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Approximation of functions by a new class of Gamma type operators; theory and applications

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Abstract

The study of the linear methods of approximation, which are given by sequences of positive and linear operators, studied extremely, in relation to different subjects of analysis, such as numerical analysis. The principal objective of this manuscript is to develop a new and more comprehensive version of Gamma type operators and presented their approximation features. For this purpose, we benefit from two sequences of functions, which are $\alpha_n(x)$ and $\beta_n(x)$, and from the function $\tau(x)$. To indicate how the function τ play a significant role in the construction of the operator, we reconstruct the mentioned operators which preserve exactly two test functions from the set $\{1, \tau, \tau^2\}$. Then we established Voronovskaya type theorem and order of approximation properties of the newly defined operators utilizing weighted modulus of continuity to show that their approximation properties. At the end of this note, we present a series of numerical results to show that the new operators are an approximation technique.

1 Introduction

It is known that Gamma and Beta functions, also known as special functions, are frequently encountered in a number of applications areas of mathematics. These functions form the basis of statistical measurement unit known as

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Gamma and Beta distributions in statistical theory and are widely used. In addition, these functions provide guidance in solving a number of boundary-value problems and computing some integrals of special type.

Approximation theory, which is one of the important areas of mathematics, has wide applications in analysis. The approximation theory basically aims to calculate the existence of the approximation and the speed of this approximation with the help of simpler and more easily calculated functions. In 1885, Weierstrass [28] proved the existence of polynomials that converge uniformly to functions which are continuous in a closed interval $[a, b]$. Several proofs of this fundamental theorem have been made so far. In 1912, Bernstein [7] gave this result using a much simpler proof method. In [18], Korovkini's theorem sometimes also called Bohman-Korovkin theorem, arose from the study of the role of Bernstein polynomials in the proof of the Weierstrass approximation theorem. After these studies, many positive linear operators were constructed and their approximation properties were investigated [4], [5], [24], [26], [6], [11].

By using the definitions of Gamma and Beta functions, Gamma and Beta operators have been defined in approximation theory and the convergence properties of these operators have been studied by some authors. During these investigations, it was seen that the conditions of Korovkin's theorem were met, and the convergence rates of the operators were also examined. Further generalizations of these operators were obtained and their convergence properties were examined. These studies are still continuing.

2 Fundamental Facts

In this section we will summarize some basic definitions and facts for use later in the article. As is known, the Gamma function defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt,$$

for $x > 0$. Note that, throughout the paper we consider the exponential function $\exp_t(x)$ defined by $\exp_t(x) = e^{-t/x}$. Using this function, the Gamma operator is seen in the paper written by Zeng [29] as follows:

$$G_n(f; x) = \frac{1}{x^n \Gamma(n)} \int_0^\infty \exp_t(x) t^{n-1} f\left(\frac{t}{n}\right) dt, \quad x > 0, \quad n \in \mathbb{N}, \tag{2.1}$$

which satisfies the conditions $|f(t)| \leq Me^{\beta t}$ where $M > 0$ and $\beta > 0$ for $t \to \infty$. On the other hand, Lupa and Mller [19] introduced the following sequence of linear positive operators,

$$G_n^*(f;x)=\frac{x^{n+1}}{\Gamma(n+1)}\int\limits_0^\infty \exp_t\left(\frac{1}{x}\right)t^n f\left(\frac{n}{t}\right)dt,\quad x>0,\quad n\in\mathbb{N}. \tag{2.2}$$

Recently, a number of new Gamma type operators have been defined and the convergence properties of these operators are examined, [8], [19], [23], [25].

On the other hand, with King's clever idea [17], a new generalization of Bernstein operators was brought to the literature in 2003. In this work, King introduced and discussed the standard approximation properties of the following operators, namely,

$$V_n(f;x):=[(B_nf)\circ r_n](x)=\sum_{k=0}^n\binom{n}{k}(r_n(x))^k(1-r_n(x))^{n-k}f\left(\frac{k}{n}\right),\quad n\in\mathbb{N},$$

for $f\in C[0,1]$ where (B_n) is classical Bernstein operators and r_n is a sequence of continuous functions defined on $[0,1]$ such that $0\le r_n(x)\le 1$. Then, in 2009 Gonska et. al. [15] introduced the following generalized sequence of King type operators as follows

$$V_n^\tau(f;x):=(B_nf)\circ\tau=(B_nf)\circ(B_n\tau)^{-1}\circ\tau,$$

where τ is a function holds $\tau(\{0,1\})=\{0,1\}$ and provide their approximation and preservation properties. In the light of these studies, Cardenas-Morales et. al. [10] generalized all these operators and brought the following operator to the literature in 2011, that is to say,

$$B_n^\tau(f;x):=[(B_n(f\circ\tau^{-1}))\circ\tau](x)=\sum_{k=0}^n\binom{n}{k}(\tau(x))^k(1-\tau(x))^{n-k}[f\circ\tau^{-1}]\left(\frac{k}{n}\right),$$

for $n\in\mathbb{N}$ and $f\in C[0,1]$ where τ is a $\infty-$times continuously differentiable function such that $\tau(\{0,1\})=\{0,1\}$ and $\tau'(x)>0$ on $[0,1]$. In this study, the authors presented their asymptotic behaviour and saturation properties and also their shape-preserving and convergence properties.

