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A simplified Proof of the Hopf Conjecture

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Abstract

The use of the *barycentre map* between two copies of $\mathbb{R}^n$, the first one with a metric without conjugate points, the second one with the canonical flat metric, allows to prove in a simplified way the fact that Riemannian tori without conjugate points are flat, as conjectured by Hopf in 1948 and proved definitively by Burago and Ivanov in 1994.

1 Introduction

The flatness of a torus without conjugate points was proposed by E. Hopf in 1948 ([4]) and established by the same author in dimension $n = 2$. D. Burago and S. Ivanov definitively proved the statement for n dimensional tori in 1994 ([1]). The main idea of the work of above cited authors is to show that the Banach norm, induced by the metric $\tilde{g}$ without conjugate points in the universal covering $(\mathbb{R}^n, \tilde{g})$ of a rational torus $(\mathbb{R}^n/\mathbb{Q}^n, g)$, holds some "roundness" property which makes flat the torus. To make so, they decomposed $(\mathbb{R}^n, \tilde{g})$ in 1-dimensional sheets getting a rational foliation of this space. The roundness of the norm comes from the use of the Birkhoff Ergodic Theorem and advanced tools of integral geometry applied to the Busemann function along the rays of this rational foliation. Classical arguments of density of $\mathbb{Q}^n$ in $\mathbb{R}^n$ complete the proof of the theorem.

In this paper we propose another proof of this theorem: we consider two copies of $\mathbb{R}^n$, one equipped by the canonical flat metric $\tilde{g}_0$, the other one

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with the metric $\tilde{g}$ without conjugate points, as in the cited work of Burago and Ivanov. We build a C^∞ equivariant map $H : (\mathbb{R}^n, \tilde{g}) \to (\mathbb{R}^n, \tilde{g}_0)$, the *barycentre map*, such that the metric tensors of both copies of $\mathbb{R}^n$ appear in its differential. We prove the constance of the tensor $\tilde{g}$ along the rays of a rational foliation using still the Birkhoff Ergodic Theorem; usual density arguments end the proof.

2 Rational foliations of $\mathbb{R}^n$

Let $(\mathbb{R}^n, \tilde{g}_0)$ and $(\mathbb{R}^n, \tilde{g})$ be two copies of $\mathbb{R}^n$, the first one equipped by the canonical flat metric, the second one with a metric $\tilde{g}$ such that the length of any geodesic is the distance between the end points, this metric *has not conjugate points*. The spaces $(\mathbb{R}^n, \tilde{g}_0)$ and $(\mathbb{R}^n, \tilde{g})$ are respectively the universale coverings of the torus $(\mathbb{R}^n/\mathbb{Q}^n, g_0)$ and of the torus $(\mathbb{R}^n/\mathbb{Q}^n, g)$; both coverings have $\mathbb{Q}^n$ as automorphisms group. Let $\mathbb{S}^n$ be the unit sphere of $(\mathbb{R}^n, \tilde{g})$, a point $\mathbf{p} \in \mathbb{S}^n$ is said to be *rational* is $t\,\mathbf{p} \in \mathbb{Q}^n$ for some positive real number t. Busemann proved (see [2]) that for every $\mathbf{p}$ rational and for every positive real number t there exists a $\mathbb{Q}^n-$ invariant vector field $\mathbf{u_p}$ such that, for every $\tilde{y} \in \mathbb{R}^n$,

- its trajectories are geodesic in the direction $\mathbf{p}$: $\exp_{\tilde{y}} t\mathbf{u_p}(\tilde{y}) = \tilde{y} + t\mathbf{p}$,
- there is a *quasi-isometry* between the two copies of $\mathbb{R}^n$, i.e. for every couple of points $\tilde{x}$ and $\tilde{y}$, if $\| \bullet \|$ is the usual distance induced by the canonical metric and $\tilde{d}$ is the distance induced by the metric $\tilde{g}$, there exists a positive constant c such that

$$\Big| \|\tilde{x} - \tilde{y}\| - \tilde{d}(\tilde{x}, \tilde{y}) \Big| \leq c \tag{2.1}$$

The foliation determined by $\mathbf{u_p}$ is called a *rational foliation* on the direction $\mathbf{p}$.

3 The barycentre map

3.1 Regularization of the distance function

A previous result of K. Grove and K. Shiohama ([3]) concerning a suitable regularization of the distance function, which will be necessary in the next developments, is presented here in such a way to define in a suitable way the barycentre map.

