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Quaternion Fractional Difference with Quaternionic Fractional Order and Applications to Fractional Difference Equation

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Abstract

In this paper, we introduce the basic notions of the fractional summation, difference and q-difference with the quaternionic fractional order for the quaternion-valued functions and establish some of their basic properties. Based on this, the summation representations of solutions for the nonlinear quaternion-valued fractional difference equation and q-difference equation are obtained. In addition, several examples are provided to illustrate the feasibility of our obtained results in each section.

1 Introduction

The notion of quaternions which is a noncommutative extension of complex numbers was initiated by Hamilton in 1843, since then quaternion theory has been widely applied in differential geometry, fuid mechanics, attitude dynamics, quantum mechanics (see [1]). In the aspect of operator theory, Colombo et al. have established the theories of noncommutative functional calculus of slice hyperholomorphic functions and S-spectrum, and introduced the notion of quaternionic evolution operator, based on this, the fractional powers and fractional diffusion processes of quaternionic operators were investigated (see

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[4, 5, 6]). With the development of the dynamical analysis on hybrid domains (see [12, 13, 14]), dynamic equations on time scales and distinguished difference equations have been deeply developed and comprehensively studied under the quaternionic background.

In [9], Li et al. studied the commutativity of quaternion-matrix-valued functions and quaternion matrix dynamic equations on time scales, the authors presented several applications including multidimensional rotations and transformations of the submarine, the gyroscope, and the planet whose dynamical behaviors are depicted by quaternion dynamics on time scales. In 2021-2022, Wang and Li et al. applied the quaternion theory to fuzzy sets and systems, the authors studied the Hyers-ulam-rassias stability of quaternion multidimensional fuzzy nonlinear difference equations with impulses (see [10]) and established the quaternionic results of the fuzzy dynamic equations on time scales (see [17]). In [15], Wang et al. developed the general theory of the higher-order quaternion linear difference equations via the complex adjoint matrix and the quaternion characteristic polynomial and these results have been applied to investigate the global behaviour of quaternion Riccati rational difference equation (see [20]). In the aspect of quaternionic dynamic equations on time scales, in [16], the authors established a theoretical framework of the quaternion hyper-complex space in which the new quaternion hyper-complex exponent, the hyper-complex logarithm are introduced on time scales, additionally, the prominent algebraic and geometric rotation features of the polar coordinates and hyper-complex space are demonstrated by comparing with the traditional quaternion theory. Moreover, the fundamental solution matrix and Cauchy properties of quaternion combined impulsive matrix dynamic equation on time scales were studied (see [18]). In [19], Wang, Qin and Agarwal introduced the notion of quaternionic exponentially dichotomous operators, through establishing S-spectral splitting and slice quaternionic Banach algebra, the authors studied the Cauchy problem of the quaternionic evolution equations.

On the other hand, the fractional difference and q-difference equations are two important distinguished discrete dynamic equations. In 1966, Al-Salam introduced some notions of fractional q-initegerals and q-derivatives, some basic properties of them were established (see [2]). In [7], Gray and Zhang introduced a new definition of the fractional difference which provides an effective methods to study the fractional difference equations under this notion. However, there is no possibility to consider fractional difference equations with quaternionic fractional order since there is no notions of the fractional summation, difference and q-difference with the quaternionic fractional order for the quaternion-valued functions. Motivated by the above, in this paper, we will address the basic notions of the fractional summation, difference and q-difference with the quaternionic fractional order for the quaternion-valued functions and establish some of their basic properties, which will be applied to study nonlinear quaternion-valued fractional difference equation and q-difference equation.

2 Quaternion Fractional Difference with Quaternionic Fractional Order

Throughout the paper, we denote the set of the quaternion numbers by $\mathbb{H}$, the set of the complex numbers by $\mathbb{C}$, the set of the real numbers by $\mathbb{R}$, the set of the integers by $\mathbb{Z}$, the set of the positive integers by $\mathbb{N}$, the set of the nonnegative integers by $\mathbb{N}_0$ and the set of the negative integers by $\mathbb{Z}^-$.

First, we will introduce some basic knowledge of quaternion algebra which is needed in our discussion. The canonical basis for the real associative algebra of quaternion $\mathbb{H}$ is given by the elements $1, i, j, k$ satisfying the relations

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

A quaternion q is denoted by $q = q_0 + q_1 i + q_2 j + q_3 k$, $q_l \in \mathbb{R}$, $l = 0, 1, 2, 3$, its conjugate is $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$, while its norm is given by $|q|^2 = q\overline{q}$. The real part and imaginary part of a quaternion will be denoted by the symbol $\mathrm{Re}[q]$ and $\mathrm{Im}[q]$, respectively. Let $\mathbb{S}$ be the 2$-$dimensional sphere of purely imaginary unit quaternions, i.e.

$$\mathbb{S} = \{q = q_1 i + q_2 j + q_3 k | q_1^2 + q_2^2 + q_3^2 = 1\}.$$

To each quaternion q it is possible to associate an element on the sphere $\mathbb{S}$:

$$I_q = \begin{cases} \frac{\mathrm{Im}[q]}{|\mathrm{Im}[q]|} & \text{if } \mathrm{Im}[q] \neq 0, \\ \text{any element of } \mathbb{S} & \text{otherwise.} \end{cases}$$

Given $I \in \mathbb{S}$ we denote by $\mathbb{C}_I$ the hyper-complex plane $\mathbb{R} + I\mathbb{R}$ containing elements of the form $x + Iy$, $x, y \in \mathbb{R}$.

Remark 2.1. *From [4], for any $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$, where $\mathrm{Im}[q] \neq 0$, we can represent q by the following equivalent form*

$$q = q_0 + \sqrt{{q_1}^2 + {q_2}^2 + {q_3}^2}\left(\frac{q_1 i + q_2 j + q_3 k}{\sqrt{{q_1}^2 + {q_2}^2 + {q_3}^2}}\right).$$

Therefore, $\mathbb{C}_{I_q} = \{\mathbb{R} + I_q\mathbb{R} | I_q \in \mathbb{S}\}$, the imaginary unit I_q determines the complex plane $\mathbb{C}_{I_q}$ containing q.

Similar to the Gamma function of the complex version (see [3]), we introduce the following notion.

Definition 2.1. *For any $q \in \mathbb{C}_I$ and $I \in \mathbb{S}$, we define the slice Gamma function by*

$$\Gamma_I(q) = \int_0^\infty y^{q-1}e^{-y}dy,\ \mathrm{Re}[q] > 0, \tag{2.1}$$

which converges for all $q \in \mathbb{C}_I$.

Definition 2.2 ([11])**.** *For any quaternion $q \in \mathbb{H}$ and $q = q_0 + q_1 i + q_2 j + q_3 k$, $q_l \in \mathbb{R}$, $l = 0, 1, 2, 3$. Let $\mathrm{sgn}(q) := \frac{\mathrm{Im}(q)}{|\mathrm{Im}(q)|} \in \mathbb{S}$, the function e^q defined by*

$$e^q := e^{q_0}\big(\cos|q| + \mathrm{sgn}(q)\sin|q|\big) \tag{2.2}$$

is called the quaternion natural exponential function.

Remark 2.2. *From Definition 2.2, we notice that for any $q \in C_I$ and $I \in \mathbb{S}$, we have*

$$|e^q| = \big|e^{q_0}\big(\cos|q| + \mathrm{sgn}(q)\sin|q|\big)\big| \leq 2e^{q_0}.$$

Lemma 2.1. *For any $q \in \mathbb{C}_I$ and $I \in \mathbb{S}$, the following property holds*

$$\Gamma_I(q+1) = q\Gamma_I(q),\quad \mathrm{Re}[q] > 0. \tag{2.3}$$

Proof. Based on integration by parts, for $\mathrm{Re}[q] > 0$, we have

$$\Gamma_I(q+1) = \int_0^\infty y^q e^{-y}dy = \big[-y^{\mathrm{Re}[q]}y^{I\mathrm{Im}[q]}e^{-y}\big]_0^\infty + \int_0^\infty qy^{q-1}e^{-y}dy.$$

By Remark 2.2, it follows that

$$\lim_{y\to\infty} -y^{\mathrm{Re}[q]}y^{I\mathrm{Im}[q]}e^{-y} = 0,$$

which implies that

$$\Gamma_I(q+1) = q\int_0^\infty y^{q-1}e^{-y}dy = q\Gamma_I(q).$$

This completes the proof. □

Remark 2.3. *We rewrite the formula* (2.3) *into the form*

$$(q-1)\Gamma_I(q-1) = \Gamma_I(q). \tag{2.4}$$

The Gamma function has poles at zero and at the negative integers. It is easy to use the integral representation (2.1) *to explicitly represent the poles and the analytic continuation of $\Gamma(q)$ for $I \in \mathbb{S}$:*

$$\Gamma_I(q) = \int_0^1 y^{q-1}e^{-y}dy + \int_1^\infty y^{q-1}e^{-y}dy$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+q)n!} + \int_1^{\infty} y^{q-1} e^{-y} dy.$$

The second function on the right-hand side is an entire function on the hypercomplex plane $\mathbb{C}_I$*, and the first shows that the poles are as claimed, with* $\frac{(-1)^n}{n!}$ *being the residue at* $q = -n$, $n = 0, 1, 2, \ldots$.

Let $q = Q_0 + IQ_1 \, (\mathrm{Re}[q] = Q_0 = 0)$*, through* (2.4)*, it follows that* $(-1 + IQ_1)\Gamma_I(-1 + IQ_1) = \Gamma_I(IQ_1)$*. If* $\mathrm{Re}[q] = Q_0 \, (0 < Q_0 \leq 1)$*, then* $-1 < Q_0 - 1 \leq 0$*, by* (2.4)*, we obtain* $(q-1)\Gamma_I(q-1) = \Gamma_I(q)$*, thus, for* $\mathrm{Re}[q] \leq 0$*,* (2.4) *is well-defined. From* (2.4) *we get*

$$\Gamma_I(q + m + 1) = q(q+1)\cdots(q+m)\Gamma_I(q), \quad m \in \mathbb{N}. \tag{2.5}$$

Definition 2.3. *Let* $\alpha, \beta \in \mathbb{C}_I$ *and* $I \in \mathbb{S}$*.* $(\alpha)_{\beta,I}$ *is defined by:*

$$(\alpha)_{\beta,I} = \begin{cases} \frac{\Gamma_I(\alpha+\beta)}{\Gamma_I(\alpha)}, & \alpha, \alpha+\beta \notin \mathbb{Z}^- \cup \{0\}, \\ 1, \quad \alpha = \beta = 0, \\ 0, \quad \alpha = 0, \, \beta \notin \mathbb{Z}^- \cup \{0\}, \\ \textit{undefined, otherwise.} \end{cases}$$

Definition 2.4. *Let* $a, t \in \mathbb{Z}$*,* $n \in \mathbb{N}_0$*,* $\Omega_{n,a,t} = [a-n, t] \cap \mathbb{Z}$ *for* $a - n \leq t$*, we define the set of all quaternion-valued functions* $F : \Omega_{n,a,t} \to \mathbb{H}$ *by*

$$\Psi_{\Omega_{n,a,t},\mathbb{H}} = \{F | F : \Omega_{n,a,t} \to \mathbb{H}\}.$$

Lemma 2.2. *Let* $\alpha \in \mathbb{C}_I \, (I \in \mathbb{S})$*,* $a, t \in \mathbb{Z}$*,* $F(\cdot) \in \Psi_{\Omega_{n,a,t},\mathbb{H}}$*,* $F(t) = f_1(t) + f_2(t)i + f_3(t)j + f_4(t)k$*,* $f_m : \Omega_{n,a,t} \to \mathbb{R}$ *for* $m = 1, 2, 3, 4$*,* $\nabla F(t) = F(t) - F(t-1)$*, then*

$$\frac{\nabla^n}{\Gamma_I(n+\alpha)} \sum_{u=a}^{t} (t-u+1)_{n+\alpha-1,I} F(u) = \frac{\nabla^{n-n_0}}{\Gamma_I(n - n_0 + \alpha)} \times \sum_{u=a}^{t} (t-u+1)_{n+\alpha-n_0-1,I} F(u), \tag{2.6}$$

where $n = \max\{0, n_0\}$*,* $n_0 \in \mathbb{Z}$ *and* $0 < \mathrm{Re}(\alpha + n_0) \leq 1$*.*

