ H = H = H \circ x$ (The latter condition is called the reproduction axiom);

3. *hypergroup* if it is a semihypergroup and a quasi-hypergroup.

$H_v$-structures were introduced by T. Vougiouklis [15, 16] as a generalization of the well-known algebraic hyperstructures. Some axioms of classical algebraic hyperstructures are replaced by their corresponding weak axioms in $H_v$-structures. Most of $H_v$-structures are used in the representation theory. A hypergroupoid $(H, \circ)$ is called an $H_v$*-semigroup* if $(x \circ (y \circ z)) \cap ((x \circ y) \circ z) \neq \emptyset$ for all $x, y, z \in H$. A hypergroupoid $(H, \circ)$ is called an $H_v$*-group* if it is a quasi-hypergroup and an $H_v$-semigroup. A multivalued system $(R, +, \cdot)$ is an $H_v$*-ring* if (1) $(R, +)$ is an $H_v$-group; (2) $(R, \cdot)$ is is an $H_v$-semigroup; (3) $\cdot$ is weak distributive with respect to $+$.
An $H_v$-ring $R$ is called an $H_v$*-field* if $R/\gamma^\star$ is a field.

**Definition 2.1.** [15] A non-empty set $M$ is an $H_v$-module over an $H_v$-ring $R$, if $(M, +)$ is a commutative $H_v$-group and there exists a map $\star : R \times M \to \mathcal{P}^*(M)$, $(r, x) \to r \star x$ such that (1) $(r \star (x + y)) \cap (r \star x + r \star y) \neq \emptyset$; (2) $((r + s) \star x) \cap (r \star x + s \star x) \neq \emptyset$; (3) $((rs) \star x) \cap (r \star (s \star x)) \neq \emptyset$.

An $H_v$-module over an $H_v$-field is called an $H_v$*-vector space*.

A non-empty subset of an $H_v$-module $M$ over an $H_v$-ring $R$ is called an $H_v$*-submodule* of $M$ if $r \star x \in N$ and $x + N = N$ for all $r \in R, x \in N$. Let $(M, +, R, \star)$ and $(N, +', R, \star')$ be two $H_v$-modules over $H_v$-ring $R, S$ respectively and $g : R \to S$ be an $H_v$-ring homomorphism. Then $f : M \to N$ is said to be an $H_v$*-module homomorphism* if $f(x + y) \subseteq f(x) +_1 f(y)$ and $f(r \star x) \subseteq g(r) \star' f(x)$ for all $x, y \in M$ and $r \in R$. If the equality holds then $f$ is called *strong* $H_v$*-module homomorphism*. Let $(M, +, R, \star)$ and $(N, +', R, \star')$ be two $H_v$-module over the $H_v$-rings $R, S$. Then $M$ and $N$ are called *isomorphic* $H_v$*-modules*, and written as $M \cong N$, if there exists an $H_v$-ring isomorphism $g : R \to S$ and bijective function $f : M \to N$ such that $f(x+y) = f(x) +_1 f(y)$ and $f(r \star x) = g(r) \star' f(x)$ for all $x, y \in M$ and $r \in R$.

The main tools connecting the class of hyperstructures with the classical algebraic structures are the fundamental relations. In [15], Vougiouklis defined the notion of fundamental relations on $H_v$-rings and $H_v$-modules.

**Definition 2.2.** [16] For all $n > 1$, we define the relation $\gamma$ on an $H_v$-ring $(R, +, \cdot)$ as follows: $a\gamma b \Leftrightarrow \{a, b\} \subseteq u$ where $u$ is finite sum of finite products of elements in $R$.

The relation $\gamma$ is reflexive and symmetric. Denote by $\gamma^*$ the transitive closure of $\gamma$. The $\gamma^\star$ is called the *fundamental equivalence relation* on $R$ and $R/\gamma^\star$ is the *fundamental ring*.

**Definition 2.3.** For all $n > 1$, we define the relation $\varepsilon$ on an $H_v$-module $(M, +, R, \star)$ over an $H_v$-ring $R$ as follows: $x\varepsilon y$ if and only if there exist $n \in \mathbb{N}$, $(m_1, \cdots, m_n) \in M^n$, $(k_1, \cdots, k_n) \in \mathbb{N}^n$, $(x_{i1}, \cdots, x_{ik}) \in R^{k_i}$ such that

$$x, y \in \sum_{i=1}^{n} m'_i, m'_i = m_i \text{ or } m'_i = \sum_{j=1}^{n_i} (\prod_{k=1}^{k_i j} x_{ijk}) \star m_i.$$

