DOI: 10.2478/auom-2024-0012
An. Şt. Univ. Ovidius Constanţa

sciendo
Vol. 32(1),2024, 221–245

Weighted MP weak group inverse

Dijana Mosić and Janko Marovt

Abstract

To extent the notion of the MP weak group inverse for square matrices, we introduce the concept of the weighted MP weak group inverse for rectangular matrices. A number of different representations and characterizations are derived for the weighted MP weak group inverse as well as limit and integral expressions. Applying the weighted MP weak group inverse, we solve some linear equations and give their solutions.

1 Introduction

According to known labels, $\operatorname{rank}(A)$, A^*, $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively, denotes the rank, conjugate-transpose, range (column space) and null space of $A \in \mathbb{C}^{m\times n}$, where $\mathbb{C}^{m\times n}$ represents the set of $m \times n$ complex matrices. The index of $A \in \mathbb{C}^{n\times n}$ will be denoted by $\operatorname{ind}(A)$, the orthogonal projector onto a subspace T by P_T, and a projector onto subspace T along subspace U by $P_{T,U}$.

Outer inverses present a broad class which involves a number of significant generalized inverses. For $A \in \mathbb{C}^{m\times n}$ of rank t and the subspaces U and V of $\mathbb{C}^n$ and $\mathbb{C}^m$ of dimensions $s \le t$ and $m - s$, respectively, $X \in \mathbb{C}^{n\times m}$ satisfying equalities

$$XAX = X, \quad \mathcal{R}(X) = U, \quad \mathcal{N}(X) = V,$$

is the unique $\{2\}$-inverse (or outer inverse) of A with the range U and null space V (denoted by $A^{(2)}_{U,V}$) [2]. Especially, for $U = \mathcal{R}(A^*)$ and $V = \mathcal{N}(A^*)$,

Key Words: weighted weak group inverse, Moore-Penrose inverse, weighted core-EP inverse, outer inverse.
2010 Mathematics Subject Classification: Primary 15A09; Secondary 15A24, 65F05.
Received: 13.11.2022
Accepted: 20.04.2023

$X = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}$ reduces to the Moore-Penrose inverse of A (marked with $A^\dagger$), for which

$$AXA = A, \qquad XAX = X, \qquad (XA)^* = XA, \qquad (AX)^* = AX.$$

When $A \in \mathbb{C}^{n\times n}$ and $k = \mathrm{ind}(A)$, $X = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)} \in \mathbb{C}^{n\times n}$ is the Drazin inverse of A (marked with A^{D}) and satisfies

$$XAX = X, \quad AX = XA, \quad A^{k+1}X = A^k.$$

For $k = 1$, $A^{\#} = A^{\mathrm{D}}$ is the group inverse of A.

As an extension of the Drazin inverse for square matrices, the weighted Drazin inverse was presented for rectangular matrices. The fact that $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}\backslash\{0\}$ and $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$ will be marked by $\{A, W\} \in \mathbb{C}^{m,n,k}$. For $\{A, W\} \in \mathbb{C}^{m,n,k}$, the W-weighted Drazin inverse of A is presented by

$$A^{\mathrm{D},W} = (WAW)^{(2)}_{\mathcal{R}(A(WA)^k),\mathcal{N}(A(WA)^k)} = A[(WA)^{\mathrm{D}}]^2.$$

Specifically, when $m = n$ and $W = I$, $A^{\mathrm{D},W}$ becomes the Drazin inverse A^{D}.

The core–EP inverse was presented in [29] for a square matrix, and extended to the weighted core–EP inverse for a rectangular matrix in [7]. For $\{A, W\} \in \mathbb{C}^{m,n,k}$, the W-weighted core–EP inverse of A is the unique matrix $X \in \mathbb{C}^{m\times n}$ (denoted by $A^{\textcircled{\dagger},W}$) such that

$$WAWX = P_{R((WA)^k)} \quad \text{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}((AW)^k).$$

Let us recall, by [23], that

$$A^{\textcircled{\dagger},W} = A[(WA)^{\textcircled{\dagger}}]^2.$$

In particular, if $m = n$ and $W = I$, $A^{\textcircled{\dagger}} = A^{\textcircled{\dagger},W}$ is the core–EP inverse of A. Some interesting properties of the (W-weighted) core–EP inverse were developed in [3, 5, 10, 11, 13, 14, 15, 16, 19, 21, 22, 24, 31, 35, 43].