Proof. Since

$$\frac{\nabla^n}{\Gamma_I(n+\alpha)} \sum_{u=a}^{t} (t-u+1)_{n+\alpha-1,I} F(u)$$

$$=\frac{\nabla^{n-n_0}}{\Gamma_I(n+\alpha)}\nabla^{n_0}\sum_{u=a}^{t}(t-u+1)_{n+\alpha-1,I}F(u).$$

By induction, we have

$$\nabla^{n_0}\sum_{u=a}^{t}(t-u+1)_{n+\alpha-1,I}F(u) =(n+\alpha-1)(n+\alpha-2)\cdots(n+\alpha-n_0) \times\sum_{u=a}^{t}(t-u+1)_{n+\alpha-n_0-1,I}F(u).$$

Therefore

$$\begin{aligned}
&\frac{\nabla^{n-n_0}}{\Gamma_I(n+\alpha)}\nabla^{n_0}\sum_{u=a}^{t}(t-u+1)_{n+\alpha-1,I}F(u)\\
=&\frac{\nabla^{n-n_0}}{\Gamma_I(n+\alpha)}(n-1+\alpha)(n-2+\alpha)\cdots(n-n_0+\alpha)\sum_{u=a}^{t}(t-u+1)_{n+\alpha-n_0-1,I}\\
&\times F(u)\\
=&\frac{\nabla^{n-n_0}(n-1+\alpha)(n-2+\alpha)\cdots(n-n_0+\alpha)}{(n-1+\alpha)(n-2+\alpha)\cdots(n-n_0+\alpha)\Gamma_I(n-n_0+\alpha)}\\
&\times\sum_{u=a}^{t}(t-u+1)_{n+\alpha-n_0-1,I}F(u)\\
=&\frac{\nabla^{n-n_0}}{\Gamma_I(n-n_0+\alpha)}\sum_{u=a}^{t}(t-u+1)_{n+\alpha-n_0-1,I}F(u).
\end{aligned}$$

This completes the proof. □

Remark 2.4. *If $n=0$ or $n=n_0$, (2.6) becomes:*

$$\frac{\nabla^{n}}{\Gamma_I(n+\alpha)}\sum_{u=a}^{t}(t-u+1)_{n+\alpha-1,I}F(u)=\frac{1}{\Gamma_I(\alpha)}\sum_{u=a}^{t}(t-u+1)_{\alpha-1,I}F(u).$$

Lemma 2.3. (2.6) *defined in Lemma 2.2 is independent of any $I\in\mathbb{S}$.*

Proof. From (2.6), we obtain

$$\frac{1}{\Gamma_I(\alpha)}\sum_{u=a}^{t}(t-u+1)_{\alpha-1,I}F(u)$$

$$
\begin{aligned}
&= \frac{1}{\Gamma_I(\alpha)}\Big[(t-a+1)_{\alpha-1,I}F(a) + (t-a)_{\alpha-1,I}F(a+1) + \cdots + (2)_{\alpha-1,I}F(t-1) \\
&\quad + (1)_{\alpha-1,I}F(t)\Big] \\
&= \frac{1}{\Gamma_I(\alpha)}\Big[\frac{\Gamma_I(t-a+\alpha)}{\Gamma_I(t-a+1)}F(a) + \frac{\Gamma_I(t-a-1+\alpha)}{\Gamma_I(t-a)}F(a+1) + \cdots + \frac{\Gamma_I(1+\alpha)}{\Gamma_I(2)} \\
&\quad \times F(t-1) + \Gamma_I(\alpha)F(t)\Big] \\
&= \frac{1}{\Gamma_I(\alpha)}\Big[\frac{(t-a-1+\alpha)(t-a-2+\alpha)\cdots(1+\alpha)\alpha\Gamma_I(\alpha)}{\Gamma_I(t-a+1)}F(a) \\
&\quad + \frac{(t-a-2+\alpha)\cdots(1+\alpha)\alpha\Gamma_I(\alpha)}{\Gamma_I(t-a)}F(a+1) + \cdots + \frac{\alpha\Gamma_I(\alpha)}{\Gamma_I(2)}F(t-1) \\
&\quad + \Gamma_I(\alpha)F(t)\Big] \\
&= \frac{(t-a-1+\alpha)(t-a-2+\alpha)\cdots(1+\alpha)\alpha}{(t-a)!}F(a) \\
&\quad + \frac{(t-a-2+\alpha)\cdots(1+\alpha)\alpha}{(t-a-1)!}F(a+1) + \cdots + \alpha F(t-1) + F(t).
\end{aligned}
$$

Hence, (2.6) is independent of any $I \in \mathbb{S}$. This completes the proof. □

By Lemmas 2.2-2.3, we can introduce the following notion of quaternionic fractional αth-order summation of the quaternion-valued function $F(t)$.

Definition 2.5. *Let $\alpha \in \mathbb{C}_I\,(I \in \mathbb{S})$ and $a, t \in \mathbb{Z}$, $F(\cdot) \in \Psi_{\Omega_{n,a,t},\mathbb{H}}$, the αth-order summation of the quaternion-valued function $F(t)$ over $\Omega_{0,a,t}$ is defined by*

$$
\underset{a}{\overset{t}{S}}{}^{\alpha}F(t) = \frac{\nabla^n}{\Gamma_I(n+\alpha)}\sum_{u=a}^{t}(t-u+1)_{n+\alpha-1,I}F(u), \tag{2.7}
$$

where $n = \max\{0, n_0\}$, $n_0 \in \mathbb{Z}$, $0 < \mathrm{Re}(\alpha + n_0) \leq 1$, $F(t) = f_1(t) + f_2(t)i + f_3(t)j + f_4(t)k$, $f_m : \Omega_{n,a,t} \to \mathbb{R}$ for $m = 1, 2, 3, 4$, $\nabla F(t) = F(t) - F(t-1)$.

Remark 2.5. *According to Lemmas 2.2-2.3, the equality* (2.7) *is well-defined.*

Example 2.1. *Let $a = 1, t = 3, \alpha = \frac{1}{2}, F(t) = t + 2ti + 3tj + 4tk$, then*

$$
\underset{1}{\overset{3}{S}}{}^{\frac{1}{2}}F(t) = \frac{35}{8}\big(1 + 2i + 3j + 4k\big).
$$

In fact, based on Definition 2.5, $n = \max\{0, n_0\} = 0$, we have

$$\begin{aligned}
\overset{3}{\underset{1}{S}}{}^{\frac{1}{2}}F(t) =& \frac{1}{\Gamma_I(\frac{1}{2})}\sum_{u=1}^{3}(4-u)_{-\frac{1}{2},I}\Big[u+2ui+3uj+4uk\Big] \\
=& \frac{1}{\Gamma_I(\frac{1}{2})}\Big[(3)_{-\frac{1}{2},I}+2(2)_{-\frac{1}{2},I}+3(1)_{-\frac{1}{2},I}\Big]\big(1+2i+3j+4k\big) \\
=& \frac{1}{\Gamma_I(\frac{1}{2})}\Big[\frac{\Gamma_I(\frac{5}{2})}{\Gamma_I(3)}+2\frac{\Gamma_I(\frac{3}{2})}{\Gamma_I(2)}+3\Gamma_I(\frac{1}{2})\Big]\big(1+2i+3j+4k\big) \\
=& \frac{1}{\Gamma_I(\frac{1}{2})}\Big[\frac{3\Gamma_I(\frac{1}{2})}{8}+\Gamma_I(\frac{1}{2})+3\Gamma_I(\frac{1}{2})\Big]\big(1+2i+3j+4k\big) \\
=& \frac{35}{8}\big(1+2i+3j+4k\big).
\end{aligned}$$

Example 2.2. *Let $a = 1, t = 3, \alpha = 1 + \frac{1}{3}i + \frac{2}{3}j + \frac{2}{3}k, F(t) = t + ti + tj + tk$, then*

$$\overset{3}{\underset{1}{S}}{}^{1+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k}F(t) = \frac{1}{3}\big(-1+20i+27j+20k\big).$$

In fact, based on Definition 2.5, $n = \max\{0, n_0\} = 0$, we obtain

$$\begin{aligned}
\overset{3}{\underset{1}{S}}{}^{1+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k}F(t) =& \frac{1}{\Gamma_I(1+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k)}\sum_{u=1}^{3}(4-u)_{\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k,I} \\
& \times\Big[u+ui+uj+uk\Big] \\
=& \frac{1}{\Gamma_I(1+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k)}\Big[(3)_{\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k,I}+2(2)_{\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k,I} \\
& +3(1)_{\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k,I}\Big]\big(1+i+j+k\big) \\
=& \frac{1}{\Gamma_I(1+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k)}\Big[\frac{\Gamma_I(\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k+3)}{\Gamma_I(3)}+2\Gamma_I(\frac{1}{3}i \\
& +\frac{2}{3}j+\frac{2}{3}k+2)+3\Gamma_I(\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k+1)\Big]\big(1+i+j+k\big) \\
=& \Big[\frac{1}{2}\big(\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k+2\big)\big(\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k+1\big) \\
& +2\big(\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k+1\big)+3\Big]\big(1+i+j+k\big) \\
=& \frac{1}{3}\big(-1+20i+27j+20k\big).
\end{aligned}$$

Example 2.3. *Let* $a = 1, t = 3, \alpha = -\frac{1}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k, F(t) = t+2ti+3tj+4tk$, *then*

$$\mathop{S}_{1}^{3}{}^{-\frac{1}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k}F(t) = \frac{1}{18}\big(-145+145i+29j+147k\big).$$

Indeed, based on Definition 2.5, $n = \max\{0, n_0\} = 1$, *we have*

$$\begin{aligned}\mathop{S}_{a}^{t}{}^{-\frac{1}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k}F(t) =& \frac{\nabla}{\Gamma_I(1-\frac{1}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k)}\sum_{u=1}^{3}(4-u)_{-\frac{1}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k,I}\\ &\times\Big[u+2ui+3uj+4uk\Big]\end{aligned}$$

Based on Remark 2.4, it follows that

$$\begin{aligned}&\mathop{S}_{a}^{t}{}^{-\frac{1}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k}F(t)\\ =&\frac{1}{\Gamma_I(-\frac{1}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k)}\sum_{u=1}^{3}(4-u)_{-\frac{4}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k,I}\Big[u+2ui+3uj+4uk\Big]\\ =&\frac{1}{\Gamma_I(-\frac{1}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k)}\Big[(3)_{-\frac{4}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k,I}+2(2)_{-\frac{4}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k,I}\\ &+3(1)_{-\frac{4}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k,I}\Big](1+2i+3j+4k)\\ =&\frac{1}{\Gamma_I(-\frac{1}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k)}\Big[\frac{\Gamma_I(\frac{5}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k)}{\Gamma_I(3)}+2\Gamma_I(\frac{2}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k)\\ &+3\Gamma_I(-\frac{1}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k)\Big](1+2i+3j+4k)\\ =&\Big[\frac{1}{2}(\frac{2}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k)(-\frac{1}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k)+(-\frac{1}{3}+\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k)+3\Big]\\ =&\frac{1}{18}\big(-145+145i+29j+147k\big).\end{aligned}$$

Definition 2.6. *Let* $\alpha \in \mathbb{C}_I\ (I \in \mathbb{S})$ *and* $a, t \in \mathbb{Z}$, $F(\cdot) \in \Psi_{\Omega_{n,a,t},\mathbb{H}}$, *the* α*th-order difference of the quaternion-valued function* $F(t)$ *over* $\Omega_{0,a,t}$ *is defined by*

$$\mathop{\nabla}_{a}^{t}{}^{\alpha}F(t) = \mathop{S}_{a}^{t}{}^{-\alpha}F(t),$$

where $F(t) = f_1(t)+f_2(t)i+f_3(t)j+f_4(t)k$, $f_m : \Omega_{n,a,t}\to\mathbb{R}$ *for* $m = 1, 2, 3, 4$.