With this development, Aral et al. [3] made a similar generalization for the Szász-Mirakyan-type operator and gave the general approximation properties of this operator. Then, in [1] Bernstein-Durrmeyer-type operators by Acar et al., in [20] Bernstein-Chlodowksi-types operators by Usta, in [21] Balázs-types operators by Usta, in [22] Baskakov-type operators by Usta, in [27] Lupaş-type operators by Qasim et al. and in [9] Meyer-König and Zeller-type operators by Cai et.al. have been introduced.

In addition to all these, Erençin and Raşa [12] provided a new generalization of Gamma type operators as follows:

$$G_n^{**}(f;x)=\frac{1}{(\tau(x))^n\Gamma(n)}\int\limits_0^\infty \exp_t(\tau(x))t^{n-1}\left(f\circ\tau^{-1}\right)\left(\frac{t}{n}\right)dt,\quad x>0,\quad n\in\mathbb{N}, \tag{2.3}$$

where, as usual, $\Gamma(\cdot)$ denotes the Gamma function, f is a function such that the integral is convergent and ρ is a function holding the following assumptions:

(A) τ is a continuously differentiable function on $\mathbb{R}^+ := [0,\infty)$,

(B) $\tau(0)=0, \inf_{x\in\mathbb{R}^+}\tau'(x)\geq 1$.

In the aforementioned study, the authors presented and proved quantitative Grss type Voronovskaya and quantitative Voronovskaya theorems via weighted modulus of continuity. It is clearly seen in this operator that the function ρ is used in three different places, which reduces the comprehensiveness of the operator. For this reason, it has been tried to construct a new type Gamma operator, aiming to use different functions say two sequences of functions, which are $\alpha_n(x)$ and $\beta_n(x)$, and function $\tau(x)$ for these three places in order to obtain a more inclusive operator.

On the other hand, we review the following fundamental facts in approximation theory.

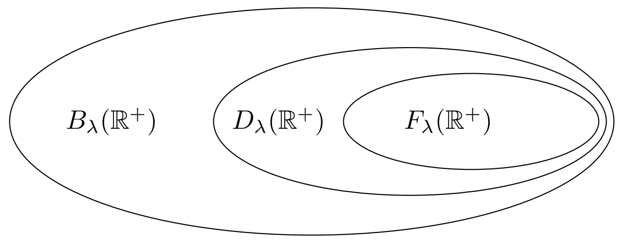
Let τ be a map that satisfies the requirements **(A)** and **(B)**. These requirements guarantee that τ is a continuous and strictly growing function on a positive real axis, and that $\lim_{x\to\infty}\tau(x)=\infty$. In addition to these, let $\lambda(x)$ be an weight functions such that $\lambda(x)=1+\tau^2(x)$. In this circumstance, the weighted space $B_\lambda(\mathbb{R}^+)$, which is a normed space with the norm

$$\|f\|=\sup_{x\in\mathbb{R}^+}\frac{|f(x)|}{\lambda(x)},$$

can be defined as follows:

$$B_\lambda(\mathbb{R}^+)=\{f:\mathbb{R}^+\to\mathbb{R}\quad \text{s.t.}\quad |f(x)|\leq C_f\lambda(x)\quad\text{and}\quad x\in\mathbb{R}^+\},$$

where C_f is a positive constant that is solely depend on f. In addition to this, the subspace of whole continuous maps in $B_\lambda(\mathbb{R}^+)$ denoted by $D_\lambda(\mathbb{R}^+)$. On the other hand, let $F_\lambda(\mathbb{R}^+)$ be the space of all maps f in $D_\lambda(\mathbb{R}^+)$ such that $\frac{f(x)}{\lambda(x)}$ is uniformly continuous. It is quite apparent that the following visual relation is valid:



It must be noted that, one can argue that $|t-x| \leq |\tau(t)-\tau(x)|$ for each $t,x \in \mathbb{R}^+$ utilizing the result of **(B)**.

Definition 1. *[16] For every $f \in D_\lambda(\mathbb{R}^+)$ and $\delta \geq 0$, Holhoş introduced the following weighted modulus of continuity*

$$\omega_\tau(f;\delta) = \sup_{\substack{|\tau(t)-\tau(x)|\leq\delta \\ t,x\in\mathbb{R}^+}} \frac{|f(t)-f(x)|}{\lambda(t)+\lambda(x)},$$

and presented the following relations.

(i) For each $f \in D_\lambda(\mathbb{R}^+)$, $t,x \geq 0$ and $\delta > 0$,

$$|f(t)-f(x)| \leq (\lambda(t)+\lambda(x))\left(2+\frac{|\tau(t)+\tau(x)|}{\delta}\right)\omega_\tau(f;\delta).$$

(ii) For each $f \in D_\lambda(\mathbb{R}^+)$ and $\gamma,\delta \geq 0$,

$$\omega_\tau(f;\gamma\delta) = (2+\gamma)\omega_\tau(f;\delta).$$

(iii) For each $f \in F_\lambda(\mathbb{R}^+)$

$$\lim_{\delta\to 0}\omega_\tau(f;\delta) = 0.$$

It is obvious that $\omega_\tau(f;0) = 0$ for all $f \in D_\lambda(\mathbb{R}^+)$ and $\omega_\tau(f;\delta)$ is an increasing and non-negative function with respect to δ.