The weak group (or WG) inverse for square matrices was given in [36] to extend the notion of the group inverse. The concept of WG inverse was generalized to rectangular matrices in [8]. For $\{A, W\} \in \mathbb{C}^{m,n,k}$, the W-weighted weak group inverse of A is expressed by

$$A^{\textcircled{w},W} = (A^{\textcircled{\dagger},W}W)^2A.$$

Especially, when $m = n$ and $W = I$, $A^{\textcircled{w},W}$ reduces to the WG inverse $A^{\textcircled{w}} = (A^{\textcircled{\dagger}})^2A$. More results about the (weighted) weak group inverse were derived in [17, 27, 28, 37, 39, 40, 41].

Utilizing the WG inverse and MP inverse, the weak core (WC) inverse and its dual were defined in [6] for complex square matrices. The weighted weak core inverse and its dual for bounded linear operators between two Hilbert spaces were presented in [25] as extensions of the weak core inverse and its dual for a complex square matrix. We give the definitions for weighted weak core inverse and its dual for complex rectangular matrices. Let $\{A, W\} \in \mathbb{C}^{m,n,k}$. The W-weighted weak core (or W-WWC) inverse of A is introduced by

$$A^{\textcircled{w},W,\dagger} = WA^{\textcircled{w},W}WAA^{\dagger}.$$

The dual W-weighted weak core (or dual W-WWC) inverse of A is given by

$$A^{\dagger,\textcircled{w},W} = A^{\dagger}AWA^{\textcircled{w},W}W.$$

For $m = n$ and $W = I$ in the definition of the W-WWC and dual W-WWC inverses, respectively, $A^{\textcircled{w},\dagger} = A^{\textcircled{w}}AA^{\dagger}$ is the weak core inverse and $A^{\dagger,\textcircled{w}} = A^{\dagger}AA^{\textcircled{w}}$ is the dual weak core inverse of A. For more information about the weak core inverse see [6, 9, 26, 42].

A new generalized inverse was recently defined in [34] based on the MP and WG inverses. The MP weak group inverse (or MPWG) of $A \in \mathbb{C}^{n\times n}$ represented in [34] by

$$A^{\dagger,WG} = A^{\dagger}A^{\textcircled{w}}A, \tag{1}$$

is a unique solution of the system

$$X = XAX, \quad AX = A^{\mathrm{D}}AA^{\textcircled{w}}A, \quad XA = A^{\dagger}A^{\textcircled{w}}A^2.$$

In order to extend the notion of the MPWG for square complex matrices to rectangular complex matrices, we define the weighted MPWG inverse. We propose a new and wider class of generalized inverses. Also, we generalize the recent results for the MPWG inverse and present some new results for the MPWG inverse. The main research directions of this paper follow.

(1) A number of representations for weighted MPWG inverse are developed.

(2) Necessary and sufficient conditions are proved for a given matrix to be the weighted MPWG inverse.

(3) The weighted MPWG inverse for block matrices is considered.

(4) Various limit and integral expressions are established for the weighted MPWG inverse.

(5) Solvability of some equations is studied applying the weighted MPWG inverse.

The paper has the next organization of sections. The definition of weighted MPWG inverse is given in Section 2 as well as its various representations and characterizations. Section 3 involves results related to the weighted MPWG inverse and block matrices. Limit and integral expressions for weighted MPWG inverse are proved in Section 4. In Section 5, we apply the weighted MPWG inverse to solve some linear equations and present their solutions.

2 Representations of weighted MPWG inverse

To generalize the notion of the MPWG inverse for complex square matrices presented in [34], we define the weighted MPWG inverse for rectangular matrices.

Theorem 2.1. *For $W \in \mathbb{C}^{n\times m}\backslash\{0\}$ and $A \in \mathbb{C}^{m\times n}$, the system of matrix equations*

$$AX = A^{ⓦ,W}WAW \quad \textit{and} \quad X = A^{\dagger}AX \tag{2}$$

has the unique solution expressed by $X = A^{\dagger}A^{ⓦ,W}WAW$.