The relation $\varepsilon$ is reflexive and symmetric. Denote by $\varepsilon^*$ the transitive closure of $\varepsilon$. The $\varepsilon^\star$ is called the *fundamental equivalence relation* on $M$ and $(M/\varepsilon^*, \oplus, R/\gamma^\star, \odot)$ is the *fundamental module*. The operations "$\oplus$" and "$\odot$" are defined as follows: For all $m, n \in M$, $r \in R$,

$$\varepsilon^\star(m) \oplus \varepsilon^\star(n) = \varepsilon^\star(s), \text{ where } s \in m + n,$$
$$\gamma^\star(r) \odot \varepsilon^\star(m) = \varepsilon^\star(s), \text{ where } s \in r \cdot m.$$

# 3 Construction of $H_v$-module of functions over $H_v$-ring of arithmetic functions

In this section, we use the $H_v$-ring of arithmetics defined by the authors in [1] to define an $H_v$-module of functions over it.

An arithmetic function is a function whose domain is the set of natural numbers and it's codomain is the set of complex numbers.
Let $I = ]0, 1[$ and $M = \{f : ]0, \infty[\to \mathbb{C}; f|_I = 0\}$ and $(G, \star, \circ)$ be the $H_v$-ring of arithmetics defined by the authors with the following hyperoperations: For all $\alpha, \beta \in G$,

$$\alpha \star \beta(n) = \{\alpha(d) + \beta(\frac{n}{d}) : d|n\},$$

$$\alpha \circ \beta(n) = \{\alpha(d)\beta(\frac{n}{d}) : d|n\}.$$

The identity "$i$" in $(G, \circ)$ is given as follows:

$$i(n) = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{otherwise.} \end{cases}$$

$(M, +)$ is an abelian group under the standard addition of functions. We define $\cdot : G \times M \to \mathcal{P}^*(M)$ as follows: for all $\alpha \in R, f \in M$,

$$\alpha \cdot f(x) = \begin{cases} 0, & \text{if } 0 < x < 1; \\ \{\alpha(n) f(\frac{x}{n}) : n \le x\}, & \text{if } x \ge 1. \end{cases}$$

It is clear that "$\cdot$" is well defined.

Example 1. Let $f(x) = \begin{cases} 0, & \text{if } 0 < x < 1; \\ x & \text{if } x \ge 1. \end{cases}$ . Then, we have

$$(1 \cdot f)(x) = \begin{cases} 0, & \text{if } 0 < x < 1; \\ x & \text{if } 1 \leq x < 2; \\ \{\frac{x}{k}, \frac{x}{k-1}, \ldots, x\} & \text{if } k \leq x < k+1. \end{cases}$$

It is easy to see that $h, f \in 1 \cdot f$ where

$$h(x) = \begin{cases} 0, & \text{if } 0 < x < 1; \\ x & \text{if } 1 \leq x < 2; \\ \frac{x}{2} & \text{if } x \geq 2. \end{cases}$$

Example 2. Let $f(x) = \begin{cases} 0, & \text{if } 0 < x < 1; \\ 3, & \text{if } x \geq 1. \end{cases}$

Then, we observe that $(i \cdot f)(x) = \begin{cases} 0, & \text{if } 0 < x < 1; \\ \{3i(n), n \leq x\}, & \text{if } 1 \leq x < 2. \end{cases}$

It is clear that

$$h(x) = \begin{cases} 0, & \text{if } 0 < x < 1; \\ 3, & \text{if } 1 \leq x < 2; \\ 0, & \text{if } x \geq 2 \end{cases} .$$

is an element of $(i \cdot f)(x)$.

Remark 1. In general, $1 \cdot f \neq f$ and $i \cdot f \neq f$ (Examples 1 and 2).

Remark 2. $1 \cdot f = f$ if and only $f(x)$ is constant when $x \geq 2$.

**Proposition 3.1.** *Let $f \in M$. Then the following are true:*

1. $f \in 1 \cdot f$,
2. $f \in i \cdot f$ ,
3. $0 \cdot f = 0$.

*Proof.* We prove (1), the other parts are similar. We have that

$$(1 \cdot f)(x) = \begin{cases} 0, & \text{if } 0 < x < 1; \\ \{f(\frac{x}{n}) : n \leq x\}, & \text{if } x \geq 1. \end{cases}$$

For all $x \geq 1$, we have that $n = 1 \leq x$. The latter implies that

$$f(x) = \begin{cases} 0, & \text{if } 0 < x < 1; \\ f(x), & \text{if } x \geq 1 \end{cases}$$

is an element of $(1 \cdot f)(x)$. □

**Lemma 3.2.** *Let $\alpha \in G$ and $f, g \in M$. Then $\alpha \cdot (f + g) \subseteq \alpha \cdot f + \alpha \cdot g$.*

*Proof.* We have

$$(\alpha\cdot(f+g))(x)=\begin{cases}0, & \text{if } 0<x<1;\\ \{\alpha(n)(f(\frac{x}{n})+g(\frac{x}{n})):n\leq x\} & \text{if } x\geq 1.\end{cases}$$