The following is a breakdown of the manuscript's structure. Section 3 covers the construction of the proposed Gamma operators and their fundamental approximation properties such as moments and central moments. Voronovskaya type theorem is established in Section 4 while order of approximation is presented in Section 5. Finally, we present a series of computational experiments in Section 6 and give a conclusion in Section 7.

3 Construction of the New Gamma Operators & Preliminary Results

Considering all the work done so far, we can introduce the following new gamma type operators and provide their fundamental approximation properties as follows. Throughout the paper, we benefit from the well-known software program that name is Maple to make a calculation.

$$G^{\tau,n,y}_{\alpha_n,\beta_n}(f;x;u,v) = \frac{1}{(\alpha_n(x))^{n+u}\Gamma(n+v)} \int\limits_0^\infty \exp_t(\beta_n(x)) t^{n-y} \left(f \circ \tau^{-1}\right)\left(\frac{t}{n}\right) dt, \tag{3.1}$$

for $x > 0$ and $n \in \mathbb{N}$ where the function τ satisfies the requirements **(A)** and **(B)**. For this newly defined Gamma operator given in (3.1) to be an approximation procedure, we must show that the following two assumptions are satisfied. First of all, we impose the condition

$$G^{\tau,n,y}_{\alpha_n,\beta_n}(1;x;u,v) = 1 + \xi_n(x),$$

where $\xi_n : \mathbb{R}^+ \to \mathbb{R}$ is a function. By using the definition of newly define Gamma operators given in (3.1), we obtain that,

$$\begin{aligned} G^{\tau,n,y}_{\alpha_n,\beta_n}(1;x;u,v) &= \frac{1}{(\alpha_n(x))^{n+u}\Gamma(n+v)} \int\limits_0^\infty \exp_t(\beta_n(x)) t^{n-y} dt, \\ &= \frac{(\beta_n(x))^{n-y+1}\Gamma(n-y+1)}{(\alpha_n(x))^{n+u}\Gamma(n+v)}, \end{aligned}$$

which yields immediately

$$\frac{(\beta_n(x))^{n-y+1}\Gamma(n-y+1)}{(\alpha_n(x))^{n+u}\Gamma(n+v)} = 1 + \xi_n(x). \tag{3.2}$$

On the other hand, we can impose the second condition such that $G^{\tau,n,y}_{\alpha_n,\beta_n}$ maps τ to the same function, to be more precise,

$$G^{\tau,n,y}_{\alpha_n,\beta_n}(\tau;x;u,v) = \tau(x) + \eta_n(x),$$

where $\eta_n : \mathbb{R}^+ \to \mathbb{R}$ is a function. Similarly,

$$\begin{aligned} G^{\tau,n,y}_{\alpha_n,\beta_n}(\tau;x;u,v) &= \frac{1}{(\alpha_n(x))^{n+u}\Gamma(n+v)} \int\limits_0^\infty \exp_t(\beta_n(x)) t^{n-y} \left(\tau \circ \tau^{-1}\right)\left(\frac{t}{n}\right) dt, \\ &= \frac{(\beta_n(x))^{n-y+2}\Gamma(n-y+2)}{n(\alpha_n(x))^{n+u}\Gamma(n+v)}. \end{aligned}$$

according to above consequence, we deduce that

$$\frac{(\beta_n(x))^{n-y+2}\Gamma(n-y+2)}{n(\alpha_n(x))^{n+u}\Gamma(n+v)} = \tau(x) + \eta_n(x). \tag{3.3}$$

From the relation (3.2) and (3.3), one can smoothly deduce that

$$\alpha_n(x) = \left[\left(\frac{n}{n-y+1}\right)^{n-y+1}\left(\frac{\Gamma(n-y+1)}{\Gamma(n+v)}\right)\left(\frac{(\tau(x)+\eta_n(x))^{n-y+1}}{(1+\xi_n(x))^{n-y+2}}\right)\right]^{\frac{1}{n+u}}, \tag{3.4}$$

and

$$\beta_n(x) = \left(\frac{n}{n-y+1}\right)\left(\frac{\tau(x)+\eta_n(x)}{1+\xi_n(x)}\right), \tag{3.5}$$

for every $x \in \mathbb{R}^+$. As a result of these relations, the newly defined Gamma operators (3.1) become

$$\begin{aligned} G^{\tau,n,y}_{\xi_n,\eta_n}(f;x) &= \left(\frac{n-y+1}{n}\right)^{n-y+1}\frac{(1+\xi_n(x))^{n-y+2}}{\Gamma(n-y+1)(\tau(x)+\eta_n(x))^{n-y+1}} \\ &\times \int_0^\infty \exp_t\left[\left(\frac{n}{n-y+1}\right)\left(\frac{\tau(x)+\eta_n(x)}{1+\xi_n(x)}\right)\right] t^{n-y}\left(f\circ\tau^{-1}\right)\left(\frac{t}{n}\right)dt, \end{aligned} \tag{3.6}$$