Proof. By [24, Theorem 2.3], $A^{\textcircled{\dagger},W} = AWA^{\mathrm{D},W}WA^{\textcircled{\dagger},W}$. Since

$$A^{ⓦ,W} = (A^{\textcircled{\dagger},W}W)^2A = AWA^{\mathrm{D},W}W(A^{\textcircled{\dagger},W}W)^2A = AWA^{\mathrm{D},W}WA^{ⓦ,W},$$

$X = A^{\dagger}A^{ⓦ,W}WAW$ satisfies (2):

$$\begin{aligned} AX &= AA^{\dagger}A^{ⓦ,W}WAW = AA^{\dagger}AWA^{\mathrm{D},W}WA^{ⓦ,W}WAW \\ &= AWA^{\mathrm{D},W}WA^{ⓦ,W}WAW = A^{ⓦ,W}WAW \end{aligned}$$

and

$$A^{\dagger}AX = A^{\dagger}AA^{\dagger}A^{ⓦ,W}WAW = A^{\dagger}A^{ⓦ,W}WAW = X.$$

If X is a solution to (2), then

$$X = A^{\dagger}AX = A^{\dagger}A^{ⓦ,W}WAW.$$

Hence, $X = A^{\dagger}A^{ⓦ,W}WAW$ presents the unique solution of (2). □

Definition 2.1. Let $W \in \mathbb{C}^{n\times m}\backslash\{0\}$ and $A \in \mathbb{C}^{m\times n}$. The W-weighted MP weak group (or W-MPWG) inverse of A is defined as

$$A^{\dagger,WG,W} = A^{\dagger}A^{ⓦ,W}WAW.$$

Obviously, in the case that $m = n$ and $W = I$, the W-weighted MP weak group inverse $A^{\dagger,WG,W}$ coincides with the MP weak group inverse $A^{\dagger,WG}$.

Example 2.1. *For the following complex matrices*

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \textit{and} \quad W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

we observe that

$$(WA)^{\textcircled{\dagger}} = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^{\textcircled{w},W} = A^{\textcircled{\dagger},W} = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$A^{\dagger} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

Hence,

$$A^{\dagger,WG,W} = A^{\dagger}A^{\textcircled{w},W}WAW = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Remark 2.1. *Recall that $A^{\textcircled{\dagger},W,\dagger} = WA^{\textcircled{\dagger},W}WAA^{\dagger}$ represents the W-CEPMP inverse considered in [33]. Because $A^{\textcircled{w},W} = (A^{\textcircled{\dagger},W}W)^2A$ and $A^{\textcircled{\dagger},W} = AWA^{\textcircled{\dagger},W}WA^{\textcircled{\dagger},W}$ imply*

$$WAWA^{\textcircled{w},W}A^{\dagger} = WAW(A^{\textcircled{\dagger},W}W)^2AA^{\dagger} = WA^{\textcircled{\dagger},W}WAA^{\dagger} = A^{\textcircled{\dagger},W,\dagger},$$

we can not study the expression $WAWA^{\textcircled{w},W}A^{\dagger}$ as a new generalized inverse of A.

We now investigate some representations of the W-weighted MP weak group inverse. Firstly, we obtain one representation of the W-MPWG inverse of A based on the MP inverse of A and weak group inverse of WA.

Corollary 2.1. *For $W \in \mathbb{C}^{n\times m}\backslash\{0\}$ and $A \in \mathbb{C}^{m\times n}$, we have*

$$A^{\dagger,WG,W} = A^{\dagger}A[(WA)^{\textcircled{w}}]^2WAW. \tag{3}$$

Proof. Using $A^{\dagger,WG,W} = A^{\dagger}A^{\textcircled{w},W}WAW$ and the representation $A^{\textcircled{w},W} = A[(WA)^{\textcircled{w}}]^2$ proved in [28, Theorem 2.5], we have that (3) is satisfied. □

We present the W-MPWG inverse of A in terms of the MP inverses of A and WA.

Theorem 2.2. *For $\{A, W\} \in \mathbb{C}^{m,n,k}$ with $l \geq k$, we have*

$$\begin{aligned} A^{\dagger,WG,W} &= A^{\dagger}A[(WA)^{\mathrm{D}}]^3(WA)^l[(WA)^l]^{\dagger}WAWAW && (4) \\ &= A^{\dagger}A(WA)^{\mathrm{D}}(WA)^{\textcircled{w}}WAW && (5) \\ &= A^{\dagger}A(WA)^l[(WA)^{l+3}]^{\dagger}WAWAW. && (6) \end{aligned}$$

Proof. By [27, Theorem 2.1], we get $(WA)^{\circledS} = [(WA)^{\mathrm{D}}]^2(WA)^l[(WA)^l]^\dagger WA$. Applying the equality (3), we observe that

$$\begin{aligned} A^{\dagger,WG,W} &= A^\dagger A[(WA)^{\circledS}]^2 WAW \\ &= A^\dagger A[(WA)^{\mathrm{D}}]^2(WA)^l[(WA)^l]^\dagger WA[(WA)^{\mathrm{D}}]^2(WA)^l \\ &\times [(WA)^l]^\dagger WAWAW \\ &= A^\dagger A[(WA)^{\mathrm{D}}]^2(WA)^l[(WA)^l]^\dagger (WA)^l(WA)^{\mathrm{D}} \\ &\times [(WA)^l]^\dagger WAWAW \\ &= A^\dagger A[(WA)^{\mathrm{D}}]^3(WA)^l[(WA)^l]^\dagger WAWAW \\ &= A^\dagger A(WA)^{\mathrm{D}}(WA)^{\circledS} WAW. \end{aligned}$$