Let $(\overline{M}, \overline{g})$ be any Riemannian manifold whose $\varrho_{\overline{M}}$ is the associated Riemannian distance and $\operatorname{inj}(\overline{M}, \overline{g})$ its injectivity radius, we denote by ω_n the volume of the canonical sphere; the result of Grove and Shiohama gives the

Lemma 3.1. *For every $r > 0$ there exists a C^∞ function $\varrho_r : \overline{M} \times \overline{M} \to \mathbb{R}^+$ such that, for every $\overline{y}, \overline{z} \in \overline{M}$*

1. $|\varrho_{\overline{M}}(\overline{y}, \overline{z}) - \varrho_r(\overline{y}, \overline{z})| \leq r$;
2. $\forall \gamma \in Isom(\overline{M}, \overline{g})$ *one has* $\varrho_r(\gamma\overline{y}, \gamma\overline{z}) = \varrho_r(\overline{y}, \overline{z})$;
3. $|\nabla \varrho_r| \leq \cosh(kr) \leq 1 + c_1 k^2 r^2$, *where $-k^2$ is a lower bound of the sectional curvature of $(\overline{M}, \overline{g})$.*

Following the above cited authors, the construction of ϱ_r is as follows.

Let $\phi : [0, +\infty[$ be a decreasing $C^\infty-$function satisfying $\phi = 1$ in a neighborhood of 0 and $\phi = 0$ on $[1, +\infty[$. For r sufficiently small one defines first ϕ_r by

$$\phi_r = \frac{\phi\left(\frac{t}{r}\right)}{\omega_{n-1} \int_0^r \phi\left(\frac{s}{r}\right) s^{n-1} ds}.$$

and one defines the *regularized distance function* ϱ_r by:

Definition 3.2.

$$\varrho_r(\overline{y}, \overline{z}) = \int_{B(\mathbf{0}_{\overline{y}}, r)} \varrho_{\overline{M}}\left(\exp_{\overline{y}}(\mathbf{v}), \overline{z}\right) \phi_r\left(\|\mathbf{v}\|\right) d\mathbf{v}$$

$$= \int_0^r \left(\int_{U_{\overline{y}}} \varrho_{\overline{y}}\left(\exp_{\overline{y}}(t\mathbf{u}), \overline{z}\right) d\mathbf{u} \right) \phi_r(t)\, t^{n-1} dt$$

where $B(\mathbf{0}_{\overline{y}}, r)$ and $U_{\overline{y}}$ are respectively the ball of radius r and the unit sphere in the tangent space $\left(T_{\overline{y}}\overline{M}, g_{\overline{y}}\right)$, both endowed with the canonical measure associated to the Euclidean structure $g_{\overline{y}}$.

We emphasize here the fact that ϱ_r no more enjoys the symmetry property (i.e. $\varrho_r(\overline{y}, \overline{z}) \neq \varrho_r(\overline{z}, \overline{y})$) but it still enjoys some of the distance function usual properties.

3.2 The barycentre of a measure in $(\mathbb{R}^n, \tilde{g}_0)$

Let μ be any positive measure on $(\mathbb{R}^n, \tilde{g}_0)$ such that $\int_{\mathbb{R}^n}(1+\|\tilde{z}\|)\,d\mu(\tilde{z}) < +\infty$, the barycentre of the measure μ is the point $bar(\mu)$ of $\mathbb{R}^n$ defined by

$$bar(\mu) = \frac{\int_{\mathbb{R}^n} \tilde{z}\, d\mu(\tilde{z})}{\int_{\mathbb{R}^n} d\mu(\tilde{z})} \tag{3.1}$$

From its definition, it follows directly that that the barycentre satisfies the equivariance property: for any isometry γ of $\mathbb{R}^n$

$$bar(\gamma_*\mu) = \gamma[\, bar\mu\,]. \tag{3.2}$$

3.3 Definition and properties of the barycentre map

Let $a > 0$ be any strictly positive constant, we define a map $\tilde{y} \to \mu_{\tilde{y}}$ from $(\mathbb{R}^n, \tilde{g})$ to the space of finite measures of $(\mathbb{R}^n, \tilde{g}_0)$ by

$$d\mu_{\tilde{y}}(\bullet) = e^{-a\varrho_r(y,\bullet)} dv_{\tilde{g}_0}(\bullet) \tag{3.3}$$

where $dv_{\tilde{g}_0}(\bullet)$ is the canonical measure on $\mathbb{R}^n$. The assumptions $a > 0$ and (2.1) imply that

$$\int_{\mathbb{R}^n} (1+\|\tilde{z}\|)\; d\mu_{\tilde{y}}(\tilde{z}) \le e^{a[c+r+\varrho(\tilde{y},0)]} \int_{\mathbb{R}^n} (1+\|\tilde{z}\|)\; e^{-a\,\|\tilde{z}\|}\, dv_{\tilde{g}_0}(\tilde{z}) < +\infty.$$