On the other hand, we get

$$(\alpha\cdot f+\alpha\cdot g)(x)=\begin{cases}0, & \text{if } 0<x<1;\\ \{\alpha(n)f(\frac{x}{n})+\alpha(m)g(\frac{x}{m}):m,n\leq x\} & \text{if } x\geq 1.\end{cases}$$

Since $\{\alpha(n)(f(\frac{x}{n})+g(\frac{x}{n})):n\leq x\}\subseteq\{\alpha(n)f(\frac{x}{n})+\alpha(m)g(\frac{x}{m}):m,n\leq x\}$ for all $x\in]0,\infty[$, it follows that $\alpha\cdot(f+g)\subseteq\alpha\cdot f+\alpha\cdot g$. □

**Lemma 3.3.** *Let $\alpha,\beta\in G$ and $f\in M$. Then $(\alpha\star\beta)\cdot f\subseteq\alpha\cdot f+\beta\cdot f$.*

*Proof.* We have

$$\begin{aligned}((\alpha\star\beta)\cdot f)(x) &= \{(\lambda\cdot f)(x):\lambda\in\alpha\star\beta\}\\ &= \begin{cases}0, & \text{if } 0<x<1;\\ \{(\alpha(d)+\beta(\frac{n}{d}))f(\frac{x}{n}):d|n,n\leq x\} & \text{if } x\geq 1.\end{cases}\end{aligned}$$

On the other hand, we obtain

$$(\alpha\cdot f+\beta\cdot f)(x)=\begin{cases}0, & \text{if } 0<x<1;\\ \{\alpha(m)f(\frac{x}{m})+\beta(n)f(\frac{x}{n}):m,n\leq x\} & \text{if } x\geq 1.\end{cases}$$

Since $\{(\alpha(d)+\beta(\frac{n}{d}))f(\frac{x}{n}):d|n,n\leq x\}\subseteq\{\alpha(m)f(\frac{x}{m})+\beta(n)f(\frac{x}{n}):m,n\leq x\}$ for all $x\in]0,\infty[$, it follows that $(\alpha\star\beta)\cdot f\subseteq\alpha\cdot f+\beta\cdot f$. □

**Lemma 3.4.** *If $\alpha,\beta\in G$ and $f\in M$, then $\alpha\cdot(\beta\cdot f)=(\alpha\circ\beta)\cdot f$.*

*Proof.* We have

$$((\alpha\circ\beta)\cdot f)(x)=\begin{cases}0, & \text{if } 0<x<1;\\ \{\alpha(d)\beta(\frac{n}{d})f(\frac{x}{n}):d|n,n\leq x)\} & \text{if } x\geq 1.\end{cases}$$

On the other hand, we observe that

$$(\alpha\cdot(\beta\cdot f))(x)=\begin{cases}0, & \text{if } 0<x<1;\\ \{\alpha(t)\beta(s)f(\frac{x}{ts}):t\leq x,s\leq\frac{x}{t})\} & \text{if } x\geq 1.\end{cases}$$

Since $\{\alpha(d)\beta(\frac{n}{d})f(\frac{x}{n}):d|n,n\leq x)\}=\{\alpha(d)\beta(\frac{n}{d})f(\frac{x}{n}):d\leq n,\frac{n}{d}\leq\frac{x}{d}\}$ for all $x\in]0,\infty[$, it follows that $((\alpha\circ\beta)\cdot f)(x)=(\alpha\cdot(\beta\cdot f))(x)$. □

**Theorem 3.5.** *$(M,+,G,\cdot)$ is an $H_v$-module.*

*Proof.* The proof follows from having "$\cdot$" a well defined map and from Lemmas 3.2, 3.3 and 3.4. □

## 4 Complete parts in the $H_v$-module of functions over $H_v$-ring of arithmetics and its fundamental module

In this section, we characterize the complete parts in the $H_v$-module of functions and find its fundamental module.

Complete parts were introduced and studied for the first time by M. Koskas [12]. Later, this topic was analyzed by P. Corsini [5] and Y. Sureau [14] mostly in the general theory of hypergroups.

**Definition 4.1.** Let $(M, +, R, \cdot)$ be an $R$-$H_v$-module and $A \subseteq M$. Then $A$ is a complete part in $M$ if the following implication holds:

$$A \cap P \neq \emptyset \Rightarrow P \subseteq A.$$

Here, $P$ is given as:

$$P = \sum_{i=1}^{n} m'_i, m'_i = m_i \text{ or } m'_i = \sum_{j=1}^{n_i} (\prod_{k=1}^{k_i j} x_{ijk}) \cdot m_i.$$

where $m_i \in M$ and $x_{ijk} \in R$.