Notice that (6) follows from

$$\begin{aligned} A^{\dagger,WG,W} &= A^\dagger A[(WA)^{\mathrm{D}}]^3(WA)^l[(WA)^l]^\dagger WAWAW \\ &= A^\dagger A[(WA)^{\mathrm{D}}]^3 P_{\mathcal{R}((WA)^l)} WAWAW \\ &= A^\dagger A[(WA)^{\mathrm{D}}]^3 P_{\mathcal{R}((WA)^{l+3})} WAWAW \\ &= A^\dagger A[(WA)^{\mathrm{D}}]^3(WA)^{l+3}[(WA)^{l+3}]^\dagger WAWAW \\ &= A^\dagger A(WA)^l[(WA)^{l+3}]^\dagger WAWAW. \end{aligned}$$

□

As a consequence of Theorem 2.2, we give the next formula for the W-MPWG inverse which generalizes the expression $A^{\dagger,WG} = A^\dagger A^{\mathrm{D}} A A^{\circledS} A$.

Corollary 2.2. *For $W \in \mathbb{C}^{n\times m}\backslash\{0\}$ and $A \in \mathbb{C}^{m\times n}$, we have*

$$A^{\dagger,WG,W} = A^\dagger A^{\mathrm{D},W} WAWA^{\circledS,W} WAW.$$

Proof. This formula is clear by (5), $(WA)^{\mathrm{D}} = WA^{D,W}$ and $(WA)^{\circledS} = WA^{\circledS,W}$ [28, Theorem 2.5]. □

In the case that $1 = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$ in Theorem 2.2, we verify the next result.

Corollary 2.3. *For $\{A, W\} \in \mathbb{C}^{m,n,1}$, we have*

$$\begin{aligned} A^{\dagger,WG,W} &= A^\dagger A(WA)^{\#} W \\ &= A^\dagger AWA[(WA)^4]^\dagger WAWAW. \end{aligned}$$

The following representations of W-MPWG inverse are verified by the relations between the W-weighted weak group inverse and W-weighted core-EP inverse.

Theorem 2.3. *For $W \in \mathbb{C}^{n\times m}\backslash\{0\}$ and $A \in \mathbb{C}^{m\times n}$, we have*

$$\begin{aligned} A^{\dagger,WG,W} &= A^{\dagger}(A^{\textcircled{\dagger},W}W)^2(AW)^2 &(7)\\ &= A^{\dagger}A[(WA)^{\textcircled{\dagger}}]^3W(AW)^2. &(8)\end{aligned}$$

Proof. By the formula $A^{\textcircled{w},W} = (A^{\textcircled{\dagger},W}W)^2A$, we obtain the expression (7). Since $A^{\textcircled{\dagger},W} = A[(WA)^{\textcircled{\dagger}}]^2$ according to [23, Theorem 2.4], we have that (8) holds:

$$\begin{aligned} A^{\dagger,WG,W} &= A^{\dagger}(A^{\textcircled{\dagger},W}W)^2(AW)^2 = A^{\dagger}A[(WA)^{\textcircled{\dagger}}]^2WA[(WA)^{\textcircled{\dagger}}]^2W(AW)^2\\ &= A^{\dagger}A[(WA)^{\textcircled{\dagger}}]^3W(AW)^2.\end{aligned}$$

□

Some expressions of the W-MPWG inverse can be written by means of projectors.

Corollary 2.4. *For $\{A, W\} \in \mathbb{C}^{m,n,k}$ with $l \geq k$, we have*

$$\begin{aligned} A^{\dagger,WG,W} &= A^{\dagger}P_{\mathcal{R}((AW)^k),\mathcal{N}([(WA)^k]^*W(AW)^2)} &(9)\\ &= P_{\mathcal{R}(A^*)}[(WA)^{\mathrm{D}}]^3P_{\mathcal{R}((WA)^l)}WAWAW &(10)\\ &= P_{\mathcal{R}(A^*)}(WA)^{\mathrm{D}}P_{\mathcal{R}((WA)^l),\mathcal{N}([(WA)^l]^*(WA)^2)}W. &(11)\end{aligned}$$

Proof. According to [28, Theorem 2.5], we have

$$A^{\textcircled{w},W}WAW = P_{\mathcal{R}(A^{\mathrm{D},W}),\mathcal{N}(A^{\textcircled{\dagger},W}W(AW)^2)} = P_{\mathcal{R}((AW)^k),\mathcal{N}([(WA)^k]^*W(AW)^2)}.$$

Now, by the definition of $A^{\dagger,WG,W}$, (9) can be obtained.