where $x > 0$ and $n \in \mathbb{N}$. In order to get a weighted approximation process from this sequence, we assume that the following expressions

$$|\xi_n(x)| \leq \xi_n(x), \quad |\eta_n(x)| \leq \eta_n(x), \tag{3.7}$$

such that

$$\lim_{n\to 0}\xi_n(x) = 0, \quad \lim_{n\to 0}\eta_n(x) = 0, \tag{3.8}$$

are fulfilled for $x \in \mathbb{R}^+$. At this stage, we emphasized that, based on weighted Korovkin theorem, the above relations α_n and β_n and values of these limits guarantee that the newly defined operator is an approximation procedure on $\mathbb{R}^+$, [2], [18]. Additionally, the appropriate choices of $\xi_n(x)$ and $\eta_n(x)$ under these given conditions give us different types of Gamma operators.

Remark 1. *We may construct some sequences of positive linear operators studied in literature from the introduced Gamma operators* (3.6)*, as shown below, by making a proper selection of* $\tau(x)$*,* $\xi_n(x)$ *and* $\eta_n(x)$*:*

1. *Choosing* $\xi_n(x) = 0$*,* $\eta_n(x) = 0$*,* $u = 0$*,* $v = 0$ *and* $y = 1$ *in* (3.6)*, then we have the operators given in* (2.3)*.*

2. *Choosing* $\tau(x) = x$, $\xi_n(x) = 0$, $\eta_n(x) = 0$, $u = 0$, $v = 0$ *and* $y = 1$ *in* (3.6), *then we deduce the operators given in* (2.1).

3. *Choosing* $\tau(x) = \frac{1}{x}$, $\xi_n(x) = 0$, $\eta_n(x) = \frac{1}{nx}$, $u = 1$, $v = 1$ *and* $y = 0$, *then we deduce the operators given in* (2.2).

Now we can present the moment and central moments values of the newly introduced Gamma type operators as follows. As the identities are conveniently deduced by direct computation, we skip the proofs.

Lemma 1. *The newly defined operators defined by* (3.6) *affirm the following identities:*

(i) $G_{\xi_n,\eta_n}^{\tau,n,y}(1;x) = 1 + \xi_n(x)$,

(ii) $G_{\xi_n,\eta_n}^{\tau,n,y}(\tau;x) = \tau(x) + \eta_n(x)$,

(iii) $G_{\xi_n,\eta_n}^{\tau,n,y}(\tau^2;x) = \left(\frac{n-y+2}{n-y+1}\right)\left(\frac{(\tau(x)+\eta_n(x))^2}{1+\xi_n(x)}\right)$,

(iv) $G_{\xi_n,\eta_n}^{\tau,n,y}(\tau^3;x) = \left(\frac{(n-y+3)(n-y+2)}{(n-y+1)^2}\right)\left(\frac{(\tau(x)+\eta_n(x))^3}{(1+\xi_n(x))^2}\right)$,

(v) $G_{\xi_n,\eta_n}^{\tau,n,y}(\tau^4;x) = \left(\frac{(n-y+4)(n-y+3)(n-y+2)}{(n-y+1)^3}\right)\left(\frac{(\tau(x)+\eta_n(x))^4}{(1+\xi_n(x))^3}\right)$.

Lemma 2. *For* $k \in \mathbb{N}$, *we can generalize the moment values of the newly defined Gamma operators as follows:*

$$G_{\xi_n,\eta_n}^{\tau,n}(\tau^k;x,y) = \left(\frac{(n-y+2)^{(k-1)}}{(n-y+1)^{k-1}}\right)\left(\frac{(\tau(x)+\eta_n(x))^k}{(1+\xi_n(x))^{k-1}}\right),$$

where $(p)^{(q)}$ *is rising factorial defined by*

$$(p)^{(q)} = p(p+1)(p+2)\cdots(p+q-1).$$

Lemma 3. *The newly defined operators defined by* (3.6) *affirm the following central moments identities:*

(i) $G_{\xi_n,\eta_n}^{\tau,n,y}(\tau(t)-\tau(x);x) = \eta_n(x) - \tau(x)\xi_n(x)$,

(ii) $G_{\xi_n,\eta_n}^{\tau,n,y}((\tau(t)-\tau(x))^2;x) = \left(\frac{n-y+2}{n-y+1}\right)\left(\frac{(\tau(x)+\eta_n(x))^2}{1+\xi_n(x)}\right)$
$+ \tau^2(x)(\xi_n(x)-1) - 2\tau(x)\eta_n(x)$,