Example 2.4. *Let* $a = 1, t = 4, \alpha = \frac{3}{2}-\frac{1}{3}i-\frac{2}{3}j-\frac{2}{3}k, F(t) = t+ti+tj+tk$, *then*

$$\mathop{\nabla}_{1}^{4}{}^{\frac{3}{2}-\frac{1}{3}i-\frac{2}{3}j-\frac{2}{3}k}F(t) = -\frac{1}{216}\big(537+195i+81j+195k\big).$$

In fact, based on Definition 2.6, we have

$$\mathop{\nabla}_1^4{}^{\frac{3}{2}-\frac{1}{3}i-\frac{2}{3}j-\frac{2}{3}k}F(t)=\mathop{S}_1^4{}^{-\frac{3}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k}F(t).$$

Based on Definition 2.5, $n=\max\{0,n_0\}=2$, *then*

$$\begin{aligned}\mathop{\nabla}_1^4{}^{\frac{3}{2}-\frac{1}{3}i-\frac{2}{3}j-\frac{2}{3}k}F(t)=&\mathop{S}_1^4{}^{-\frac{3}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k}F(t)\\ =&\frac{\nabla^2}{\Gamma_I(2-\frac{3}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k)}\sum_{u=1}^{4}(5-u)_{-\frac{1}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k,I}\\ &\times\Big[u+ui+uj+uk\Big].\end{aligned}$$

Then by Remark 2.4, we have

$$\begin{aligned}\mathop{\nabla}_1^4{}^{\frac{3}{2}-\frac{1}{3}i-\frac{2}{3}j-\frac{2}{3}k}F(t)=&\mathop{S}_1^4{}^{-\frac{3}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k}F(t)\\ =&\frac{1}{\Gamma_I(-\frac{3}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k)}\sum_{u=1}^{4}(5-u)_{-\frac{5}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k,I}\\ &\times\Big[u+ui+uj+uk\Big]\\ =&\frac{1}{\Gamma_I(-\frac{3}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k)}\Big[(4)_{-\frac{5}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k,I}\\ &+2(3)_{-\frac{5}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k,I}+3(2)_{-\frac{5}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k,I}\\ &+4(1)_{-\frac{5}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k,I}\Big](1+i+j+k)\\ =&\frac{1}{\Gamma_I(-\frac{3}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k)}\Big[\frac{\Gamma_I(\frac{3}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k)}{\Gamma_I(4)}\\ &+\frac{2\Gamma_I(\frac{1}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k)}{\Gamma_I(3)}+3\Gamma_I(-\frac{1}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k)\\ &+4\Gamma_I(-\frac{3}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k)\Big](1+i+j+k)\\ =&\Big[\frac{1}{6}(\frac{1}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k)(-\frac{1}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k)(-\frac{3}{2}+\frac{1}{3}i\\ &+\frac{2}{3}j+\frac{2}{3}k)+(-\frac{1}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k)(-\frac{3}{2}+\frac{1}{3}i+\frac{2}{3}j\\ &+\frac{2}{3}k)+3(-\frac{3}{2}+\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k)+4\Big](1+i+j+k)\end{aligned}$$

$$= -\frac{1}{216}\big(537 + 195i + 81j + 195k\big).$$

Remark 2.6. *Let* $\alpha \in \mathbb{C}_I\ (I \in \mathbb{S})$, $H(t) := \sum_{u=a}^{t} (t-u+1)_{n+\alpha-1,I} F(u)$. *Then*

$$\nabla^n H(t) = (n+\alpha-1)(n+\alpha-2)\cdots(\alpha+1)\alpha \sum_{u=a}^{t} (t-u+1)_{\alpha-1,I} F(u).$$

In fact, since $\nabla H(t) = H(t) - H(t-1)$ *and*

$$\begin{aligned}
&(t-u+1)_{n+\alpha-m,I} - (t-u)_{n+\alpha-m,I}\\
=&\frac{\Gamma_I(t-u+1+n+\alpha-m)}{\Gamma_I(t-u+1)} - \frac{\Gamma_I(t-u+n+\alpha-m)}{\Gamma_I(t-u)}\\
=&\frac{(t-u+n+\alpha-m)\Gamma_I(t-u+n+\alpha-m) - (t-u)\Gamma_I(t-u+n+\alpha-m)}{(t-u)\Gamma_I(t-u)}\\
=&\frac{(n+\alpha-m)\Gamma_I(t-u+n+\alpha-m)}{\Gamma_I(t-u+1)} = (n+\alpha-m)(t-u+1)_{n+\alpha-m-1,I},
\end{aligned}$$

for $1 \le m \le n$, *we have*

$$\begin{aligned}
\nabla H(t) =&H(t) - H(t-1)\\
=& \sum_{u=a}^{t}(t-u+1)_{n+\alpha-1,I}F(u) - \sum_{u=a}^{t-1}(t-u)_{n+\alpha-1,I}F(u)\\
=&(t-t+1)_{n+\alpha-1,I}F(t) + \sum_{u=a}^{t-1}\Big[(t-u+1)_{n+\alpha-1,I} - (t-u)_{n+\alpha-1,I}\Big]\\
&\times F(u)\\
=&\Gamma_I(n+\alpha)F(t) + (n+\alpha-1)\sum_{u=a}^{t-1}(t-u+1)_{n+\alpha-2,I}F(u)\\
=&(n+\alpha-1)\Big[\Gamma_I(n+\alpha-1)F(t) + \sum_{u=a}^{t-1}(t-u+1)_{n+\alpha-2,I}F(u)\Big]\\
=&(n+\alpha-1)\sum_{u=a}^{t}(t-u+1)_{n+\alpha-2,I}F(u)
\end{aligned}$$

and

$$\nabla^2 H(t) =\nabla(\nabla H(t)) = \nabla H(t) - \nabla H(t-1)$$

$$
\begin{aligned}
=&(n+\alpha-1)\Big[\sum_{u=a}^{t}(t-u+1)_{n+\alpha-2,I}F(u)-\sum_{u=a}^{t-1}(t-u)_{n+\alpha-2,I}F(u)\Big]\\
=&(n+\alpha-1)\Big\{\Gamma_I(n+\alpha-1)F(t)+\sum_{u=a}^{t-1}\Big[(t-u+1)_{n+\alpha-2,I}\\
&-(t-u)_{n+\alpha-2,I}\Big]F(u)\Big\}\\
=&(n+\alpha-1)(n+\alpha-2)\Big[\Gamma_I(n+\alpha-2)F(t)+\sum_{u=a}^{t-1}(t-u+1)_{n+\alpha-3,I}\\
&\times F(u)\Big]\\
=&(n+\alpha-1)(n+\alpha-2)\sum_{u=a}^{t}(t-u+1)_{n+\alpha-3,I}F(u).
\end{aligned}
$$

By induction, one has

$$
\nabla^{n-1}H(t)=(n+\alpha-1)(n+\alpha-2)\cdots(\alpha+1)\sum_{u=a}^{t}(t-u+1)_{\alpha,I}F(u),
$$

thus

$$
\begin{aligned}
\nabla^{n}H(t)=&\nabla(\nabla^{n-1}H(t))=\nabla^{n-1}H(t)-\nabla^{n-1}H(t-1)\\
=&(n+\alpha-1)(n+\alpha-2)\cdots(\alpha+1)\Big[\sum_{u=a}^{t}(t-u+1)_{\alpha,I}F(u)\\
&-\sum_{u=a}^{t-1}(t-u)_{\alpha,I}F(u)\Big]\\
=&(n+\alpha-1)(n+\alpha-2)\cdots(\alpha+1)\Big\{(t-t+1)_{\alpha,I}F(t)\\
&+\sum_{u=a}^{t-1}\Big[(t-u+1)_{\alpha,I}-(t-u)_{\alpha,I}\Big]F(u)\Big\}\\
=&(n+\alpha-1)(n+\alpha-2)\cdots(\alpha+1)\alpha\Big[\Gamma_I(\alpha)F(t)\\
&+\sum_{u=a}^{t-1}(t-u+1)_{\alpha-1,I}F(u)\Big]\\
=&(n+\alpha-1)(n+\alpha-2)\cdots(\alpha+1)\alpha\sum_{u=a}^{t}(t-u+1)_{\alpha-1,I}F(u).
\end{aligned}
$$

By Remark 2.6, the following result can be established.

Remark 2.7. *Let* $\alpha \in \mathbb{C}_I\,(I \in \mathbb{S})$ *and* $a, t \in \mathbb{Z}$, $F(\cdot) \in \Psi_{\Omega_{n,a,t},\mathbb{H}}$, $F(t) = f_1(t) + f_2(t)i + f_3(t)j + f_4(t)k$, $f_m : \Omega_{n,a,t} \to \mathbb{R}$ *for* $m = 1, 2, 3, 4$, *then*

$$\overset{t}{\underset{a}{S}}{}^{\alpha} F(t) = \frac{\sum\limits_{u=a}^{t} (t-u+1)_{\alpha-1,I} F(u)}{\Gamma_I(\alpha)}, \quad \overset{t}{\underset{a}{\nabla}}{}^{\alpha} F(t) = \frac{\sum\limits_{u=a}^{t} (t-u+1)_{-\alpha-1,I} F(u)}{\Gamma_I(-\alpha)}.$$

In fact, by Definition 2.5, we have

$$\overset{t}{\underset{a}{S}}{}^{\alpha} F(t) = \frac{\nabla^n}{\Gamma_I(n+\alpha)} \sum_{u=a}^{t} (t-u+1)_{n+\alpha-1,I} F(u).$$

By Remark 2.6, one has

$$\begin{aligned}
\overset{t}{\underset{a}{S}}{}^{\alpha} F(t) =& \frac{\nabla^n}{\Gamma_I(n+\alpha)} \sum_{u=a}^{t} (t-u+1)_{n+\alpha-1,I} F(u) = \frac{\nabla^n H(t)}{\Gamma_I(n+\alpha)} \\
=& \frac{(n+\alpha-1)(n+\alpha-2)\cdots(\alpha+1)\alpha \sum\limits_{u=a}^{t} (t-u+1)_{\alpha-1,I} F(u)}{\Gamma_I(n+\alpha)} \\
=& \frac{(n+\alpha-1)(n+\alpha-2)\cdots(\alpha+1)\alpha \sum\limits_{u=a}^{t} (t-u+1)_{\alpha-1,I} F(u)}{(n+\alpha-1)\Gamma_I(n+\alpha-1)} \\
=& \frac{(n+\alpha-2)\cdots(\alpha+1)\alpha \sum\limits_{u=a}^{t} (t-u+1)_{\alpha-1,I} F(u)}{\Gamma_I(n+\alpha-1)} \\
=& \ldots = \frac{\alpha \sum\limits_{u=a}^{t} (t-u+1)_{\alpha-1,I} F(u)}{\alpha \Gamma_I(\alpha)} \\
=& \frac{\sum\limits_{u=a}^{t} (t-u+1)_{\alpha-1,I} F(u)}{\Gamma_I(\alpha)}.
\end{aligned}$$

Similarly, the result of the $\alpha th-$*order difference of* $F(t)$ *can be obtained.*

3 Properties of Fractional Difference for Quaternion-valued Functions

In 1988, Gray and Zhang established the following results of the real-valued function with the fractional difference.