It is well-known that $A^{\dagger}A = P_{\mathcal{R}(A^*)}$, $(WA)^l[(WA)^l]^{\dagger} = P_{\mathcal{R}((WA)^l)}$ and

$$(WA)^{\textcircled{w}}WA = P_{\mathcal{R}((WA)^l),\mathcal{N}([(WA)^l]^*(WA)^2)}.$$

Applying (4) and (5), we can get (10) and (11), respectively. □

Theorem 2.4 contains equivalent conditions for a rectangular matrix to be the W-MPWG inverse.

Theorem 2.4. *For $\{A, W\} \in \mathbb{C}^{m,n,k}$ and $X \in \mathbb{C}^{n\times m}$, the following statements are equivalent:*

(i) $X = A^{\dagger,WG,W}$;

(ii) $XAX = X$, $AX = A^{\textcircled{w},W}WAW$ *and* $XA = A^{\dagger}A^{\textcircled{w},W}(WA)^2$;

(iii) $X(AW)^k = A^{\dagger}(AW)^k$ *and* $XA^{\textcircled{w},W}WAW = X$;

(iv) $X(AW)^k = A^\dagger(AW)^k$ *and* $XA^{\mathrm{D},W}WAWA^{\textcircled{w},W}WAW = X$*;*

(v) $A^\dagger AXA^{\textcircled{w},W}WAW = X$ *and* $AX = A^{\textcircled{w},W}WAW$*;*

(vi) $A^\dagger AXA^{\textcircled{w},W}WAW = X$ *and* $AXA^{\textcircled{w},W} = A^{\textcircled{w},W}$*;*

(vii) $XA^{\textcircled{w},W}(WA)^2X = X$, $A^{\textcircled{w},W}(WA)^2X = A^{\textcircled{w},W}WAW$ *and* $XA^{\textcircled{w},W}(WA)^2 = A^\dagger A^{\textcircled{w},W}(WA)^2$*;*

(viii) $XA^{\textcircled{w},W}(WA)^2X = X$, $A^{\textcircled{w},W}(WA)^2XA^{\textcircled{w},W}(WA)^2 = A^{\textcircled{w},W}(WA)^2$, $A^{\textcircled{w},W}(WA)^2X = A^{\textcircled{w},W}WAW$ *and* $XA^{\textcircled{w},W}(WA)^2 = A^\dagger A^{\textcircled{w},W}(WA)^2$.

Proof. (i) $\Rightarrow$ (ii): From $X = A^{\dagger,WG,W} = A^\dagger A^{\textcircled{w},W}WAW$ and (2), we get $XA = A^\dagger A^{\textcircled{w},W}WAWA$ and $AX = A^{\textcircled{w},W}WAW$. Also,

$$\begin{aligned} XAX &= XA^{\textcircled{w},W}WAW = A^\dagger(A^{\textcircled{w},W}WAWA^{\textcircled{w},W})WAW \\ &= A^\dagger A^{\textcircled{w},W}WAW = X. \end{aligned}$$

(ii) $\Rightarrow$ (iii): The assumptions $XAX = X$ and $AX = A^{\textcircled{w},W}WAW$ give $X = X(AX) = XA^{\textcircled{w},W}WAW$. Using [24, Theorem 2.3], $A^{\textcircled{\dagger},W}WA(WA)^{\mathrm{D}} = A^{\mathrm{D},W}$ which implies

$$A^{\textcircled{\dagger},W}WAW(AW)^{\mathrm{D}} = A^{\textcircled{\dagger},W}WA(WA)^{\mathrm{D}}W = A^{\mathrm{D},W}W = (AW)^{\mathrm{D}}$$

and so $A^{\textcircled{\dagger},W}W(AW)^{k+1} = A^{\textcircled{\dagger},W}WAW(AW)^{\mathrm{D}}(AW)^{k+1} = (AW)^{\mathrm{D}}(AW)^{k+1} = (AW)^k$. From $XA = A^\dagger A^{\textcircled{w},W}(WA)^2$ and

$$A^{\textcircled{w},W}W(AW)^{k+1} = (A^{\textcircled{\dagger},W}W)^2AW(AW)^{k+1} = A^{\textcircled{\dagger},W}W(AW)^{k+1} = (AW)^k,$$

we show that

$$\begin{aligned} X(AW)^k &= (XA)W(AW)^{k-1} = A^\dagger A^{\textcircled{w},W}(WA)^2W(AW)^{k-1} \\ &= A^\dagger A^{\textcircled{w},W}W(AW)^{k+1} = A^\dagger(AW)^k. \end{aligned}$$