(iii) $G_{\xi_n,\eta_n}^{\tau,n,y}((\tau(t)-\tau(x))^4;x) = \left(\frac{(n-y+2)^{(3)}}{(n-y+1)^3}\right)\left(\frac{(\tau(x)+\eta_n(x))^4}{(1+\xi_n(x))^3}\right)$

$$-4\left(\frac{(n-y+2)^{(2)}}{(n-y+1)^2}\right)\left(\frac{(\tau(x)+\eta_n(x))^3}{(1+\xi_n(x))^2}\right)\tau(x)+6\left(\frac{(n-y+2)}{(n-y+1)}\right)\left(\frac{(\tau(x)+\eta_n(x))^2}{(1+\xi_n(x))}\right)\tau^2(x)$$

$$-4(\tau(x)+\eta_n(x))\tau^3(x)+(1+\xi_n(x))\tau^4(x),$$

(iv) $G_{\xi_n,\eta_n}^{\tau,n,y}((\tau(t)-\tau(x))^6;x) = \left(\frac{(n-y+2)^{(5)}}{(n-y+1)^5}\right)\left(\frac{(\tau(x)+\eta_n(x))^6}{(1+\xi_n(x))^5}\right)$

$$-6\left(\frac{(n-y+2)^{(4)}}{(n-y+1)^4}\right)\left(\frac{(\tau(x)+\eta_n(x))^5}{(1+\xi_n(x))^4}\right)\tau(x)$$

$$+15\left(\frac{(n-y+2)^{(3)}}{(n-y+1)^3}\right)\left(\frac{(\tau(x)+\eta_n(x))^4}{(1+\xi_n(x))^3}\right)\tau^2(x)$$

$$-20\left(\frac{(n-y+2)^{(2)}}{(n-y+1)^2}\right)\left(\frac{(\tau(x)+\eta_n(x))^3}{(1+\xi_n(x))^2}\right)\tau^3(x)$$

$$+15\left(\frac{(n-y+2)}{(n-y+1)}\right)\left(\frac{(\tau(x)+\eta_n(x))^2}{(1+\xi_n(x))}\right)\tau^4(x)$$

$$-6(\tau(x)+\eta_n(x))\tau^5(x)+(1+\xi_n(x))\tau^6(x)\ .$$

Remark 2. *Since any linear positive operators K_n act from $B_\lambda(\mathbb{R}^+)$ to $D_\lambda(\mathbb{R}^+)$ iff $|K_n(\lambda,x)| \le M_n\lambda(x)$ for $n \in \mathbb{N}$ such that M_n is a positive constant [13], [14], one can argue that the newly defined Gamma operators $G_{\xi_n,\eta_n}^{\tau,n,y}$ act from $B_\lambda(\mathbb{R}^+)$ to $D_\lambda(\mathbb{R}^+)$ utilizing Lemma 2.*

4 Voronovskaya Type Theorem

Now we will focus on pointwise convergences of the operators introduced by (3.6). For this purpose, we show Voronovskaya type theorems in this section. However, first of all, we review the following lemma introduced by Holhoş.

Lemma 4. *[16] For $\delta > 0$, for every $f \in B_\lambda(\mathbb{R}^+)$ and for all $s, x \ge 0$,*

$$|f(s)-f(x)| \le (\lambda(s)-\lambda(x))\left(2+\frac{|\tau(s)-\tau(x)|}{\delta}\right)\omega_\tau(f,\delta),$$

holds.

Theorem 1. *Assume that* $(f \circ \tau^{-1})'$ *and* $(f \circ \tau^{-1})''$ *exists at* $\tau(x)$ *and let* $f \in B_\lambda(\mathbb{R}^+)$. *If*

$$\lim_{n\to\infty} n\xi_n(x) = z_1 \qquad \textit{and} \qquad \lim_{n\to\infty} n\eta_n(x) = z_2,$$

and $(f \circ \tau^{-1})''$ *is bounded on* $\mathbb{R}^+$, *then we get*

$$\begin{aligned}\lim_{n\to\infty} n[G^{\tau,n,y}_{\xi_n,\eta_n}(f;x) - f(x)] &= z_1 f(x) + (z_2 - \tau(x) z_1)(f \circ \tau^{-1})'(\tau(x)) \\ &\quad + \frac{1}{2}\tau^2(x)(f \circ \tau^{-1})''(\tau(x)).\end{aligned}$$

Proof. Utilizing Taylor expansion of $f \circ \tau^{-1}$ at the point $\tau(x)$, we can deduce the following expression,

$$\begin{aligned}f(t) = (f \circ \tau^{-1})(\tau(t)) &= (f \circ \tau^{-1})(\tau(x)) + (f \circ \tau^{-1})'(\tau(x))(\tau(t) - \tau(x)) \\ &\quad + \frac{1}{2}(f \circ \tau^{-1})''(\tau(x))(\tau(t) - \tau(x))^2 + R_2(f, x, t),\end{aligned}$$

where $R_2(f, x, t) = h(x, t)(\tau(t) - \tau(x))^2$ and $h(x, t)$ is a function such that

$$\lim_{t\to x} h(x, t) = h(x, x) = 0.$$

Then applying the newly defined operator $G^{\tau,n,y}_{\xi_n,\eta_n}$ and with the help of Lemma 3-*i* and Lemma 3-*ii*, we deduce that