Let $K = [1, 2[$, $g : K \to \mathbb{C}$ and $A_g = \{f \in M : f|_K = g\}$.

**Lemma 4.2.** *$A_g$ is a complete part in $M$.*

*Proof.* Let $P = \prod_{j=1}^{k_1} \alpha_{1,j} \cdot f_1 + \ldots + \prod_{j=1}^{k_m} \alpha_{m,j} \cdot f_m$ and $f \in A_g \cap P$. For every $h \in P$, we have that $h|_K = \prod_{j=1}^{k_1} \alpha_{1,j}(1) f_1|_K + \ldots + \prod_{j=1}^{k_m} \alpha_{m,j}(1) f_m|_K = f|_K = g$. Thus, $P \subseteq A_g$. □

**Lemma 4.3.** *Let $L = \{g : K \to \mathbb{C}\}$ and $S \neq \emptyset \subseteq G$. Then $A = \bigcup_{g \in S} A_g$ is a complete part in $M$.*

*Proof.* Let $f \in A \cap P$. Then there exists $g \in S$ such that $f|_K = g$. The latter implies that $f \in A_g \cap P$. Lemma 5.4 asserts that $P \subseteq A_g$. Therefore, $P \subseteq A$. □

**Lemma 4.4.** *Let $A$ be a complete part in $M$. Then there exists $S \subseteq L$ such that $A = \bigcup_{g \in S} A_g$.*

*Proof.* If $S = L$ then $M = \bigcup_{g \in S} A_g$. Let $A$ be a complete part in $M$. Then there exists $S \subseteq L$ such that $A \subseteq \bigcup_{g \in S} A_g$. Suppose, to get contradiction, that there is no $S \subseteq G$ satisfying $A = \bigcup_{g \in S} A_g$. Then there exist $g \in G$, $f, h \in M$ such that $f|_K = h|_K = g$ such $f \in A$ and $h$ is not in $A$. Let $\alpha(n) = n$ and $\beta(n) = \begin{cases} 0, & \text{if } n = 2 \text{ ;} \\ n, & \text{otherwise.} \end{cases}$

It is easy to see that $f \in \alpha \cdot h + \beta \cdot (f - h)$. Having $f \in A \cap (\alpha \cdot h + \beta \cdot (f - h))$ and $A$ a complete part in $M$ imply that $f \in \alpha \cdot h + \beta \cdot (f - h) \subseteq A$. We have that

$$h(x) = \begin{cases} 0, & \text{if } 0 < x < 1; \\ \alpha(1)h(x) + \beta(1)(f - h)(x) = f(x) & \text{if } x \in K; \\ \alpha(1)h(x) + \beta(2)(f - h)(\frac{x}{2}) & \text{otherwise.} \end{cases}$$

is an element of $(h + \beta \cdot (f - h))(x)$. We get now that $h \in h + \beta \cdot (f - h) \subseteq A$. The latter is a contradiction. □

**Theorem 4.5.** *Let $A \subseteq M$. Then $A$ is a complete part in $M$ if and only if there exists $S \subseteq L$ such that $A = \bigcup_{g \in S} A_g$.*

*Proof.* The proof follows from Lemmas 4.3 and 4.4. □

Next, we identify the fundamental module of the $H_v$-module of functions.

**Definition 4.6.** Let $(M, +, R, \cdot)$ be an $R$-$H_v$-module. The heart of $M$, denoted as $w_M$, is defined as follows:

$$w_M = \{m \in M : \varepsilon^\star(m) = 0\},$$

where 0 is the zero of the module $M/\varepsilon^\star$.

Let $I = ]0, 1[$, $J = ]0, 2[$ and $F = \{g : J \to \mathbb{C} : g|_I = 0\}$.

**Proposition 4.7.** *Let $f, g \in M$ and $\rho$ be a relation on $M$ defined as follows:*

$$f \rho g \Leftrightarrow f|_J = g|_J.$$

*Then $\rho$ is an equivalence relation on $M$.*

*Proof.* The proof is straightforward. □

**Theorem 4.8.** *The fundamental relation $\varepsilon$ on $M$ coincides with $\rho$. Moreover, $\varepsilon = \varepsilon^\star$.*

*Proof.* Let $f|_J = g|_J$, $\alpha(n) = n$ and

$$\beta(n) = \begin{cases} 0, & \text{if } n = 2 \ ; \\ n, & \text{otherwise.} \end{cases}$$

It is easy to see that $f, g \in \alpha \cdot h + \beta \cdot (f - h)$. Thus, $f \varepsilon g$.