(iii) $\Rightarrow$ (iv): It is evident by

$$A^{\textcircled{w},W} = (A^{\textcircled{\dagger},W}W)^2A = A^{\mathrm{D},W}WAW(A^{\textcircled{\dagger},W}W)^2A = A^{\mathrm{D},W}WAWA^{\textcircled{w},W}.$$

(iv) $\Rightarrow$ (i): Because $X(AW)^k = A^\dagger(AW)^k$ and $XA^{\mathrm{D},W}WAWA^{\textcircled{w},W}WAW = X$, we check that

$$\begin{aligned} X &= XA^{\mathrm{D},W}WAWA^{\textcircled{w},W}WAW = X(AW)^k(A^{\mathrm{D},W}W)^kA^{\textcircled{w},W}WAW \\ &= A^\dagger(AW)^k(A^{\mathrm{D},W}W)^kA^{\textcircled{w},W}WAW = A^\dagger(AWA^{\mathrm{D},W}WA^{\textcircled{w},W})WAW \\ &= A^\dagger A^{\textcircled{w},W}WAW. \end{aligned}$$

(i) $\Rightarrow$ (v): This implication follows by (i) $\Leftrightarrow$ (iii) and Theorem 2.1.

(v) $\Rightarrow$ (vi): It is obvious by $A^{ⓦ,W}WAWA^{ⓦ,W} = A^{ⓦ,W}$.

(vi) $\Rightarrow$ (i): Notice that $X = A^{\dagger}(AXA^{ⓦ,W})WAW = A^{\dagger}A^{ⓦ,W}WAW = A^{\dagger,WG,W}$.

Similarly, we verify the rest. $\square$

Recall that $X \in \mathbb{C}^{n\times m}$ satisfying $AXA = A$, is an $\{1\}$-inverse (or inner inverse) of $A \in \mathbb{C}^{m\times n}$. Denote by $A^{(1,2)}_{U,V}$ the outer inverse $A^{(2)}_{U,V}$ of A for which $AA^{(2)}_{U,V}A = A$ holds.

By Theorem 2.4, we observe that $A^{\dagger,WG,W}$ is an outer inverse of A and both outer and inner inverse of $A^{ⓦ,W}(WA)^2$. We determine the ranges and null spaces of $A^{\dagger,WG,W}$ and projectors involving $A^{\dagger,WG,W}$.

Theorem 2.5. *For $\{A, W\} \in \mathbb{C}^{m,n,k}$ and $X \in \mathbb{C}^{n\times m}$, the following statements hold:*

(i) $AA^{\dagger,WG,W} = P_{\mathcal{R}((AW)^k),\mathcal{N}([(WA)^k]^*(WA)^2W)}$;

(ii) $A^{\dagger,WG,W}A = P_{\mathcal{R}(A^{\dagger}(AW)^k),\mathcal{N}([(WA)^k]^*(WA)^3)}$;

(iii) $A^{\dagger,WG,W} = A^{(2)}_{\mathcal{R}(A^{\dagger}(AW)^k),\mathcal{N}([(WA)^k]^*(WA)^2W)}$;

(iv) $A^{\dagger,WG,W} = \left(A^{ⓦ,W}(WA)^2\right)^{(1,2)}_{\mathcal{R}(A^{\dagger}(AW)^k),\mathcal{N}([(WA)^k]^*(WA)^2W)}$.