Theorem 3.1 ([7]). *Let $f, g : \Omega_{n,a,t} \to \mathbb{R}$, $a, t \in \mathbb{Z}$, then one has the following results.*

(i) For any $\alpha \in \mathbb{C}$, $p \in \mathbb{N}_0$ and $p - \alpha \in \mathbb{C} \backslash (\mathbb{Z}^- \cup \{0\})$, one has

$$\mathop{\nabla}_{a}^{t}{}^{\alpha} f(t) = \frac{\nabla^p}{\Gamma(p-\alpha)} \sum_{u=a}^{t} (t-u+1)_{p-\alpha-1} f(u).$$

(ii) For $\alpha, \beta \in \mathbb{C}$, one obtain the following results:

(1) *if $\alpha, \beta \in \mathbb{N}_0$, then*

$$\mathop{\nabla}_{a}^{t}{}^{\alpha} \mathop{\nabla}_{a}^{t}{}^{\beta} f(t) = \mathop{\nabla}_{a}^{t}{}^{\alpha+\beta} f(t);$$

(2) *if $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C} \backslash \mathbb{N}$, then*

$$\mathop{\nabla}_{a}^{t}{}^{\alpha} \mathop{\nabla}_{a}^{t}{}^{\beta} f(t) = \mathop{\nabla}_{a}^{t}{}^{\alpha+\beta} f(t);$$

(3) *if $\alpha \in \mathbb{C} \backslash \mathbb{N}_0$ and $\beta \in \mathbb{N}$, then*

$$\mathop{\nabla}_{a}^{t}{}^{\alpha} \mathop{\nabla}_{a}^{t}{}^{\beta} f(t) = \mathop{\nabla}_{a}^{t}{}^{\alpha+\beta} f(t) + \frac{1}{\Gamma(-\alpha)} \sum_{l=1}^{\beta} \sum_{v=a-l}^{a-1} (-1)^l \binom{\beta}{l} (t-l-v+1)_{-\alpha-1} f(v),$$

where $\binom{\beta}{l} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-l+1)\Gamma(l+1)}$.

(iii) For $\alpha \in \mathbb{C} \backslash \mathbb{Z}^-$, one has

$$\mathop{\nabla}_{a}^{t}{}^{\alpha} \mathop{\nabla}_{a}^{t}{}^{-\alpha} f(t) = f(t).$$

(iv) For $c, \alpha \in \mathbb{C}$, one has

$$\mathop{\nabla}_{a}^{t}{}^{\alpha} \big[c f(t) + g(t) \big] = c \mathop{\nabla}_{a}^{t}{}^{\alpha} f(t) + \mathop{\nabla}_{a}^{t}{}^{\alpha} g(t).$$

*(v) **(Leibniz Rule)** If $m \in \mathbb{N}_0$, then*

$$\nabla^{m} \big[f(t) g(t) \big] = \sum_{n=0}^{m} \binom{m}{n} [\nabla^{m-n} f(t-n)][\nabla^{n} g(t)];$$

if $\alpha \in \mathbb{C} \backslash \mathbb{N}_0$, then

$$\mathop{\nabla}_{a}^{t}{}^{\alpha} \big[f(t) g(t) \big] = \sum_{n=0}^{t-a} \binom{\alpha}{n} \left[\mathop{\nabla}_{a}^{t-n}{}^{\alpha-n} f(t-n) \right] [\nabla^{n} g(t)].$$

Now we establish the following properties of fractional difference with quaternionic fractional order which is essentially different from Theorem 3.1.

Theorem 3.2. *Let $F, G : \Omega_{n,a,t} \to \mathbb{H}$, $a, t \in \mathbb{Z}$, then the following results hold.*

(i) For any $\alpha \in \mathbb{C}_I\,(I \in \mathbb{S})$, $p \in \mathbb{N}_0$ and $p - \alpha \in \mathbb{C}_I \backslash (\mathbb{Z}^- \cup \{0\})$, one has

$$\underset{a}{\overset{t}{\nabla}}{}^{\alpha} F(t) = \frac{\nabla^p}{\Gamma_I(p-\alpha)} \sum_{u=a}^{t} (t-u+1)_{p-\alpha-1,I} F(u).$$

(ii) For $\alpha, \beta \in \mathbb{C}_I\,(I \in \mathbb{S})$, the following results hold:

(1) *if $\alpha, \beta \in \mathbb{N}_0$, then*

$$\underset{a}{\overset{t}{\nabla}}{}^{\alpha} \underset{a}{\overset{t}{\nabla}}{}^{\beta} F(t) = \underset{a}{\overset{t}{\nabla}}{}^{\alpha+\beta} F(t);$$

(2) *if $\alpha \in \mathbb{C}_I$ and $\beta \in \mathbb{C}_I \backslash \mathbb{N}$, then*

$$\underset{a}{\overset{t}{\nabla}}{}^{\alpha} \underset{a}{\overset{t}{\nabla}}{}^{\beta} F(t) = \underset{a}{\overset{t}{\nabla}}{}^{\alpha+\beta} F(t);$$

(3) *if $\alpha \in \mathbb{C}_I \backslash \mathbb{N}_0$ and $\beta \in \mathbb{N}$, then*

$$\begin{aligned}\underset{a}{\overset{t}{\nabla}}{}^{\alpha} \underset{a}{\overset{t}{\nabla}}{}^{\beta} F(t) =& \underset{a}{\overset{t}{\nabla}}{}^{\alpha+\beta} F(t) \\ &+ \frac{1}{\Gamma_I(-\alpha)} \sum_{l=1}^{\beta} \sum_{v=a-l}^{a-1} (-1)^l \binom{\beta}{l} (t-l-v+1)_{-\alpha-1,I} F(v).\end{aligned}$$

(iii) For $\alpha \in \mathbb{C}_I \backslash \mathbb{Z}^-\,(I \in \mathbb{S})$, we have

$$\underset{a}{\overset{t}{\nabla}}{}^{\alpha} \underset{a}{\overset{t}{\nabla}}{}^{-\alpha} F(t) = F(t).$$

(iv) For $C \in \mathbb{H}$, $\alpha \in \mathbb{C}_I\,(I \in \mathbb{S})$, we have

$$\begin{aligned}\underset{a}{\overset{t}{\nabla}}{}^{\alpha} \big[CF(t) + G(t)\big] =& \underset{a}{\overset{t}{\nabla}}{}^{\alpha} \Big[(c_1 \quad c_2 \quad c_3 \quad c_4) \begin{pmatrix} f_1(t) \\ -f_2(t) \\ -f_3(t) \\ -f_4(t) \end{pmatrix} \\ &+ (c_1 \quad c_2 \quad c_3 \quad c_4) \begin{pmatrix} f_2(t) \\ f_1(t) \\ f_4(t) \\ -f_3(t) \end{pmatrix} i\end{aligned}$$

$$+ (c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} f_3(t) \\ -f_4(t) \\ f_1(t) \\ f_2(t) \end{pmatrix} j$$

$$+ (c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} f_4(t) \\ f_3(t) \\ -f_2(t) \\ f_1(t) \end{pmatrix} k\Big] + \mathop{\nabla}_{a}^{t}{}^{\alpha} G(t).$$

Proof. Let $C = c_1 + c_2 i + c_3 j + c_4 k$, $F(t) = f_1(t) + f_2(t)i + f_3(t)j + f_4(t)k$, $G(t) = g_1(t) + g_2(t)i + g_3(t)j + g_4(t)k$, similar to the proof of Theorem 3.1, the results of (i)-(iii) can be proved on the hyper-complex plane $\mathbb{C}_I$ immediately. Now we only prove (iv), it follows that

$$\begin{aligned}
&\mathop{\nabla}_{a}^{t}{}^{\alpha}\big[CF(t) + G(t)\big] \\
=&\mathop{\nabla}_{a}^{t}{}^{\alpha}\big[(c_1 + c_2 i + c_3 j + c_4 k)(f_1(t) + f_2(t)i + f_3(t)j + f_4(t)k) + g_1(t) + g_2(t)i \\
&+ g_3(t)j + g_4(t)k\big] \\
=&\mathop{\nabla}_{a}^{t}{}^{\alpha}\Big[(c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} f_1(t) \\ -f_2(t) \\ -f_3(t) \\ -f_4(t) \end{pmatrix} + (c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} f_2(t) \\ f_1(t) \\ f_4(t) \\ -f_3(t) \end{pmatrix} i \\
&+ (c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} f_3(t) \\ -f_4(t) \\ f_1(t) \\ f_2(t) \end{pmatrix} j + (c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} f_4(t) \\ f_3(t) \\ -f_2(t) \\ f_1(t) \end{pmatrix} k\Big] \\
&+ \mathop{\nabla}_{a}^{t}{}^{\alpha}\big[g_1(t) + g_2(t)i + g_3(t)j + g_4(t)k\big] \\
=&\mathop{\nabla}_{a}^{t}{}^{\alpha}\Big[(c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} f_1(t) \\ -f_2(t) \\ -f_3(t) \\ -f_4(t) \end{pmatrix} + (c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} f_2(t) \\ f_1(t) \\ f_4(t) \\ -f_3(t) \end{pmatrix} i \\
&+ (c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} f_3(t) \\ -f_4(t) \\ f_1(t) \\ f_2(t) \end{pmatrix} j + (c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} f_4(t) \\ f_3(t) \\ -f_2(t) \\ f_1(t) \end{pmatrix} k\Big] \\
&+ \mathop{\nabla}_{a}^{t}{}^{\alpha} G(t).
\end{aligned}$$

The proof is completed. □

Particularly, if $C \in \mathbb{R}$, we have

$$\mathop{\nabla}_{a}^{t}{}^{\alpha}\big[CF(t)+G(t)\big] = C\mathop{\nabla}_{a}^{t}{}^{\alpha}F(t) + \mathop{\nabla}_{a}^{t}{}^{\alpha}G(t).$$

Theorem 3.3. *Let $F(\cdot), G(\cdot) \in \Psi_{\Omega_{n,a,t},\mathbb{H}}$, $a, t \in \mathbb{Z}$, $F(t) = f_1(t) + if_2(t) + jf_3(t) + kf_4(t)$ and $G(t) = g_1(t) + ig_2(t) + jg_3(t) + kg_4(t)$, where $f_l, g_l : \Omega_{n,a,t} \to \mathbb{R}$, $1 \le l \le 4$.*

(i) If $m \in \mathbb{N}_0$, then

$$\begin{aligned}
&\nabla^{m}\big[F(t)G(t)\big]\\
=&\sum_{n=0}^{m}\binom{m}{n}[\nabla^{m-n}f_1(t-n)][\nabla^{n}g_1(t)] - \sum_{n=0}^{m}\binom{m}{n}[\nabla^{m-n}f_2(t-n)]\\
&\times[\nabla^{n}g_2(t)] - \sum_{n=0}^{m}\binom{m}{n}[\nabla^{m-n}f_3(t-n)][\nabla^{n}g_3(t)]\\
&-\sum_{n=0}^{m}\binom{m}{n}[\nabla^{m-n}f_4(t-n)][\nabla^{n}g_4(t)]\\
&+\bigg\{\sum_{n=0}^{m}\binom{m}{n}[\nabla^{m-n}f_1(t-n)][\nabla^{n}g_2(t)] + \sum_{n=0}^{m}\binom{m}{n}[\nabla^{m-n}f_2(t-n)]\\
&\times[\nabla^{n}g_1(t)] + \sum_{n=0}^{m}\binom{m}{n}[\nabla^{m-n}f_3(t-n)][\nabla^{n}g_4(t)] - \sum_{n=0}^{m}\binom{m}{n}\\
&\times[\nabla^{m-n}f_4(t-n)][\nabla^{n}g_3(t)]\bigg\}i\\
&+\bigg\{\sum_{n=0}^{m}\binom{m}{n}[\nabla^{m-n}f_1(t-n)][\nabla^{n}g_3(t)] + \sum_{n=0}^{m}\binom{m}{n}[\nabla^{m-n}f_3(t-n)]\\
&\times[\nabla^{n}g_1(t)] - \sum_{n=0}^{m}\binom{m}{n}[\nabla^{m-n}f_2(t-n)][\nabla^{n}g_4(t)]\\
&+\sum_{n=0}^{m}\binom{m}{n}[\nabla^{m-n}f_4(t-n)][\nabla^{n}g_2(t)]\bigg\}j\\
&+\bigg\{\sum_{n=0}^{m}\binom{m}{n}[\nabla^{m-n}f_1(t-n)][\nabla^{n}g_4(t)] + \sum_{n=0}^{m}\binom{m}{n}[\nabla^{m-n}f_4(t-n)]\\
&\times[\nabla^{n}g_1(t)] + \sum_{n=0}^{m}\binom{m}{n}[\nabla^{m-n}f_2(t-n)][\nabla^{n}g_3(t)]
\end{aligned}$$

$$-\sum_{n=0}^{m}\binom{m}{n}[\nabla^{m-n}f_3(t-n)][\nabla^{n}g_2(t)]\Big\}k. \tag{3.8}$$