For the converse, suppose that $f \varepsilon g$. Then $f, g \in f_1 + \ldots + f_m$ or $f, g \in \prod_{j=1}^{k_1} \alpha_{1,j} \cdot f_1 + \ldots + \prod_{j=1}^{k_m} \alpha_{m,j} \cdot f_m$. If $f, g \in f_1 + \ldots + f_m$ then $f = g = f_1 + \ldots + f_m$ and hence, $f|_J = g|_J$. If $f, g \in \prod_{j=1}^{k_1} \alpha_{1,j} \cdot f_1 + \ldots + \prod_{j=1}^{k_m} \alpha_{m,j} \cdot f_m$ then it is easy to see that $f|_J = g|_J = \prod_{j=1}^{k_1} \alpha_{1,j}(1) f_1|_J + \ldots + \prod_{j=1}^{k_m} \alpha_{m,j}(1) f_m|_J$. Proposition 4.7 asserts that $\varepsilon$ is transitive. Thus, $\varepsilon^\star = \varepsilon$. $\square$

**Proposition 4.9.** *The heart of $M$, $w_M = \varepsilon(0) = A_0$. Moreover, it is an $H_v$-submodule of $M$.*

*Proof.* Since $\varepsilon^\star(f) \oplus \varepsilon^\star(0) = \varepsilon^\star(f)$, it follows that $\varepsilon^\star(0)$ is the zero of the module $M/\varepsilon^\star$. We have that $w_M = \{f \in M : \varepsilon^\star(f) = \varepsilon^\star(0)\} = \{f \in M : f|_J = 0\}$. Since $f|_J = 0$ and $K = [1, 2[ \subset J = ]0, 2[$, it follows that $w_M = A_0$. Since $(w_M, +)$ is a subgroup of $(M, +)$, it suffices to show that $\alpha \cdot w_M \subseteq w_M$ for all $\alpha \in R$. Let $\alpha \in R, f \in w_M$. We have that $\alpha \cdot f|_I = 0$ and $\alpha \cdot f|_{J \setminus I} = \alpha(1) f|_{J \setminus I} = 0$. $\square$

The authors in [2] proved that the fundamental relation $\gamma = \gamma^\star$ on the $H_v$-ring $(G, \star, \circ)$ is given as follows: For all $\alpha, \beta \in G$,

$$\alpha \gamma \beta \Leftrightarrow \alpha(1) = \beta(1).$$

Moreover, they showed that the fundamental ring of the $H_v$-ring $(G, \star, \circ)$ is, up to isomorphism, the ring of complex numbers $(\mathbb{C}, +, \cdot)$ under standard addition and multiplication. Thus, $(G, \star, \circ)$ is an $H_v$-field.

Remark 3. $(M, +, G, \cdot)$ is an $H_v$-vector space over the $H_v$-field $(G, \star, \circ)$.

**Theorem 4.10.** *$(F, +, \mathbb{C}, \cdot)$ is the fundamental module of $(M/\varepsilon^\star, \oplus, G/\gamma^\star, \odot)$ (up to isomorphism).*

*Proof.* Let $\psi : G/\gamma^\star \to \mathbb{C}$ be the ring isomorphism defined by the authors in [2] as $\psi(\alpha) = \alpha(1)$ and $\phi : (M/\varepsilon, \oplus, G/\gamma^\star, \odot) \to (F, +, \mathbb{C}, \cdot)$ be defined as $\phi(\varepsilon^\star(f)) = f|_J$. We prove that $\phi$ is a module isomorphism.
Theorem 4.8 asserts that $\phi$ is well defined and one-to-one. For every $g \in F$, we define

$$f(x) = \begin{cases} g(x), & x \in J; \\ 0, & \text{otherwise.} \end{cases}$$

Now, we conclude that $f \in M$ and that $\phi(f) = g$. Thus, $\phi$ is onto. We have that $\phi(\varepsilon^\star(f) \oplus \varepsilon^\star(g)) = \phi(\varepsilon^\star(f+g)) = (f+g)|_J = \phi(\varepsilon^\star(f)) + \phi(\varepsilon^\star(g))$ and $\phi(\gamma^\star(\alpha) \odot \varepsilon^\star(f)) = \phi(\varepsilon^\star(h)) = h|_J$ with $h \in \alpha \cdot f$. Since $h|_J = \alpha(1)f|_J$, it follows that $\phi(\gamma^\star(\alpha) \odot \varepsilon^\star(f)) = \psi(\gamma^\star(\alpha)) \cdot \phi(\varepsilon^\star(f))$. □

## 5 Strongly regular relations on the $H_v$-module of functions

In this section, we classify the strongly regular relations on $M$.

**Definition 5.1.** Let $R$ be an equivalence relation on an $H_v$-module $(M, +, S, \cdot)$, $A, B \subseteq M$. Then

1. $A\overline{R}B$ means that for every $a \in A$, there exists $b \in B$ such that $aRb$ and for every $b' \in B$, there exists $a' \in A$ such that $a'Rb'$;
2. $A\overline{\overline{R}}B$ means that for every $a \in A$ and for every $b \in B$, we have $aRb$.