Proof. (i) Using Theorem 2.4, notice that $A^{\dagger,WG,W} = A^{\dagger,WG,W}AA^{\dagger,WG,W}$ and so $AA^{\dagger,WG,W}$ is a projector. By Theorem 2.1 and the equality (4),

$$AA^{\dagger,WG,W} = A^{ⓦ,W}WAW = A[(WA)^{\mathrm{D}}]^3(WA)^k[(WA)^k]^{\dagger}(WA)^2W.$$

According to [28, Lemma 2.2], $\mathcal{R}(AA^{\dagger,WG,W}) = \mathcal{R}(A^{ⓦ,W}WAW) = \mathcal{R}(A^{\mathrm{D},W}) = \mathcal{R}((AW)^k)$. It is clear that $\mathcal{N}([(WA)^k]^{\dagger}(WA)^2W) = \mathcal{N}([(WA)^k]^*(WA)^2W) \subseteq \mathcal{N}(AA^{\dagger,WG,W})$. Now, by

$$\begin{aligned}
[(WA)^k]^*(WA)^2W &= [(WA)^k]^*(WA)^k[(WA)^k]^{\dagger}(WA)^2W \\
&= [(WA)^k]^*(WA)^2WA[(WA)^{\mathrm{D}}]^3(WA)^k \\
&\times [(WA)^k]^{\dagger}(WA)^2W \\
&= [(WA)^k]^*(WA)^2WAA^{\dagger,WG,W}, \qquad (12)
\end{aligned}$$

we see that $\mathcal{N}(AA^{\dagger,WG,W}) \subseteq \mathcal{N}([(WA)^k]^*(WA)^2W)$. Hence, $\mathcal{N}(AA^{\dagger,WG,W}) = \mathcal{N}([(WA)^k]^*(WA)^2W)$.

(ii) Theorem 2.4 gives that $A^{\dagger,WG,W}A$ is a projector. Because

$$A^{\dagger,WG,W}A = A^{\dagger}A[(WA)^{\mathrm{D}}]^3(WA)^k[(WA)^k]^{\dagger}(WA)^3,$$

$\mathcal{R}(A^{\dagger,WG,W}A) \subseteq \mathcal{R}(A^{\dagger}A(WA)^k)$ and $\mathcal{N}([(WA)^k]^*(WA)^3) \subseteq \mathcal{N}(A^{\dagger,WG,W}A)$. From

$$\begin{aligned} A^{\dagger}A(WA)^k &= A^{\dagger}A[(WA)^{\mathrm{D}}]^3(WA)^{k+3} \\ &= A^{\dagger}A[(WA)^{\mathrm{D}}]^3(WA)^k[(WA)^k]^{\dagger}(WA)^3(WA)^k \\ &= A^{\dagger,WG,W}A(WA)^k, \end{aligned}$$

we deduce that $\mathcal{R}(A^{\dagger,WG,W}A) = \mathcal{R}(A^{\dagger}A(WA)^k) = \mathcal{R}(A^{\dagger}(AW)^k)$. Applying the equality (12), we get $[(WA)^k]^*(WA)^3 = [(WA)^k]^*(WA)^2WAA^{\dagger,WG,W}A$ and $\mathcal{N}(A^{\dagger,WG,W}A) = \mathcal{N}([(WA)^k]^*(WA)^3)$.

(iii) Using parts (i) and (ii), it follows that $\mathcal{R}(A^{\dagger,WG,W}) = \mathcal{R}(A^{\dagger,WG,W}A) = \mathcal{R}(A^{\dagger}(AW)^k)$ and $\mathcal{N}(A^{\dagger,WG,W}) = \mathcal{N}(AA^{\dagger,WG,W}) = \mathcal{N}([(WA)^k]^*(WA)^2W)$.

(iv) It is clear by Theorem 2.4 and part (iii). □

Consequently, for $1 = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$ in Theorem 2.5, we obtain the following result.

Corollary 2.5. *For $\{A, W\} \in \mathbb{C}^{m,n,1}$, we have*

(i) $AA^{\dagger,WG,W} = P_{\mathcal{R}(AW),\mathcal{N}(AW)}$;

(ii) $A^{\dagger,WG,W}A = P_{\mathcal{R}(A^{\dagger}AW),\mathcal{N}(WA)}$;

(iii) $A^{\dagger,WG,W} = A^{(2)}_{\mathcal{R}(A^{\dagger}AW),\mathcal{N}(AW)}$.

Proof. Since $1 = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$, we have $\mathcal{N}([(WA)^k]^*(WA)^2W) = \mathcal{N}((WA)^2W) = \mathcal{N}(WAW) = \mathcal{N}(AW)$ and $\mathcal{N}([(WA)^k]^*(WA)^3) = \mathcal{N}((WA)^3) = \mathcal{N}(WA)$. The rest is evident by Theorem 2.5. □

By Theorem 2.5, we establish the next representations for the W-MPWG inverse.