(ii) If $\alpha\in\mathbb{C}_I\backslash\mathbb{N}_0\,(I\in\mathbb{S})$, *then*

$$\begin{aligned}
&\mathop{\nabla}_{a}^{t}{}^{\alpha}\big[F(t)G(t)\big]\\
&=\sum_{n=0}^{t-a}\binom{\alpha}{n}\mathop{\nabla}_{a}^{t-n}{}^{\alpha-n}f_1(t-n)\nabla^{n}g_1(t)-\sum_{n=0}^{t-a}\binom{\alpha}{n}\mathop{\nabla}_{a}^{t-n}{}^{\alpha-n}f_2(t-n)\nabla^{n}g_2(t)\\
&\quad-\sum_{n=0}^{t-a}\binom{\alpha}{n}\mathop{\nabla}_{a}^{t-n}{}^{\alpha-n}f_3(t-n)\nabla^{n}g_3(t)-\sum_{n=0}^{t-a}\binom{\alpha}{n}\mathop{\nabla}_{a}^{t-n}{}^{\alpha-n}f_4(t-n)\\
&\quad\times\nabla^{n}g_4(t)+\Big\{\sum_{n=0}^{t-a}\binom{\alpha}{n}\mathop{\nabla}_{a}^{t-n}{}^{\alpha-n}f_1(t-n)\nabla^{n}g_2(t)+\sum_{n=0}^{t-a}\binom{\alpha}{n}\\
&\quad\times\mathop{\nabla}_{a}^{t-n}{}^{\alpha-n}f_2(t-n)\nabla^{n}g_1(t)+\sum_{n=0}^{t-a}\binom{\alpha}{n}\mathop{\nabla}_{a}^{t-n}{}^{\alpha-n}f_3(t-n)\nabla^{n}g_4(t)\\
&\quad-\sum_{n=0}^{t-a}\binom{\alpha}{n}\mathop{\nabla}_{a}^{t-n}{}^{\alpha-n}f_4(t-n)\nabla^{n}g_3(t)\Big\}i\\
&\quad+\Big\{\sum_{n=0}^{t-a}\binom{\alpha}{n}\mathop{\nabla}_{a}^{t-n}{}^{\alpha-n}f_1(t-n)\nabla^{n}g_3(t)+\sum_{n=0}^{t-a}\binom{\alpha}{n}\mathop{\nabla}_{a}^{t-n}{}^{\alpha-n}f_3(t-n)\\
&\quad\times\nabla^{n}g_1(t)-\sum_{n=0}^{t-a}\binom{\alpha}{n}\mathop{\nabla}_{a}^{t-n}{}^{\alpha-n}f_2(t-n)\nabla^{n}g_4(t)+\sum_{n=0}^{t-a}\binom{\alpha}{n}\\
&\quad\times\mathop{\nabla}_{a}^{t-n}{}^{\alpha-n}f_4(t-n)\nabla^{n}g_2(t)\Big\}j+\Big\{\sum_{n=0}^{t-a}\binom{\alpha}{n}\mathop{\nabla}_{a}^{t-n}{}^{\alpha-n}f_1(t-n)\nabla^{n}g_4(t)\\
&\quad+\sum_{n=0}^{t-a}\binom{\alpha}{n}\mathop{\nabla}_{a}^{t-n}{}^{\alpha-n}f_4(t-n)\nabla^{n}g_1(t)+\sum_{n=0}^{t-a}\binom{\alpha}{n}\mathop{\nabla}_{a}^{t-n}{}^{\alpha-n}f_2(t-n)\\
&\quad\times\nabla^{n}g_3(t)-\sum_{n=0}^{t-a}\binom{\alpha}{n}\mathop{\nabla}_{a}^{t-n}{}^{\alpha-n}f_3(t-n)\nabla^{n}g_2(t)\Big\}k.
\end{aligned}\tag{3.9}$$

Proof. Since $F(t)=f_1(t)+if_2(t)+jf_3(t)+kf_4(t)$ and $G(t)=g_1(t)+ig_2(t)+jg_3(t)+kg_4(t)$, one has

$$\begin{aligned}F(t)G(t)=&f_1(t)g_1(t)-f_2(t)g_2(t)-f_3(t)g_3(t)-f_4(t)g_4(t)+[f_1(t)g_2(t)\\&+f_2(t)g_1(t)+f_3(t)g_4(t)-f_4(t)g_3(t)]i+[f_1(t)g_3(t)+f_3(t)g_1(t)\end{aligned}$$

$$- f_2(t)g_4(t) + f_4(t)g_2(t)]j + [f_1(t)g_4(t) + f_4(t)g_1(t) + f_2(t)g_3(t) - f_3(t)g_2(t)]k := R.$$

Similar to the proof of (iv) and (v) in Theorem 3.1, we have $\nabla^m\big[F(t)G(t)\big] = \nabla^m R$ and $\overset{t}{\underset{a}{\nabla}}{}^{\alpha}\big[F(t)G(t)\big] = \overset{t}{\underset{a}{\nabla}}{}^{\alpha} R$, then the results follow. The proof is completed. □

In what follows, some properties of the fractional difference of the quaternion-valued functions for the limit case will be established.

Theorem 3.4 ([7]). *Let $f, g : \Omega_{0,-\infty,t} \to \mathbb{R}$, $t \in \mathbb{Z}$, where $\Omega_{0,-\infty,t} = (-\infty, t] \cap \mathbb{Z}$.*

(i) *For any $\alpha \in \mathbb{C}$, $p \in \mathbb{N}_0$ and $p - \alpha \in \mathbb{C}\backslash(\mathbb{Z}^- \cup \{0\})$, if $\overset{t}{\underset{-\infty}{\nabla}}{}^{\alpha} f(t)$ exists, then*

$$\overset{t}{\underset{-\infty}{\nabla}}{}^{\alpha} f(t) = \lim_{a \to -\infty} \frac{\nabla^p}{\Gamma(p-\alpha)} \sum_{u=a}^{t} (t-u+1)_{p-\alpha-1} f(u).$$

(ii) *If one of the following conditions $(C_1) - (C_3)$ holds, then*

$$\overset{t}{\underset{-\infty}{\nabla}}{}^{\alpha} \overset{t}{\underset{-\infty}{\nabla}}{}^{\beta} f(t) = \overset{t}{\underset{-\infty}{\nabla}}{}^{\alpha+\beta} f(t) \quad \text{and} \quad \overset{t}{\underset{-\infty}{\nabla}}{}^{\alpha} \overset{t}{\underset{-\infty}{\nabla}}{}^{-\alpha} f(t) = f(t).$$

(C_1) *$\alpha, \beta \in \mathbb{N}_0$;*

(C_2) *$\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}\backslash\mathbb{N}_0$, $\overset{t}{\underset{-\infty}{\nabla}}{}^{\beta} f(t)$, $\lim\limits_{\substack{a\to-\infty \\ b\to-\infty}} \overset{t}{\underset{a}{\nabla}}{}^{\alpha} \overset{t}{\underset{b}{\nabla}}{}^{\beta} f(t)$ and $\overset{t}{\underset{-\infty}{\nabla}}{}^{\alpha+\beta-n} f(t)$ exist, where $n = \max\{0, n_1, n_2\}$ for $0 < \mathrm{Re}(n_1 - \alpha) \le 1$ and $0 < \mathrm{Re}(n_2 - \alpha - \beta) \le 1$;*

(C_3) *$\alpha \in \mathbb{C}\backslash\mathbb{N}_0$ and $\beta \in \mathbb{N}$, $\overset{t}{\underset{-\infty}{\nabla}}{}^{\alpha} f(t)$ and $\overset{t}{\underset{-\infty}{\nabla}}{}^{\alpha} \overset{t}{\underset{-\infty}{\nabla}}{}^{\beta} f(t)$ exist.*

(iii) *For $c, \alpha \in \mathbb{C}$, if $\overset{t}{\underset{-\infty}{\nabla}}{}^{\alpha} f(t)$ and $\overset{t}{\underset{-\infty}{\nabla}}{}^{\alpha} g(t)$ exist, then*

$$\overset{t}{\underset{-\infty}{\nabla}}{}^{\alpha} [cf(t) + g(t)] = c \overset{t}{\underset{-\infty}{\nabla}}{}^{\alpha} f(t) + \overset{t}{\underset{-\infty}{\nabla}}{}^{\alpha} g(t).$$

(iv) *If $\alpha \in \mathbb{C}\backslash\mathbb{N}_0$, $\lim\limits_{\substack{a\to-\infty \\ b\to-\infty}} \sum_{n=0}^{t-a} \binom{\alpha}{n} \Big[\overset{t-n}{\underset{b}{\nabla}}{}^{\alpha-n} f(t-n)\Big] \nabla^n g(t)$ and $\lim\limits_{b\to-\infty} \overset{t}{\underset{b}{\nabla}}{}^{\alpha-n} f(t)$ exist for any fixed t, then*

$$\overset{t}{\underset{-\infty}{\nabla}}{}^{\alpha} f(t)g(t) = \sum_{n=0}^{\infty} \binom{\alpha}{n} \Big[\overset{t-n}{\underset{-\infty}{\nabla}}{}^{\alpha-n} f(t-n)\Big] [\nabla^n g(t)].$$

Theorem 3.5. *Let* $F, G : \Omega_{0,-\infty,t} \to \mathbb{H}$, $t \in \mathbb{Z}$, $F(t) = f_1(t) + i f_2(t) + j f_3(t) + k f_4(t)$ *and* $G(t) = g_1(t) + i g_2(t) + j g_3(t) + k g_4(t)$, *where* $f_l, g_l : \Omega_{0,-\infty,t} \to \mathbb{R}$, $1 \le l \le 4$.