**Definition 5.2.** Let $R$ be an equivalence relation on an $H_v$-module $(M, +, S, \cdot)$, $a, b, c \in M$ and $r \in S$. Then $R$ is called:

1. *regular relation on* $M$ if $aRb$ implies that $(a+c)\overline{R}(b+c)$ and $r \cdot a\overline{R}r \cdot b$;
2. *strongly regular relation on* $M$ if $aRb$ implies that $(a+c)\overline{\overline{R}}(b+c)$ and $r \cdot a\overline{\overline{R}}r \cdot b$.

**Proposition 5.3.** *Let $R$ be a strongly regular relation on $M$. If $0_R = A_0$ then $R = \varepsilon$.*

*Proof.* Let $f, g \in M$ such that $fRg$. Since $R$ is a strongly regular relation on $M$, it follows that $(f-g)R0$. The latter implies that $f - g \in 0_R = A_0$. We get now that $(f-g)|_J = 0$ and by applying Theorem 4.8, we deduce that $R \subseteq \varepsilon$.

For the converse, let $f, g \in M$ such that $f\varepsilon g$. Then $f|_J = g|_J$. The latter implies that $(f-g)|_J = 0 = 0|_J$. We get now that $f - g \in \varepsilon(0) = A_0 = 0_R$. Thus, $(f-g)R0$. Since $R$ is strongly regular relation on $M$, it follows that $fRg$. Thus, $\varepsilon \subseteq R$. □

**Proposition 5.4.** *Let $N \subseteq F$ be a fixed non-empty set and $R$ be the relation defined by:*

$$fRg \Leftrightarrow f|_J = g|_J + \sum_{f_\lambda \in N, c_\lambda \in \mathbb{C}} c_\lambda f_\lambda.$$

*Then $R$ is strongly regular relation on $M$ containing $\varepsilon$.*

*Proof.* It is easy to see that $R$ is an equivalence relation on $M$. Let $fRg$, $h \in M$ and $\alpha \in R$. Then $(f+h)|_J = f|_J + h|_J = g|_J + \sum_{f_\lambda \in N, c_\lambda \in \mathbb{C}} c_\lambda f_\lambda + h|_J = (g+h)|_J + \sum_{f_\lambda \in N, c_\lambda \in \mathbb{C}} c_\lambda f_\lambda$. Thus, $(f+h)\overline{\overline{R}}(g+h)$. Let $u \in \alpha \cdot f, v \in \alpha \cdot g$. Then $u|_J = \alpha(1)f|_J$ and $v|_J = \alpha(1)g|_J$. We get now that $u|_J = v|_J + \alpha(1) \sum_{f_\lambda \in N, c_\lambda \in \mathbb{C}} c_\lambda f_\lambda = v|_J + \sum_{f_\lambda \in N, m_\lambda = \alpha(1)c_\lambda \in \mathbb{C}} m_\lambda f_\lambda$. Thus, $\alpha \cdot f \overline{\overline{R}} \alpha \cdot g$. Therefore, $R$ is a strongly regular relation on $M$. □

**Proposition 5.5.** *Let $R$ be a strongly regular relation on $M$. Then there exists $N \subseteq F$ such that*

$$fRg \Leftrightarrow f|_J = g|_J + \sum_{f_\lambda \in N, c_\lambda \in \mathbb{C}} c_\lambda f_\lambda.$$

*Proof.* If $0_R = A_0$ then $R = \varepsilon$ by Proposition 5.3. Thus, $N = \{0\}$. Suppose $0_R \neq A_0$. Set $N = \{f_\lambda = h_\lambda|_J : h_\lambda \in 0_R\} \neq \emptyset$. Since $R$ is a strongly regular relation on $M$, it follows that $\sum h_\lambda \in 0_R$ and $\alpha \cdot h_\lambda \subseteq 0_R$. It is easy to see that $\sum c_\lambda h_\lambda \in 0_R$ for all $c_\lambda \in \mathbb{C}$. We can write $0_R$ as $0_R = \{\sum c_\lambda h_\lambda : c_\lambda \in \mathbb{C}\}$ where $h_\lambda \in 0_R$. Let $fRg$. Then $(f-g)R0$. We get that $(f-g)|_J = \sum c_\lambda h_\lambda|_J = \sum c_\lambda f_\lambda$. □

**Theorem 5.6.** *Let $R$ be an equivalence relation on $M$. Then $R$ is a strongly regular relation on $M$ if and only if there exists $N \subseteq F$ such that*

$$fRg \Leftrightarrow f|_J = g|_J + \sum_{f_\lambda \in N, c_\lambda \in \mathbb{C}} c_\lambda f_\lambda.$$