$$\begin{aligned}G^{\tau,n,y}_{\xi_n,\eta_n}(f;x) - f(x) &= f(x)\xi_n(x) + (f \circ \tau^{-1})'(\tau(x)) G^{\tau,n,y}_{\xi_n,\eta_n}(\tau(t) - \tau(x); x) \\ &\quad + \frac{1}{2}(f \circ \tau^{-1})''(\tau(x)) G^{\tau,n,y}_{\xi_n,\eta_n}((\tau(t) - \tau(x))^2; x) \\ &\quad + G^{\tau,n,y}_{\xi_n,\eta_n}(R_2(f, x, t); x),\end{aligned}$$

which yields

$$\begin{aligned}n[G^{\tau,n,y}_{\xi_n,\eta_n}(f;x) - f(x)] &= n f(x)\xi_n(x) + (f \circ \tau^{-1})'(\tau(x)) n G^{\tau,n,y}_{\xi_n,\eta_n}(\tau(t) - \tau(x); x) \\ &\quad + \frac{1}{2}(f \circ \tau^{-1})''(\tau(x)) n G^{\tau,n,y}_{\xi_n,\eta_n}((\tau(t) - \tau(x))^2; x) \\ &\quad + n G^{\tau,n,y}_{\xi_n,\eta_n}(R_2(f, x, t); x).\end{aligned}$$

On the other hand, by using Lemma 1-*i*, Lemma 3-*i* and Lemma 3-*ii*, we deduce the followings

$$\begin{aligned}\lim_{n\to\infty} n G^{\tau,n,y}_{\xi_n,\eta_n}(\tau(t) - \tau(x); x) &= z_2 - \tau(x) z_1, \\ \lim_{n\to\infty} n G^{\tau,n,y}_{\xi_n,\eta_n}((\tau(t) - \tau(x))^2; x) &= \tau^2(x).\end{aligned}$$

So we deduce that

$$\lim_{n\to\infty} n[G^{\tau,n,y}_{\xi_n,\eta_n}(f,x) - f(x)] = z_1 f(x) + (z_2 - \tau(x) z_1)(f\circ\tau^{-1})'(\tau(x)) \\ + \frac{1}{2}\tau^2(x)(f\circ\tau^{-1})''(\tau(x)) + \lim_{n\to\infty} nG^{\tau,n,y}_{\xi_n,\eta_n}(R_2(f,x,t);x).$$

As a last step of the proof, we need to show that

$$\lim_{n\to\infty} nG^{\tau,n,y}_{\xi_n,\eta_n}(R_2(f,x,t);x) = \lim_{n\to\infty} nG^{\tau,n,y}_{\xi_n,\eta_n}(h(x,t)(\tau(t)-\tau(x))^2;x) = 0.$$

Let $\varepsilon > 0$. From the continuity feature of the function h, one can find a small enough $\delta > 0$ such that for every $t \in \mathbb{R}^+$ with the property $|t - x| < \delta$, we get

$$|h(x,t)| < \varepsilon.$$

Consider $t \in \mathbb{R}^+$ such that $|t - x| \geq \delta$. Because τ is monotonic, we deduce that

$$\inf_{|t-x|\geq\delta} |\tau(t) - \tau(x)| \geq \min(|\tau(x+\delta) - \tau(x)|, |\tau(x) - \tau(x-\delta)|) > 0.$$

On the other hand, since f and τ^2 belong to $B_\lambda(\mathbb{R}^+)$, we deduce that $R_2(f,x,\cdot) \in B_\lambda(\mathbb{R}^+)$. Then

$$|h(x,t)| = \frac{|R_2(f,x,t)|}{(\tau(t)-\tau(x))^2} \leq \frac{\|R_2(f,x,\cdot)\|_\lambda \lambda(t)}{\inf\limits_{|t-x|\geq\delta} |\tau(t)-\tau(x)|^2} \leq H_x \lambda(t) |\tau(t) - \tau(x)|,$$

where

$$H_x = \frac{\|R_2(f,x,\cdot)\|_\lambda}{\min(|\tau(x+\delta)-\tau(x)|^3, |\tau(x)-\tau(x-\delta)|^3)}.$$

We have just proved that

$$|h(x,t)| < \varepsilon + H_x \lambda(t) |\tau(t) - \tau(x)|,$$

for every $t \in \mathbb{R}^+$. We deduce that

$$\begin{aligned} n|G^{\tau,n,y}_{\xi_n,\eta_n}(R_2(f,x,t);x)| &\leq n|G^{\tau,n,y}_{\xi_n,\eta_n}(|h(x,t)|(\tau(t)-\tau(x))^2;x)|, \\ &< \varepsilon n G^{\tau,n,y}_{\xi_n,\eta_n}((\tau(t)-\tau(x))^2;x) \\ &+ H_x n G^{\tau,n,y}_{\xi_n,\eta_n}(\lambda(t)|\tau(t)-\tau(x)|^3;x), \\ &\leq \varepsilon n G^{\tau,n,y}_{\xi_n,\eta_n}((\tau(t)-\tau(x))^2;x) \\ &+ H_x \left(G^{\tau,n,y}_{\xi_n,\eta_n}(\lambda^4;x)\right)^{1/4} n \left(G^{\tau,n,y}_{\xi_n,\eta_n}((\tau(t)-\tau(x))^4;x)\right)^{3/4}. \end{aligned}$$