(*i*) *For any* $\alpha \in \mathbb{C}_I\,(I \in \mathbb{S})$, $p \in \mathbb{N}_0$ *and* $p - \alpha \in \mathbb{C}_I \backslash (\mathbb{Z}^- \cup \{0\})$, *if* $\underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha} f_l(t)$ *exists for* $1 \le l \le 4$, *then*

$$\underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha} F(t) = \lim_{a \to -\infty} \frac{\nabla^p}{\Gamma_I(p-\alpha)} \sum_{u=a}^{t} (t-u+1)_{p-\alpha-1,I} F(u).$$

(*ii*) *If one of the following conditions* $(D_1) - (D_3)$ *holds, then*

$$\underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha} \underset{-\infty}{\overset{t}{\nabla}}{}^{\beta} F(t) = \underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha+\beta} F(t) \quad \text{and} \quad \underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha} \underset{-\infty}{\overset{t}{\nabla}}{}^{-\alpha} F(t) = F(t).$$

(D_1) $\alpha, \beta \in \mathbb{N}_0$;

(D_2) $\alpha \in \mathbb{C}_I\,(I \in \mathbb{S})$ *and* $\beta \in \mathbb{C}_I \backslash \mathbb{N}_0\,(I \in \mathbb{S})$, $\underset{-\infty}{\overset{t}{\nabla}}{}^{\beta} f_l(t)$, $\lim\limits_{\substack{a \to -\infty \\ b \to -\infty}} \underset{a}{\overset{t}{\nabla}}{}^{\alpha} \underset{b}{\overset{t}{\nabla}}{}^{\beta} f_l(t)$ *and* $\underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha+\beta-n} f_l(t)$ *exist for* $1 \le l \le 4$, *where* $n = \max\{0, n_1, n_2\}$ *for* $0 < \mathrm{Re}(n_1 - \alpha) \le 1$ *and* $0 < \mathrm{Re}(n_2 - \alpha - \beta) \le 1$;

(D_3) $\alpha \in \mathbb{C}_I \backslash \mathbb{N}_0\,(I \in \mathbb{S})$ *and* $\beta \in \mathbb{N}$, $\underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha} f_l(t)$ *and* $\underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha} \underset{-\infty}{\overset{t}{\nabla}}{}^{\beta} f_l(t)$ *exist for* $1 \le l \le 4$.

(*iii*) *For* $C \in \mathbb{H}$, $\alpha \in \mathbb{C}_I\,(I \in \mathbb{S})$, *if* $\underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha} f_l(t)$ *and* $\underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha} g_l(t)$ *for* $1 \le l \le 4$, *then*

$$\begin{aligned}
&\underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha} \big[C F(t) + G(t) \big] \\
&= \underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha} \Big[(c_1 \quad c_2 \quad c_3 \quad c_4) \begin{pmatrix} f_1(t) \\ -f_2(t) \\ -f_3(t) \\ -f_4(t) \end{pmatrix} + (c_1 \quad c_2 \quad c_3 \quad c_4) \begin{pmatrix} f_2(t) \\ f_1(t) \\ f_4(t) \\ -f_3(t) \end{pmatrix} i \\
&\quad + (c_1 \quad c_2 \quad c_3 \quad c_4) \begin{pmatrix} f_3(t) \\ -f_4(t) \\ f_1(t) \\ f_2(t) \end{pmatrix} j + (c_1 \quad c_2 \quad c_3 \quad c_4) \begin{pmatrix} f_4(t) \\ f_3(t) \\ -f_2(t) \\ f_1(t) \end{pmatrix} k \Big] \\
&\quad + \underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha} G(t).
\end{aligned}$$

Proof. Let $C = c_1 + c_2 i + c_3 j + c_4 k$, $F(t) = f_1(t) + f_2(t)i + f_3(t)j + f_4(t)k$, $G(t) = g_1(t) + g_2(t)i + g_3(t)j + g_4(t)k$, similar to the proof of Theorem 3.4, the results of (i)-(ii) can be proved on the hyper-complex plane $\mathbb{C}_I$ immediately. Now we only prove (iii), it follows that

$$\begin{aligned}
&\underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha}\big[CF(t) + G(t)\big] \\
=&\underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha}\big[(c_1 + c_2 i + c_3 j + c_4 k)(f_1(t) + f_2(t)i + f_3(t)j + f_4(t)k) \\
&+ g_1(t) + g_2(t)i + g_3(t)j + g_4(t)k\big] \\
=&\underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha}\Big[(c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} f_1(t) \\ -f_2(t) \\ -f_3(t) \\ -f_4(t)\end{pmatrix} + (c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} f_2(t) \\ f_1(t) \\ f_4(t) \\ -f_3(t)\end{pmatrix} i \\
&+ (c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} f_3(t) \\ -f_4(t) \\ f_1(t) \\ f_2(t)\end{pmatrix} j + (c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} f_4(t) \\ f_3(t) \\ -f_2(t) \\ f_1(t)\end{pmatrix} k\Big] \\
&+ \underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha}\big[g_1(t) + g_2(t)i + g_3(t)j + g_4(t)k\big] \\
=&\underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha}\Big[(c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} f_1(t) \\ -f_2(t) \\ -f_3(t) \\ -f_4(t)\end{pmatrix} + (c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} f_2(t) \\ f_1(t) \\ f_4(t) \\ -f_3(t)\end{pmatrix} i \\
&+ (c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} f_3(t) \\ -f_4(t) \\ f_1(t) \\ f_2(t)\end{pmatrix} j + (c_1 \quad c_2 \quad c_3 \quad c_4)\begin{pmatrix} f_4(t) \\ f_3(t) \\ -f_2(t) \\ f_1(t)\end{pmatrix} k\Big] \\
&+ \underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha} G(t).
\end{aligned}$$

This completes the proof. □

Particularly, for $C \in \mathbb{R}$, we have

$$\underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha}\big[CF(t) + G(t)\big] = C \underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha} F(t) + \underset{-\infty}{\overset{t}{\nabla}}{}^{\alpha} G(t).$$

Theorem 3.6. *Let* $\alpha \in \mathbb{C}_I \backslash \mathbb{N}_0$ $(I \in \mathbb{S})$, $t \in \mathbb{Z}$, $F(\cdot), G(\cdot) \in \Psi_{\Omega_{0,-\infty,t},\mathbb{H}}$, $F(t) = f_1(t) + i f_2(t) + j f_3(t) + k f_4(t)$ *and* $G(t) = g_1(t) + i g_2(t) + j g_3(t) + k g_4(t)$, *where*

$f_l, g_l : \Omega_{0,-\infty,t} \to \mathbb{R}$, $1 \le l \le 4$. *If*

$$\lim_{\substack{a\to-\infty\\ b\to-\infty}} \sum_{n=0}^{t-a} \binom{\alpha}{n} \mathop{\nabla}_{b}^{t-n}{}^{\alpha-n} f_u(t-n)\nabla^n g_v(t) \text{ and } \lim_{b\to-\infty} \mathop{\nabla}_{b}^{t}{}^{\alpha-n} f_u(t)$$

exist for any fixed t *and* $1 \le u, v \le 4$, *then*

$$\begin{aligned}
&\mathop{\nabla}_{-\infty}^{t}{}^{\alpha}\big[F(t)G(t)\big]\\
=&\sum_{n=0}^{\infty}\binom{\alpha}{n}\mathop{\nabla}_{-\infty}^{t-n}{}^{\alpha-n}f_1(t-n)\nabla^n g_1(t)-\sum_{n=0}^{\infty}\binom{\alpha}{n}\mathop{\nabla}_{-\infty}^{t-n}{}^{\alpha-n}f_2(t-n)\nabla^n g_2(t)\\
&-\sum_{n=0}^{\infty}\binom{\alpha}{n}\mathop{\nabla}_{-\infty}^{t-n}{}^{\alpha-n}f_3(t-n)\nabla^n g_3(t)-\sum_{n=0}^{\infty}\binom{\alpha}{n}\mathop{\nabla}_{-\infty}^{t-n}{}^{\alpha-n}f_4(t-n)\nabla^n g_4(t)\\
&+\Bigg\{\sum_{n=0}^{\infty}\binom{\alpha}{n}\mathop{\nabla}_{-\infty}^{t-n}{}^{\alpha-n}f_1(t-n)\nabla^n g_2(t)+\sum_{n=0}^{\infty}\binom{\alpha}{n}\mathop{\nabla}_{-\infty}^{t-n}{}^{\alpha-n}f_2(t-n)\nabla^n g_1(t)\\
&+\sum_{n=0}^{\infty}\binom{\alpha}{n}\mathop{\nabla}_{-\infty}^{t-n}{}^{\alpha-n}f_3(t-n)\nabla^n g_4(t)-\sum_{n=0}^{\infty}\binom{\alpha}{n}\mathop{\nabla}_{-\infty}^{t-n}{}^{\alpha-n}f_4(t-n)\nabla^n g_3(t)\Bigg\}i\\
&+\Bigg\{\sum_{n=0}^{\infty}\binom{\alpha}{n}\mathop{\nabla}_{-\infty}^{t-n}{}^{\alpha-n}f_1(t-n)\nabla^n g_3(t)+\sum_{n=0}^{\infty}\binom{\alpha}{n}\mathop{\nabla}_{-\infty}^{t-n}{}^{\alpha-n}f_3(t-n)\nabla^n g_1(t)\\
&-\sum_{n=0}^{\infty}\binom{\alpha}{n}\mathop{\nabla}_{-\infty}^{t-n}{}^{\alpha-n}f_2(t-n)\nabla^n g_4(t)+\sum_{n=0}^{\infty}\binom{\alpha}{n}\mathop{\nabla}_{-\infty}^{t-n}{}^{\alpha-n}f_4(t-n)\nabla^n g_2(t)\Bigg\}j\\
&+\Bigg\{\sum_{n=0}^{\infty}\binom{\alpha}{n}\mathop{\nabla}_{-\infty}^{t-n}{}^{\alpha-n}f_1(t-n)\nabla^n g_4(t)+\sum_{n=0}^{\infty}\binom{\alpha}{n}\mathop{\nabla}_{-\infty}^{t-n}{}^{\alpha-n}f_4(t-n)\nabla^n g_1(t)\\
&+\sum_{n=0}^{\infty}\binom{\alpha}{n}\mathop{\nabla}_{-\infty}^{t-n}{}^{\alpha-n}f_2(t-n)\nabla^n g_3(t)-\sum_{n=0}^{\infty}\binom{\alpha}{n}\mathop{\nabla}_{-\infty}^{t-n}{}^{\alpha-n}f_3(t-n)\nabla^n g_2(t)\Bigg\}k.
\end{aligned}$$

Proof. Similar to the proof process of (*iii*) and (*iv*) in Theorem 3.4, the desired result is immediate, we omit the proof here. □

4 Solution Representation of Nonlinear Quaternion-valued Fractional Difference Equation

Consider the following nonlinear quaternion-valued fractional difference equation

$$\mathop{\nabla}_{a}^{t}{}^{-\alpha}y(t) = F\big(t-1, y(t-1)\big) \tag{4.10}$$

with the initial condition:

$$y(a-1) = y_0 \in \mathbb{H}, \tag{4.11}$$

where $\alpha \in \mathbb{C}_I \backslash \mathbb{Z}^-$ $(I \in \mathbb{S})$, $a, t \in \mathbb{Z}$.