*Proof.* The proof follows from Propositions 5.4 and 5.5. □

**Proposition 5.7.** *Let $N \subseteq F$ be a fixed non-empty set $Q = \sum_{f_\lambda \in N, c_\lambda \in \mathbb{C}} c_\lambda f_\lambda$. Then $Q$ is a submodule of $F$. Moreover, $(F/Q, +, \mathbb{C}, \cdot)$ is a module.*

*Proof.* The proof is straightforward. □

We define the operations on $(M/R, \oplus, G/\gamma^\star, \odot)$ as follows: For all $\alpha \in G, f, g \in M$,

$$R(f) \oplus R(g) = R(f+g) \text{ and } \gamma^\star(\alpha) \odot R(f) = R(h) \text{ such that } h \in \alpha.f.$$

**Proposition 5.8.** *Let $N \subseteq F$ be a fixed non-empty set and and $R$ be the strongly relation defined by:*

$$fRg \Leftrightarrow f|_J = g|_J + \sum_{f_\lambda \in N, c_\lambda \in \mathbb{C}} c_\lambda f_\lambda.$$

*Then $(M/R, \oplus, G/\gamma^\star, \odot)$ is a module.*

*Proof.* The proof is straightforward. $\square$

**Theorem 5.9.** *Let $N \subseteq F$ be a fixed non-empty set and and $R$ be the strongly relation defined by:*

$$fRg \Leftrightarrow f|_J = g|_J + \sum_{f_\lambda \in N, c_\lambda \in \mathbb{C}} c_\lambda f_\lambda.$$

*Then $(M/R, \oplus, G/\gamma^\star, \odot) \cong (F/Q, +, \mathbb{C}, \cdot)$*

*Proof.* Let $\psi : G/\gamma^\star \longrightarrow \mathbb{C}$ be the ring isomorphism defined by the authors in [2] as $\psi(\gamma^\star(\alpha)) = \alpha(1)$ and $\chi : M/R \longrightarrow F/Q$ be the map such that $\chi(R(f)) = f|_J + Q$. Theorem 5.6 asserts that $\chi$ is well defined and one-to-one. First, we show that $\chi$ is an onto map. Let $h + Q \in F/Q$. Then $h : J \to \mathbb{C}$. It is easy to see that $\chi(R(f)) = h + Q$ where $f \in M$ is defined as follows:

$$f(x) = \begin{cases} h(x), & x \in J; \\ 0, & \text{otherwise.} \end{cases}$$

Finally, we need to show that $\chi$ is module homomorphism. We have that $\chi(R(f) \oplus R(g)) = \chi(R(f+g)) = (f+g)|_J + Q = f|_J + Q + g|_J + Q = \chi(R(f)) + \chi(R(g))$. Moreover, $\chi(\gamma^\star(\alpha) \odot R(f)) = \chi(R(\alpha \cdot f)) = \alpha(1) f|_J + Q$. Since $\alpha(1) f|_J + Q = \alpha(1)(f|_J + Q)$, it follows that $\chi(\gamma^\star(\alpha) \odot R(f)) = \psi(\gamma^\star(\alpha))\chi(R(f))$. $\square$

Next, we present two examples of different strongly regular relations on $M$.

Example 3. Let $h(x) = \begin{cases} 0, & 0 < x < 1; \\ 1, & 1 \leq x < 2 \end{cases}$ and $N = \{h\} \subset F$. We define $R$ on $M$ as follows:

$$fRg \Leftrightarrow f|_{[1,2[} = g|_{[1,2[} + c; c \in \mathbb{C}.$$

Then $R$ is a strongly regular relation on $M$. Moreover, $0_R = \{f \in M : f|_{[1,2[} = c, c \in \mathbb{C}\} \neq A_0$ and $(M/R, \oplus, R/\gamma^\star, \odot) \cong (F/ < h >, +, \mathbb{C}, \cdot)$.

Example 4. Let $h(x) = \begin{cases} 0, & 0 < x < 1; \\ x^2, & 1 \leq x < 2 \end{cases}$, $k(x) = \begin{cases} 0, & 0 < x < 1; \\ x, & 1 \leq x < 2 \end{cases}$ and $N = \{h, k\} \subset F$. We define $R$ on $M$ as follows:

$$fRg \Leftrightarrow f|_{[1,2[} = g|_{[1,2[} + ax + bx^2; a, b \in \mathbb{C}.$$

Then $R$ is a strongly regular relation on $M$. Moreover, $0_R = \{f \in M : f|_{[1,2[} = ax + bx^2; a, b \in \mathbb{C}\} \neq A_0$ and $(M/R, \oplus, R/\gamma^\star, \odot) \cong (F/ < h, k >, +, \mathbb{C}, \cdot)$.