Applying Lemma 3, one can easily deduce that

$$\lim_{n\to\infty} n\left(G^{\tau,n,y}_{\xi_n,\eta_n}((\tau(t)-\tau(x))^4;x)\right)^{3/4}=0.$$

Since $G^{\tau,n,y}_{\xi_n,\eta_n}(\lambda^4;x)$ exists for all $n\geq n_0$ for some $n_0\in\mathbb{N}$ and is bounded with respect to n, one deduce that

$$\left(G^{\tau,n,y}_{\xi_n,\eta_n}(\lambda^4;x)\right)^{1/4}\leq G(x),$$

where $G(x)>0$ does not depend on n. As a result, all these above prove that

$$\lim_{n\to\infty} nG^{\tau,n,y}_{\xi_n,\eta_n}(R_2(f,x,t);x)=0$$

which concludes the proof. □

5 Order of Approximation

In this section, we present a quantitative type theorem for the newly defined Gamma operators. in line with this objective, we use the weighted modulus of continuity given in Definition 1. In addition to this, we need to recall the following theorem presented by Holhoş [16] for the main result of this section.

Theorem 2. *Let $P_n: D_\lambda(\mathbb{R}^+)\to B_\lambda(\mathbb{R}^+)$ be a sequence of linear and positive operators with*

$$\|P_n(\tau^0)-\tau^0\|_{\lambda^0}=\theta_1,\ \ \|P_n(\tau)-\tau\|_{\lambda^{1/2}}=\theta_2,\ \ \|P_n(\tau^2)-\tau^2\|_{\lambda}=\theta_3,\ \ \|P_n(\tau^3)-\tau^3\|_{\lambda^{3/2}}=\theta_4$$

where θ_1, θ_2, θ_3 and θ_4 tends to zero in the case of $n\to 0$. In this circumstance, one has

$$\|P_n(f)-f\|_{\lambda^{3/2}}\leq(7+4\theta_1+2\theta_3)\omega_\tau(f;\delta_n)+\|f\|_\lambda\theta_1,$$

for all $f\in D_\lambda(\mathbb{R}^+)$, where

$$\delta_n=2\sqrt{(1+\theta_1)(\theta_1+2\theta_2+\theta_3)}+\theta_1+3\theta_2+3\theta_3+\theta_4.$$

Theorem 3. *For all $f\in D_\lambda(\mathbb{R}^+)$, we get*

$$\|G^{\tau,n,y}_{\xi_n,\eta_n}(f)-f\|_{\lambda^{3/2}}\leq\left(7+4\xi_n+\left(\frac{n-y+2}{n-y+1}\right)(4\eta_n+2\eta_n^2)\right)\omega_\tau(f;\delta_n)+\|f\|_\lambda\xi_n,$$

where

$$\delta_n=\xi_n+3\eta_n+\left(\frac{n-y+2}{n-y+1}\right)(12\eta_n+6\eta_n^2)+\left(\frac{(n-y+3)(n-y+2)}{(n-y+1)^2}\right)(3\eta_n+3\eta_n^2+\eta_n^3)$$
$$+2\sqrt{(1+\xi_n)\left(\xi_n+2\eta_n+\left(\frac{n-y+2}{n-y+1}\right)(4\eta_n+2\eta_n^2)\right)}.$$

Proof. Throughout the proof we take $G^{\tau,n,y}_{\xi_n,\eta_n}(f;x) = G^{\tau,n,y}_{\xi_n,\eta_n}(f)$. Then thanks to the Theorem 2, Lemma 1-*i* and Lemma 1-*ii*, we deduce that

$$\|G^{\tau,n,y}_{\xi_n,\eta_n}(\tau^0) - \tau^0\|_{\lambda^0} = \sup_{x\in\mathbb{R}^+} \xi_n(x) \leq \xi_n = \theta_1$$

and

$$\|G^{\tau,n,y}_{\xi_n,\eta_n}(\tau) - \tau\|_{\lambda^{1/2}} = \sup_{x\in\mathbb{R}^+} \frac{\eta_n(x)}{\sqrt{1+\lambda^2(x)}} \leq \eta_n = \theta_2,$$

Then using the Lemma 1-*iii*, we deduce

$$\|G^{\tau,n,y}_{\xi_n,\eta_n}(\tau^2) - \tau^2\|_{\lambda} \leq \left(\frac{n-y+2}{n-y+1}\right)(2\eta_n + \eta_n^2) = \theta_3.$$

On the other hand, from Lemma 1-*iv*, we get

$$\|G^{\tau,n,y}_{\xi_n,\eta_n}(\tau^3) - \tau^3\|_{\lambda^{3/2}} \leq \left(\frac{(n-y+3)(n-y+2)}{(n-y+1)^2}\right)(3\eta_n + 3\eta_n^2 + \eta_n^3) = \theta_4.$$

As a result, by using the above findings and the result of Theorem 2, we get the desired results which completes the proof.

Remark 3. *According to result of Theorem 3 and using* $\lim_{\delta\to 0}\omega_\tau(f;\delta) = 0$, *we deduce that*

$$\lim_{n\to\infty} \|G^{\tau,n,y}_{\xi_n,\eta_n}(f) - f\|_{\lambda^{3/2}} = 0,$$

for $f \in F_\lambda(\mathbb{R}^+)$.