Theorem 4.1. *Let $a, t \in \mathbb{Z}$, $a < t$. The solution of* (4.10) *with the initial condition* (4.11) *has the following summation representation*

$$\begin{aligned} y(t) = \sum_{u=a+1}^{t} & \frac{(t-u-1-\alpha)(t-u-2-\alpha)\cdots(-\alpha)}{(t-u)!} F(u-1, y(u-1)) \\ & + \frac{(t-a-1-\alpha)(t-a-2-\alpha)\cdots(-\alpha)}{(t-a)!} F(a-1, y_0). \end{aligned}$$

Proof. By Theorem 3.2 (iii), one has

$$\mathop{\nabla}_a^t{}^{\alpha} \mathop{\nabla}_a^t{}^{-\alpha} y(t) = \mathop{\nabla}_a^t{}^{\alpha} F(t-1, y(t-1))$$

and by Remark 2.7, we have

$$y(t) = \frac{\sum\limits_{u=a}^{t} (t-u+1)_{-\alpha-1,I}}{\Gamma_I(-\alpha)} F(u-1, y(u-1)).$$

On the other hand, since

$$\begin{aligned} y(t) =& \frac{1}{\Gamma_I(-\alpha)} \Bigg[\sum_{u=a+1}^{t} (t-u+1)_{-\alpha-1,I} F(u-1, y(u-1)) + (t-a+1)_{-\alpha-1,I} \\ & \times F(a-1, y(a-1)) \Bigg] \\ =& \frac{1}{\Gamma_I(-\alpha)} \Bigg[\sum_{u=a+1}^{t} \frac{\Gamma_I(t-u-\alpha)}{\Gamma_I(t-u+1)} F(u-1, y(u-1)) + \frac{\Gamma_I(t-a-\alpha)}{\Gamma_I(t-a+1)} \\ & \times F(a-1, y_0) \Bigg] \\ =& \frac{1}{\Gamma_I(-\alpha)} \Bigg[\sum_{u=a+1}^{t} \frac{(t-u-1-\alpha)(t-u-2-\alpha)\cdots(-\alpha)\Gamma_I(-\alpha)}{\Gamma_I(t-u+1)} \\ & \times F(u-1, y(u-1)) + \frac{(t-a-1-\alpha)(t-a-2-\alpha)\cdots(-\alpha)\Gamma_I(-\alpha)}{\Gamma_I(t-a+1)} \end{aligned}$$

$$\begin{aligned}&\times F(a-1,y_0)\Big]\\ =&\sum_{u=a+1}^{t}\frac{(t-u-1-\alpha)(t-u-2-\alpha)\cdots(-\alpha)}{(t-u)!}F(u-1,y(u-1))\\ &+\frac{(t-a-1-\alpha)(t-a-2-\alpha)\cdots(-\alpha)}{(t-a)!}F(a-1,y_0),\end{aligned}$$

the solution of (4.10) with the initial condition (4.11) has the following summation representation

$$\begin{aligned}y(t)=&\sum_{u=a+1}^{t}\frac{(t-u-1-\alpha)(t-u-2-\alpha)\cdots(-\alpha)}{(t-u)!}F(u-1,y(u-1))\\ &+\frac{(t-a-1-\alpha)(t-a-2-\alpha)\cdots(-\alpha)}{(t-a)!}F(a-1,y_0).\end{aligned}$$

The proof is completed. □

Remark 4.1. *In Theorem 4.1, the solution of* (4.10) *can be obtained by the following iteration algorithm. Since the initial condition is provided on the point $a-1$, we obtain the solution at the point a,*

$$\begin{aligned}y(a)=&\frac{1}{\Gamma_I(-\alpha)}\sum_{u=a}^{a}\frac{\Gamma_I(a-u-\alpha)}{\Gamma_I(a-u+1)}F(u-1,y(u-1))\\ =&\frac{1}{\Gamma_I(-\alpha)}\left[\frac{\Gamma_I(-\alpha)}{\Gamma_I(1)}F(a-1,y(a-1))\right]\\ =&F(a-1,y_0)\end{aligned}$$

and then the solution at the point $a+1$ is obtained by

$$\begin{aligned}y(a+1)=&\frac{1}{\Gamma_I(-\alpha)}\sum_{u=a}^{a+1}\frac{\Gamma_I(a+1-u-\alpha)}{\Gamma_I(a+1-u+1)}F(u-1,y(u-1))\\ =&\frac{1}{\Gamma_I(-\alpha)}\left[\frac{\Gamma_I(1-\alpha)}{\Gamma_I(2)}F(a-1,y_0)+\frac{\Gamma_I(-\alpha)}{\Gamma_I(1)}F(a,y(a))\right]\\ =&\frac{1}{\Gamma_I(-\alpha)}\Big[-\alpha\Gamma_I(-\alpha)F(a-1,y_0)+\Gamma_I(-\alpha)F(a,y(a))\Big]\\ =&-\alpha F(a-1,y_0)+F(a,y(a)).\end{aligned}$$

By iteration process, the solution at the point $a+2$ is given by

$$y(a+2)=\frac{1}{\Gamma_I(-\alpha)}\sum_{u=a}^{a+2}\frac{\Gamma_I(a+2-u-\alpha)}{\Gamma_I(a+2-u+1)}F(u-1,y(u-1))$$

$$
\begin{aligned}
=&\frac{1}{\Gamma_I(-\alpha)}\left[\frac{\Gamma_I(2-\alpha)}{\Gamma_I(3)}F(a-1,y_0)+\frac{\Gamma_I(1-\alpha)}{\Gamma_I(2)}F(a,y(a))\right.\\
&\left.+\frac{\Gamma_I(-\alpha)}{\Gamma_I(1)}F(a+1,y(a+1))\right]\\
=&\frac{1}{\Gamma_I(-\alpha)}\left[\frac{(1-\alpha)(-\alpha)\Gamma_I(-\alpha)}{2!}F(a-1,y_0)+(-\alpha)\Gamma_I(-\alpha)F(a,y(a))\right.\\
&\left.+\Gamma_I(-\alpha)F(a+1,y(a+1))\right]\\
=&\frac{(1-\alpha)(-\alpha)}{2!}F(a-1,y_0)+(-\alpha)F(a,y(a))+F(a+1,y(a+1)).
\end{aligned}
$$

Repeating the same process, we have

$$
\begin{aligned}
y(a+n)=&\frac{1}{\Gamma_I(-\alpha)}\sum_{u=a}^{a+n}\frac{\Gamma_I(a+n-u-\alpha)}{\Gamma_I(a+n-u+1)}F(u-1,y(u-1))\\
=&\frac{1}{\Gamma_I(-\alpha)}\left[\frac{\Gamma_I(n-\alpha)}{\Gamma_I(n+1)}F(a-1,y_0)+\frac{\Gamma_I(n-1-\alpha)}{\Gamma_I(n)}F(a,y(a))\right.\\
&\left.+\cdots+\frac{\Gamma_I(-\alpha)}{\Gamma_I(1)}F(a+n-1,y(a+n-1))\right]\\
=&\frac{1}{\Gamma_I(-\alpha)}\left[\frac{(n-1-\alpha)(n-2-\alpha)\cdots(-\alpha)\Gamma_I(-\alpha)}{n!}F(a-1,y_0)\right.\\
&+\frac{(n-2-\alpha)(n-3-\alpha)\cdots(-\alpha)}{(n-1)!}\Gamma_I(-\alpha)F(a,y(a))\\
&\left.+\cdots+\Gamma_I(-\alpha)F(a+n-1,y(a+n-1))\right]\\
=&\frac{(n-1-\alpha)(n-2-\alpha)\cdots(-\alpha)}{n!}F(a-1,y_0)\\
&+\frac{(n-2-\alpha)(n-3-\alpha)\cdots(-\alpha)}{(n-1)!}F(a,y(a))\\
&+\cdots+F(a+n-1,y(a+n-1)).
\end{aligned}
$$

Example 4.1. *Consider the following quaternion fractional difference equation with the initial value* $y(1)=1+i+j+k$,

$$
{}_{2}^{t}\nabla^{-\frac{1}{2}-\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k}y(t)=(t-1)y(t-1)(1+2i+3j+4k).
$$

By Theorem 4.1, we have the following summation representation of its solu-

tion

$$y(t)=\sum_{u=3}^{t}\frac{(t-u-\frac{1}{2}+\frac{1}{3}i-\frac{2}{3}j-\frac{2}{3}k)\cdots(\frac{1}{2}+\frac{1}{3}i-\frac{2}{3}j-\frac{2}{3}k)}{(t-u)!}(u-1)y(u-1)$$
$$\times(1+2i+3j+4k)+\frac{(t-\frac{5}{2}+\frac{1}{3}i-\frac{2}{3}j-\frac{2}{3}k)\cdots(\frac{1}{2}+\frac{1}{3}i-\frac{2}{3}j-\frac{2}{3}k)}{(t-u)!}$$
$$\times(-8+4i+3j+6k).$$

In fact, for $F(t-1,y(t-1))=(t-1)y(t-1)(1+2i+3j+4k)$, $a=2$, $\alpha=-\frac{1}{2}-\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k$, $y_0=1+i+j+k$, *by Theorem 4.1 we have*

$$\begin{aligned}y(t)=&\sum_{u=3}^{t}\frac{(t-u-\frac{1}{2}+\frac{1}{3}i-\frac{2}{3}j-\frac{2}{3}k)\cdots(\frac{1}{2}+\frac{1}{3}i-\frac{2}{3}j-\frac{2}{3}k)}{(t-u)!}F(u-1,y(u-1))\\&+\frac{(t-\frac{5}{2}+\frac{1}{3}i-\frac{2}{3}j-\frac{2}{3}k)\cdots(\frac{1}{2}+\frac{1}{3}i-\frac{2}{3}j-\frac{2}{3}k)}{(t-u)!}F(1,y(1))\\=&\sum_{u=3}^{t}\frac{(t-u-\frac{1}{2}+\frac{1}{3}i-\frac{2}{3}j-\frac{2}{3}k)\cdots(\frac{1}{2}+\frac{1}{3}i-\frac{2}{3}j-\frac{2}{3}k)}{(t-u)!}(u-1)y(u-1)\\&\times(1+2i+3j+4k)+\frac{(t-\frac{5}{2}+\frac{1}{3}i-\frac{2}{3}j-\frac{2}{3}k)\cdots(\frac{1}{2}+\frac{1}{3}i-\frac{2}{3}j-\frac{2}{3}k)}{(t-u)!}\\&\times(-8+4i+3j+6k).\end{aligned}$$

5 Solution Representation of Nonlinear Quaternion-valued Fractional q-Difference Equation

Definition 5.1 ([2])**.** *Let* $0<q<1$, $\overline{q^{\mathbb{Z}}}=\{q^v: v\in\mathbb{Z}\}\cup\{0\}$, $f:\overline{q^{\mathbb{Z}}}\to\mathbb{R}$. *Then a q-analogue of the integral is defined by*

$$\int_x^{\infty}f(t)\mathrm{d}(t;q)=x(1-q)\sum_{k=1}^{\infty}q^{-k}f(xq^{-k})\quad \text{for } x\in\overline{q^{\mathbb{Z}}}.$$

Moreover, the q-derivative of f is defined by

$$D_qf(t)=\begin{cases}\frac{f(t)-f(qt)}{t-qt}, & t\neq 0,\\ \lim\limits_{t\to 0}(D_qf)(t), & t=0,\end{cases}\qquad D_q^0f=f,\quad D_q^nf=D_q(D_q^{n-1}f).$$

According to (1.11) in [2], we introduce the q-Gamma function of the quaternionic version.

Definition 5.2. *Let $\alpha \in \mathbb{C}_I \backslash (\mathbb{Z}^- \cup \{0\})$, $I \in \mathbb{S}$. The q-Gamma function is defined by*

$$\Gamma_{q,I}(\alpha) = \frac{e(q^\alpha)}{e(q)(1-q)^{\alpha-1}},$$

where $e(u) = \prod_{n=0}^{\infty} (1 - uq^n)^{-1}$ and this function $\Gamma_{q,I}(\alpha)$ satisfies $\Gamma_{q,I}(\alpha+1) = [\alpha]\Gamma_{q,I}(\alpha)$ for $[\alpha] = \frac{1-q^\alpha}{1-q}$; $\Gamma_{q,I}(\alpha+1) = [1][2]\ldots[\alpha]$ if $\alpha \in \mathbb{N}$.

According to (2.1) in [2], we introduce the notion of the αth-order fractional q-difference of the quaternion-valued function $F(x)$.