Following the relation (3.1), previous inequality gives sense to the definition:

$$\widetilde{H}(\tilde{y}) = bar(\mu_{\tilde{y}}) = \frac{\int_{\mathbb{R}^n} \tilde{z}\, e^{-a\varrho_r(\tilde{y},\tilde{z})}\; dv_{\tilde{g}_0}(\tilde{z})}{\int_{\mathbb{R}^n} e^{-a\varrho_r(\tilde{y},\tilde{z})}\; dv_{\tilde{g}_0}(\tilde{z})} \tag{3.4}$$

which is valid for every $\tilde{y} \in (\mathbb{R}^n, \tilde{g})$ and thus defines a map: $\widetilde{H} : (\mathbb{R}^n, \tilde{g}) \to (\mathbb{R}^n, \tilde{g}_0)$ called the "barycentre map". The regularity of ϱ_r implies the

Lemma 3.3. *The regularized barycentre map is $C^\infty\left((\mathbb{R}^n, \tilde{g}), (\mathbb{R}^n, \tilde{g}_0)\right)$; moreover we have:*

$$\tilde{g}_0\left(\nabla_{\tilde{y}}\widetilde{H}(\mathbf{u}), \mathbf{v}\right) \int_{\mathbb{R}^n} d\mu_{\tilde{y}}(\tilde{z}) = a \int_{\mathbb{R}^n} \tilde{g}_0\left(\widetilde{H}(\tilde{y}) - \tilde{z}, \mathbf{v}\right) \cdot \tilde{g}\left(\nabla\varrho_r(\tilde{y},\tilde{z}), \mathbf{u}\right) d\mu_{\tilde{y}}(\tilde{z}) \tag{3.5}$$

Proof: The functions $\tilde{y} \mapsto \tilde{z}\ e^{-a\varrho_r(\tilde{y},\tilde{z})}$ and $\tilde{y} \mapsto e^{-a\varrho_r(\tilde{y},\tilde{z})}$ are C^∞ (for *any* fixed value of $\tilde{z}$). Moreover, for every point $\tilde{y}_0 \in (\mathbb{R}^n, \tilde{g})$, the derivatives of these functions are uniformly bounded on the unit ball $B(\tilde{y}_0, 1) \subset (\mathbb{R}^n, \tilde{g}_0)$ by functions which are both independent from $\tilde{y} \in B_{\mathbb{R}^n}(\tilde{y}_0, 1)$ and integrable with respect to $\tilde{z}$. For any unit tangent vector $\mathbf{u} \in T(\mathbb{R}^n, \tilde{g})$ we have, indeed,

$$\left\| -a\,\tilde{z}\,d\varrho_r(\mathbf{u})\ e^{-a\varrho_r(\tilde{y},\tilde{z})}\ \right\| \leq a\|\tilde{z}\|e^{-a\tilde{\varrho}_r(\tilde{y},\tilde{z})} \leq a\|\tilde{z}\|e^{-a[\tilde{\varrho}(\tilde{y},\tilde{z})-r]} \leq$$

$$a\|\tilde{z}\|e^{-a[\tilde{\varrho}(\tilde{y}_0,\tilde{z})-r-1]} \leq a\|\tilde{z}\|e^{-a[\|\tilde{y}_0-\tilde{z}\|-c-r-1]} = a\|\tilde{z}\|e^{a(c+r+1)}e^{-\|\tilde{y}_0-\tilde{z}\|}$$

The use of the Lebesgue-dominate-derivability theorem to the function $\widetilde{H}$ defined by the equality (3.4) proves that $\widetilde{H}$ does admit derivatives in any direction $\mathbf{u}$ and that

$$\tilde{g}_0\left(\nabla_{\tilde{y}}\widetilde{H}(\mathbf{u}), \mathbf{v}\right)\int_{\mathbb{R}^n} d\mu_{\tilde{y}}(\tilde{z}) = \frac{a}{\int_{\mathbb{R}^n} d\mu_{\tilde{y}}(\tilde{z})}\ \cdot \tag{3.6}$$