□

6 Numerical Tests

In this part, we provide a series of computational experiments for the newly defined Gamma operators in order to support theoretical results and to illustrate the efficiency of them. At the same time, we compared the classical Gamma type operators [19], which are widely used in the literature, with the new operator that we defined for the sake of consistent comparison. In all examples, the package of MATLAB 2020a has been performed to compute the test functions considered in this study.

Example 1. *First of all, we consider the following test function*

$$f(x) = x^4\ln(1+x^4),$$

on $[2,4]$. *In this example, we take*

$$\tau(x) = e^x - 1, \quad \xi_n(x) = \frac{x}{n^2}, \quad \eta_n(x) = \frac{x}{n^3}, \quad \textit{and} \quad y = 2,$$

for $n = 15$ *with equally spaced grid.*

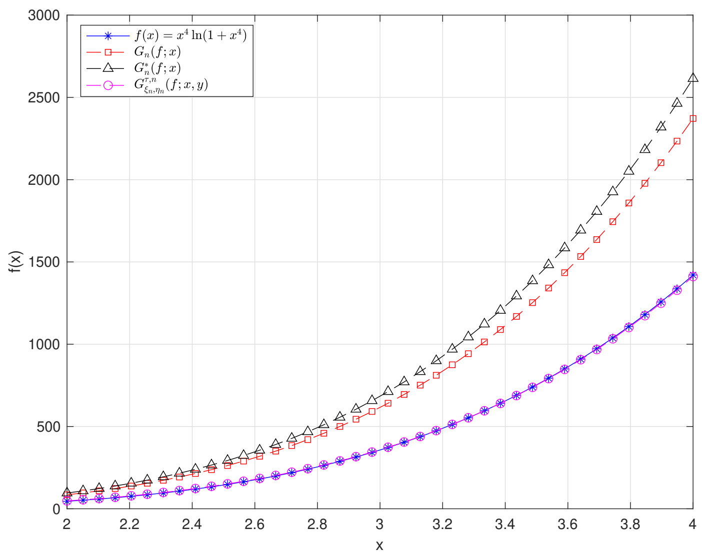


Figure 1: Test function (Blue-Star), Gamma operator given in (2.1) (Red - Squared), Gamma operator given in (2.2) (Black - Triangle) and new construction of Gamma operators (Magenta-Circle) with the given parameters on uniform grid.

Figure 1 indicates that the new construction of the Gamma operators performs better approximation behaviour for the test function in comparison to their classical correspondence. Naturally, the choosing of the above functions play a crucial role to obtain good approximation performance.

Example 2. *Now, let discuss another test function given by*

$$f(x) = e^{-x}(16x^3 - 24x + 5),$$

on $[2,5]$. *In this experiments, we choose*

$$\tau(x) = e^x - 1, \quad \xi_n(x) = \frac{x}{n^3}, \quad \eta_n(x) = \frac{x}{n^5}, \quad \textit{and} \quad y = 1,$$

for $n = 15$ with equally spaced grid.

Similar result can be observed in Figure 2 which demonstrates the good approximation performance of newly defined Gamma operators. All these three experiments confirm that new construction of the Gamma operators present note-worthy performance with the suitable selection of the above functions.

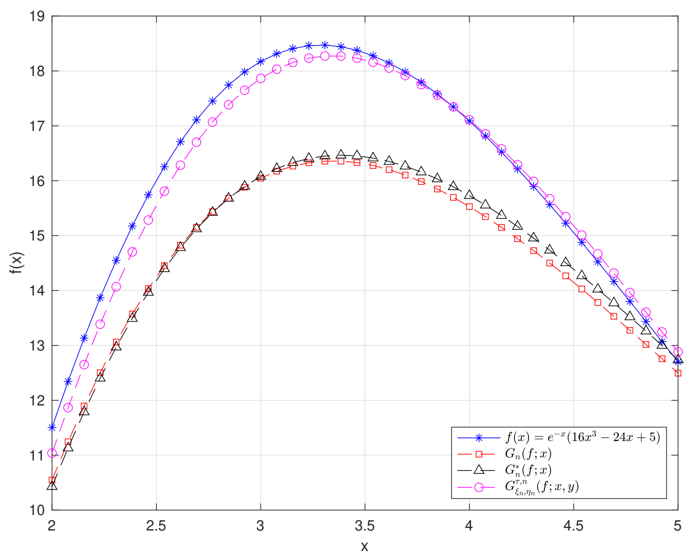


Figure 2: Test function (Blue-Star), Gamma operator given in (2.1) (Red - Squared), Gamma operator given in (2.2) (Black - Triangle) and new construction of Gamma operators (Magenta-Circle) with the given parameters on uniform grid.

7 Concluding Remarks

The present paper deal with the construction of more comprehensive class of Gamma-type operator and their approximation properties. This newly defined Gamma operator also includes classical Gamma operators existing in the literature with appropriate choices of parameters. Both the theoretical and numerical results of this operator are examined and it is observed that it is an approximation procedure.

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