Definition 5.3. *Let $\alpha \in \mathbb{C}_I \backslash (\mathbb{Z}^- \cup \{0\})$, $0 < q < 1$, $F : \overline{q^{\mathbb{Z}}} \to \mathbb{H}$, $F(x) = f_1(x) + f_2(x)i + f_3(x)j + f_4(x)k$, $f_m : \overline{q^{\mathbb{Z}}} \to \mathbb{R}$, $m = 1, 2, 3, 4$, the αth-order fractional q-difference of the quaternion-valued function $F(x)$ is defined by*

$$S_q^0 F(x) = F(x), \;\; S_q^{-\alpha} F(x) = \frac{q^{-\frac{1}{2}\alpha(\alpha-1)}}{\Gamma_{q,I}(\alpha)} \int_x^\infty (t-x)_{\alpha-1} F(tq^{1-\alpha}) d(t;q),$$

where

$$(t-x)_{\alpha-1} = t^{\alpha-1} \frac{e\left(\frac{q^{\alpha-1}x}{t}\right)}{e\left(\frac{x}{t}\right)}.$$

Theorem 5.1 ([2])**.** *Let $0 < q < 1$.*

(i) If $c_1, c_2 \in \mathbb{C}$, $\alpha \in \mathbb{C} \backslash (\mathbb{Z}^- \cup \{0\})$, $f_1, f_2 : \overline{q^{\mathbb{Z}}} \to \mathbb{R}$, then

$$S_q^\alpha \big(c_1 f_1(x) + c_2 f_2(x)\big) = c_1 S_q^\alpha f_1(x) + c_2 S_q^\alpha f_2(x).$$

(ii) If $\alpha, \beta \in \mathbb{C} \backslash (\mathbb{Z}^- \cup \{0\})$, $f : \overline{q^{\mathbb{Z}}} \to \mathbb{R}$, then

$$S_q^\alpha S_q^\beta f(x) = S_q^{\alpha+\beta} f(x).$$

Similar to the proof of Theorem 5.1, the properties can be extended to the quaternionic version immediately.

Theorem 5.2. *Let $0 < q < 1$.*

(i) If $c_1, c_2 \in \mathbb{C}_I$, $\alpha \in \mathbb{C}_I \backslash (\mathbb{Z}^- \cup \{0\})$, $F, G : \overline{q^{\mathbb{Z}}} \to \mathbb{H}$, then

$$S_q^\alpha \big(c_1 F(x) + c_2 G(x)\big) = c_1 S_q^\alpha F(x) + c_2 S_q^\alpha G(x).$$

(ii) If $\alpha, \beta \in \mathbb{C}_I \backslash (\mathbb{Z}^- \cup \{0\})$, $F : \overline{q^{\mathbb{Z}}} \to \mathbb{R}$, then

$$S_q^\alpha S_q^\beta F(x) = S_q^{\alpha+\beta} F(x).$$

Consider the following nonlinear quaternion-valued fractional q-difference equation

$$S_q^{\alpha} y(t) = F\big(t, y(t)\big) \tag{5.12}$$

with the initial condition:

$$y(a) = q_0 \in \mathbb{H}. \tag{5.13}$$

where $\alpha \in \mathbb{C}_I \backslash (\mathbb{Z}^- \cup \{0\})$, $0 < q < 1$, $t < a < \infty$ and $F : \overline{q^{\mathbb{Z}}} \to \mathbb{H}$.

Theorem 5.3. *Let $\alpha \in \mathbb{C}_I \backslash (\mathbb{Z}^- \cup \{0\})$, $0 < q < 1$, $t < a < \infty$ and $F : \overline{q^{\mathbb{Z}}} \to \mathbb{H}$, the summation representation of solution of* (5.12) *with the initial condition* (5.13) *is given by*

$$\begin{aligned} y(t) =& \frac{q^{-\frac{1}{2}\alpha(\alpha-1)}}{\Gamma_{q,I}(\alpha)} a(1-q) \sum_{k=0}^{\frac{\ln t - \ln a}{\ln q} - 1} q^k (tq^k - t)_{(\alpha-1)} F(tq^{1-\alpha+k}, y(tq^{1-\alpha+k})) \\ &+ \frac{\prod_{n=0}^{\frac{\ln t - \ln a}{\ln q} - 1} \big(1 - (q^{\alpha-1}\tau/t) q^n\big)^{-1}}{\prod_{n=0}^{\frac{\ln t - \ln a}{\ln q} - 1} \big(1 - (\tau/t) q^n\big)^{-1}} q_0. \end{aligned}$$

Proof. By Theorem 5.2 (ii), we have

$$\begin{aligned} y(t) =& S_q^{-\alpha} S_q^{\alpha} y(t) \\ =& S_q^{-\alpha} F(t, y(t)) \\ =& \frac{q^{-\frac{1}{2}\alpha(\alpha-1)}}{\Gamma_{q,I}(\alpha)} \int_t^{\infty} (\tau - t)_{\alpha-1} F(\tau q^{1-\alpha}, y(\tau q^{1-\alpha})) d(\tau; q) \\ =& \frac{q^{-\frac{1}{2}\alpha(\alpha-1)}}{\Gamma_{q,I}(\alpha)} \int_t^{a} (\tau - t)_{\alpha-1} F(\tau q^{1-\alpha}, y(\tau q^{1-\alpha})) d(\tau; q) \\ &+ \frac{q^{-\frac{1}{2}\alpha(\alpha-1)}}{\Gamma_{q,I}(\alpha)} \int_a^{\infty} (\tau - t)_{\alpha-1} F(\tau q^{1-\alpha}, y(\tau q^{1-\alpha})) d(\tau; q). \end{aligned}$$

On the other hand, since

$$(\tau - t)_{\alpha-1} = \tau^{\alpha-1} \frac{\prod_{n=0}^{\infty} \big(1 - (q^{\alpha-1}\tau/t) q^n\big)^{-1}}{\prod_{n=0}^{\infty} \big(1 - (\tau/t) q^n\big)^{-1}}$$

$$
=\tau^{\alpha-1}\frac{\prod_{n=0}^{\infty}\left(1-(q^{\alpha-1}\tau/a)q^n\right)^{-1}}{\prod_{n=0}^{\infty}\left(1-(\tau/a)q^n\right)^{-1}}\frac{\prod_{n=0}^{\frac{\ln t-\ln a}{\ln q}-1}\left(1-(q^{\alpha-1}\tau/t)q^n\right)^{-1}}{\prod_{n=0}^{\frac{\ln t-\ln a}{\ln q}-1}\left(1-(\tau/t)q^n\right)^{-1}}
$$

$$
=(\tau-a)_{\alpha-1}\frac{\prod_{n=0}^{\frac{\ln t-\ln a}{\ln q}-1}\left(1-(q^{\alpha-1}\tau/t)q^n\right)^{-1}}{\prod_{n=0}^{\frac{\ln t-\ln a}{\ln q}-1}\left(1-(\tau/t)q^n\right)^{-1}},
$$

through using the initial condition $y(a)=q_0$ and $t<a$, we have

$$
\begin{aligned}
y(t)=&\frac{q^{-\frac{1}{2}\alpha(\alpha-1)}}{\Gamma_{q,I}(\alpha)}\int_t^a(\tau-t)_{(\alpha-1)}F(\tau q^{1-\alpha},y(\tau q^{1-\alpha}))d(\tau;q)\\
&+\frac{\prod_{n=0}^{\frac{\ln t-\ln a}{\ln q}-1}\left(1-(q^{\alpha-1}\tau/t)q^n\right)^{-1}}{\prod_{n=0}^{\frac{\ln t-\ln a}{\ln q}-1}\left(1-(\tau/t)q^n\right)^{-1}}q_0\\
=&\frac{q^{-\frac{1}{2}\alpha(\alpha-1)}}{\Gamma_{q,I}(\alpha)}a(1-q)\sum_{k=0}^{\frac{\ln t-\ln a}{\ln q}-1}q^k(tq^k-t)_{(\alpha-1)}F(tq^{1-\alpha+k},y(tq^{1-\alpha+k}))\\
&+\frac{\prod_{n=0}^{\frac{\ln t-\ln a}{\ln q}-1}\left(1-(q^{\alpha-1}\tau/t)q^n\right)^{-1}}{\prod_{n=0}^{\frac{\ln t-\ln a}{\ln q}-1}\left(1-(\tau/t)q^n\right)^{-1}}q_0.
\end{aligned}
$$

The proof is completed. □

6 Conclusion and Further Discussion

The real, associative algebra of quaternions $\mathbb{H}=\mathbb{H}_{\mathbb{R}}$ is given by

$$
\mathbb{H}_{\mathbb{R}}=\left\{a+\sum_{l=1}^{3}v_le_l:\ a,v_1,v_2,v_3\in\mathbb{R}\right\},
$$

where the imaginary units e_1, e_2, e_3 satisfy $e_1^2=e_2^2=e_3^2=-1,\ \ e_1e_2=-e_2e_1=e_3,\ \ e_2e_3=-e_3e_2=e_1,\ \ e_3e_1=-e_1e_3=e_2$.

Each quaternion $q = a + \sum_{l=1}^{3} v_l e_l$ may be decomposed as $q = \mathrm{Sc}(q) + \mathrm{Vec}(q)$, where $\mathrm{Sc}(q) = a$ is the scalar part of q and $\mathrm{Vec}(q) = v = \sum_{l=1}^{3} v_l e_l$ is the vector part of q.

For $q = a + v = a + \sum_{l=1}^{3} v_l e_l \in \mathbb{H}_{\mathbb{R}}$ and $z \in \mathbb{C}$, we define the quaternionic power $z^q \in \mathbb{H}_{\mathbb{C}}$ by

$$z^q := z^a \left[\cos(|v| \log z) + \frac{v}{|v|} \sin(|v| \log z) \right], \tag{6.14}$$

where

$$\mathbb{H}_{\mathbb{C}} = \left\{ a + \sum_{l=1}^{3} v_l e_l : \ a, v_1, v_2, v_3 \in \mathbb{C} \right\}.$$

This definition of z^q allows for the usual differentiation and integration rules:

$$\frac{d}{dz} z^q = q z^{q-1} \quad \text{and} \quad \int z^q dz = \frac{z^{q+1}}{q+1} + const., \quad q \neq -1. \tag{6.15}$$

Now, we introduce the quaternionic Gamma function Γ by setting

$$\Gamma(q) := \int_0^\infty t^{a-1} \cos(|v| \log t) e^{-t} dt + \frac{v}{|v|} \int_0^\infty t^{a-1} \sin(|v| \log t) e^{-t} dt \tag{6.16}$$

Based on (6.14), we can write (6.16) into the following more succinct from

$$\Gamma(q) := \int_0^\infty t^{q-1} e^{-t} dt, \quad \mathrm{Sc}(q) > 0. \tag{6.17}$$

Indeed, based on (6.14), we have

$$z^{a-1} z^q z^{-a} e^{-z} = \left[z^{a-1} \cos(|v| \log z) e^{-z} + \frac{v}{|v|} z^{a-1} \sin(|v| \log z) e^{-z} \right].$$

Integrating the both sides of above equation, we have

$$\begin{aligned} \int_0^\infty z^{a-1} z^q z^{-a} e^{-z} dz &= \int_0^\infty z^{a-1} \cos(|v| \log z) e^{-z} dz \\ &\quad + \frac{v}{|v|} \int_0^\infty z^{a-1} \sin(|v| \log z) e^{-z} dz \\ \int_0^\infty z^{q-1} e^{-z} dz &= \int_0^\infty z^{a-1} \cos(|v| \log z) e^{-z} dz \end{aligned}$$

$$
\begin{aligned}
&+\frac{v}{|v|}\int_0^\infty z^{a-1}\sin(|v|\log z)e^{-z}dz \\
=&\Gamma(q).
\end{aligned}
$$

i.e $\Gamma(q) = \int_0^\infty t^{q-1}e^{-t}dt$.

Moreover, integration by parts in the integral (6.17), where we use the differentiation formula (6.15), produces the functional equation

$$\Gamma(q+1) = q\Gamma(q), \quad \mathrm{Sc}(q) > 0.$$

Therefore, by using the properties of the usual quaternion Gamma function in [8] or the slice Gamma function of this paper on the hyper-complex plane $\mathbb{C}_I$, the basic notions of the fractional summation, difference and q-difference with the quaternionic fractional order for the quaternion-valued functions and their basic properties are consistent. Moreover, the summation representations of solutions for the nonlinear quaternion-valued fractional difference equation and q-difference equation can be used to transform these two types of fractional difference equation into the corresponding sum equation, for which we can use the methods of functional analysis such as the fixed point theorems and Lyapunov functionals to study the qualitative properties of solutions in the future.

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