$$\cdot\left(-\int_{\mathbb{R}^n}\tilde{g}_0\left(\widetilde{H}(\tilde{y}), \mathbf{v}\rangle\right)\cdot\tilde{g}\left(\nabla\varrho_r(\tilde{y},\tilde{z}), \mathbf{u}\right)d\mu_{\tilde{y}}(\tilde{z})\int_{\mathbb{R}^n} d\mu_{\tilde{y}}(\tilde{z})\right.$$

$$\left.+\int_{\mathbb{R}^n}\tilde{g}_0\left(\tilde{z}, \mathbf{v}\right)d\mu_{\tilde{y}}(\tilde{z})\cdot\int_{\mathbb{R}^n}\tilde{g}\left(\nabla\varrho_r(\tilde{y},\tilde{z}), \mathbf{u}\right)d\mu_{\tilde{y}}(\tilde{z})\right) =$$

$$a\int_{\mathbb{R}^n}\tilde{g}_0\left(\widetilde{H}(\tilde{y}) - \tilde{z}, \mathbf{v}\right)\cdot\tilde{g}\left(\nabla\varrho_r(\tilde{y},\tilde{z}), \mathbf{u}\right)d\mu_{\tilde{y}}(\tilde{z})$$

The same kind of argument proves that, for any vector field $\mathbf{U}$ on $B_{\widetilde{M}}(\tilde{y}_0, 1)$ the function $\tilde{y} \mapsto d\widetilde{H}\Big|_{\tilde{y}}(\mathbf{U})$ is continuous (by Lebesgue dominated continuity theorem) and thus that $\widetilde{H}$ is C^1. The proof that the barycentre map is C^∞ follows iterating the previous argument. □

4 Proof of the *Hopf Conjecture*

Let $\mathbf{p}$ be a rational point of the unite sphere and $\mathbf{u_p}$ the corresponding vector field and let $\gamma_{\tilde{y}}$ be the corresponding geodesic starting from $\tilde{y}$ for which $\exp_{\tilde{y}} t\mathbf{u_p}(\tilde{y}) = \tilde{y} + t\cdot\mathbf{u_p}$. The invariance of the barycentre map implies that

$$\widetilde{H}(\tilde{y} + t\cdot\mathbf{u_p}) = \widetilde{H}(\tilde{y}) + t\cdot\mathbf{u_p}$$

and it represents also the geodesic in $(\mathbb{R}^n, \tilde{g}_0)$ starting from the barycenter. We go to evaluate the average value along the geodesics starting form $\tilde{y}$ and $\widetilde{H}(\tilde{y})$ of (3.6) of the derivatives to respect of $\tilde{y}$ of both sides as the limits

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^{+\infty} D\left[\int_{\mathbb{R}^n} F_i(\tilde{y}+\mathbf{u_p}\cdot t,\tilde{H}(\tilde{y})+\mathbf{u_p}\cdot t)dv_{\tilde{g}_0}\right]dt$$

being F_i , $i=1,\,2$ the integrand functions of the equality (3.6). The regularity of the barycentre map and of the integrand functions of both sides of (3.6) allows to jump the limits within the integrals getting the following limits

$$\lim_{T\to+\infty}\frac{F_i(\tilde{y}+\mathbf{u_p}\cdot T,\tilde{H}(\tilde{y})+\mathbf{u_p}\cdot T)-F_i(\tilde{y},\tilde{H}(\tilde{y}))}{T}\;;$$

the direct computation proves that both limits are equal to zero. From the Birkhoff Ergodic theorem it follows that the derivatives of these functions are identically zero along the above defined geodesics, so the integrand functions are constant along them. If $\mathbf{u_p}$ and $\mathbf{v}$ are not orthogonal via the metric $\tilde{g}_0$, we deduce that

$$\tilde{g}(\mathbf{p},\mathbf{u_p})=\tilde{g}_0\left(\nabla_{\tilde{y}}\widetilde{H}(\mathbf{u_p}),\mathbf{v}\right)\cdot\tilde{g}_0\left(\mathbf{u_p},\mathbf{v}\right)^{-1}$$

which is constant along the ray of the rational foliation induced by the direction $\mathbf{p}$. Taking a system of rational unit vectors $\mathbf{p}_i$, $i=1,2,\cdots,n$ as a base of $(\mathbb{R}^n,\tilde{g})$, the density of $\mathbb{Q}^n$ in $\mathbb{R}^n$ gets that the tensor $\tilde{g}$ is constant almost everywhere, and thanks to the equivariance of the map that the torus $(\mathbb{R}^n/\mathbb{Q}^n,g)$ is flat. □

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