Finally, we present a regular relation on $M$ that is not a strongly regular relation.

**Proposition 5.10.** *Let $R$ be the relation defined by:*

$$fRg \Leftrightarrow f|_{[1,3[} = g|_{[1,3[}.$$

*Then $R$ is a regular relation on $M$ that is not a strongly regular relation.*

*Proof.* It is clear that $R$ is an equivalence relation on $M$. Let $fRg$, $h \in M, \alpha \in G$. It is clear that $f + h\overline{R}g + h$. We need to show that $\alpha \cdot f\overline{R}\alpha \cdot g$. For every $f_1 \in \alpha \cdot f$, $f_1|_{[1,3[} = \alpha(1)f_{[1,3[}$ or $f_1|_{[1,2[} = \alpha(1)f_{[1,2[}$ and $f_1|_{[2,3[} = \alpha(2)f_{[1,\frac{3}{2}[}$. Similarly, for $g_1 \in \alpha \cdot g$ we have: $g_1|_{[1,3[} = \alpha(1)g_{[1,3[}$ or $g_1|_{[1,2[} = \alpha(1)g_{[1,2[}$ and $g_1|_{[2,3[} = \alpha(2)g_{[1,\frac{3}{2}[}$. It is easy to see that $\alpha \cdot f\overline{R}\alpha \cdot g$. Thus, $R$ is a regular relation on $M$. □

Remark 4. The regular relation defined in Proposition 5.10 is not a strongly regular relation on $M$. Let

$$\alpha(n) = \begin{cases} 1, & n = 1; \\ 2, & \text{otherwise} \end{cases}, \ f(x) = \begin{cases} 0, & 0 < x < 1; \\ 1, & x \geq 1 \end{cases} \quad \text{and}$$

$$f_1(x) = \begin{cases} 0, & 0 < x < 1; \\ 1, & 1 \leq x < 2; \\ 2, & x \geq 2. \end{cases}$$

We observe that $\{f, f_1\} \subseteq \alpha \cdot f$. Since $f|_{[1,3[} \neq f_1|_{[1,3[}$, it follows that $\alpha \cdot f\overline{\overline{R}}\alpha \cdot f$ is not satisfied. Thus, $R$ is not a strongly regular relation on $M$.

## References

[1] M. Al-Tahan and B. Davvaz, *On the existence of hyperrings associated to arithmetic functions*, J. Number Theory, **174** (2017), 136-149.

[2] M. Al-Tahan and and B. Davvaz, *Strongly regular relations of arithmetic functions*, J. Number Theory, **187** (2018), 391-402.

[3] M. Asghari-Larimi and B. Davvaz, *Hyperstructures associated to arithmetic functions*, Ars Combin., **97** (2010), 51-63.

[4] P. Corsini, *Prolegomena of Hypergroup Theory*, Udine, Tricesimo, Italy: Second edition, Aviani editore, 1993.

[5] P. Corsini, *Contributo alla teoria degli ipergruppi*, Atti Soc. Pelor. Sc. Mat. Fis. Nat. Messina, Messina, Italy, (1980), 1-22.

[6] P. Corsini and and V. Leoreanu, *Applications of Hyperstructures Theory*, Advances in Mathematics, Kluwer Academic Publisher, 2003.

[7] B. Davvaz, *Polygroup Theory and Related Systems*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013, viii+200 pp.

[8] B. Davvaz, *Semihypergroup Theory*, Elsevier/Academic Press, London, 2016, viii+156 pp.

[9] B. Davvaz and and V. Leoreanu-Fotea, *Hyperring Theory and Applications*, International Academic Press, USA, 2007.

[10] B. Davvaz and T. Vougiouklis, *A walk through weak hyperstructures: $H_v$-structures*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2019, xi+334 pp.

[11] D. Freni, *A note on the core of a hypergroup and the transitive closure $\beta^*$ of $\beta$*, Riv. Mat. Pura Appl., **8** (1991), 153-156.

[12] M. Koskas, *Groupoides, demi-hypergroupes et hypergroupes*, J. Math. Pure Appl., **(9) 49** (1970), 155-192.

[13] F. Marty, *Sur une generalization de la notion de group*, In 8*th* Congress Math. Scandenaves, (1934), pp. 45-49.

[14] Y. Sureau, *Contribution a la thorie des hypergroupes et hypergroupes opérant transivement sur un ensemble*, Doctoral Thesis, (1980).

[15] T. Vougiouklis, *Hyperstructures and Their Representations*, Hadronic Press, Inc, 115, Palm Harber, USA, 1994.

[16] T. Vougiouklis, *The fundamental relation in hyperrings. The general hyperfield*, In Proc. Fourth Int. Congress on Algebraic Hyperstructures and Appl. (AHA 1990), World Scientific, 1991, 203-211.

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