Corollary 2.6. *For $\{A, W\} \in \mathbb{C}^{m,n,k}$ with $1 \leq k$,*

$$A^{\dagger,WG,W} = A^{\dagger}(AW)^k\left([(WA)^k]^*(WA)^{k+2}W\right)^{\dagger}[(WA)^k]^*(WA)^2W.$$

Proof. Utilizing $A^{\dagger,WG,W} = A^{(2)}_{\mathcal{R}(A^{\dagger}(AW)^k),\mathcal{N}([(WA)^k]^*(WA)^2W)}$ and the Urquhart formula [2], we get

$$\begin{aligned} A^{\dagger,WG,W} &= A^{\dagger}(AW)^k\left([(WA)^k]^*(WA)^2WAA^{\dagger}(AW)^k\right)^{\dagger}[(WA)^k]^*(WA)^2W \\ &= A^{\dagger}(AW)^k\left([(WA)^k]^*(WA)^{k+2}W\right)^{\dagger}[(WA)^k]^*(WA)^2W. \end{aligned}$$

□

Corollary 2.7. *For* $\{A, W\} \in \mathbb{C}^{m,n,k}$,

$$A^{\dagger,WG,W} = A^{(2)}_{\mathcal{R}(A^\dagger(AW)^k A[(WA)^k]^*(WA)^2W),\mathcal{N}(A^\dagger(AW)^k A[(WA)^k]^*(WA)^2W)}.$$

Proof. Since $\mathcal{R}(A^\dagger(AW)^k A[(WA)^k]^*(WA)^2W) \subseteq \mathcal{R}(A^\dagger(AW)^k)$ and

$$\begin{aligned}
\mathcal{R}(A^\dagger(AW)^k) &= \mathcal{R}(A^\dagger(AW)^{k+1}(AW)^{\mathrm{D}}) \subseteq \mathcal{R}(A^\dagger A(WA)^k) \\
&= \mathcal{R}(A^\dagger A(WA)^k[(WA)^k]^*([(WA)^k]^\dagger)^*) \\
&= \mathcal{R}(A^\dagger A(WA)^k[(WA)^k]^*(WA)^k[(WA)^k]^\dagger([(WA)^k]^\dagger)^*) \\
&\subseteq \mathcal{R}(A^\dagger A(WA)^k[(WA)^k]^*(WA)^{k+3}[(WA)^{\mathrm{D}}]^3) \\
&\subseteq \mathcal{R}(A^\dagger(AW)^k A[(WA)^k]^*(WA)^2W),
\end{aligned}$$

we deduce that $\mathcal{R}(A^\dagger(AW)^k) = \mathcal{R}(A^\dagger(AW)^k A[(WA)^k]^*(WA)^2W)$. From $\mathcal{N}([(WA)^k]^*(WA)^2W) \subseteq \mathcal{N}(A^\dagger(AW)^k A[(WA)^k]^*(WA)^2W)$ and

$$\begin{aligned}
\mathcal{N}([(WA)^k]^*(WA)^2W) &= \mathcal{N}([(WA)^k]^\dagger(WA)^k[(WA)^k]^*(WA)^2W) \\
&\supseteq \mathcal{N}((WA)^k[(WA)^k]^*(WA)^2W) \\
&= \mathcal{N}((WA)^{\mathrm{D}}WA(WA)^k[(WA)^k]^*(WA)^2W) \\
&\supseteq \mathcal{N}((AW)^k A[(WA)^k]^*(WA)^2W) \\
&= \mathcal{N}(AA^\dagger(AW)^k A[(WA)^k]^*(WA)^2W) \\
&\supseteq \mathcal{N}(A^\dagger(AW)^k A[(WA)^k]^*(WA)^2W),
\end{aligned}$$

we get $\mathcal{N}([(WA)^k]^*(WA)^2W) = \mathcal{N}(A^\dagger(AW)^k A[(WA)^k]^*(WA)^2W)$. The rest is evident by $A^{\dagger,WG,W} = A^{(2)}_{\mathcal{R}(A^\dagger(AW)^k),\mathcal{N}([(WA)^k]^*(WA)^2W)}$. $\square$

The W-MPWG inverse can also be characterized by the following statements.

Theorem 2.6. *For* $\{A, W\} \in \mathbb{C}^{m,n,k}$ *with* $1 \leq k$,

(i) $A^{\dagger,WG,W}$ *is the unique solution of*

$$AX = P_{\mathcal{R}((AW)^k),\mathcal{N}([(WA)^k]^*(WA)^2W)} \quad \textit{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^*); \tag{13}$$

(ii) $A^{\dagger,WG,W}$ *is the unique solution of*

$$XA = P_{\mathcal{R}(A^\dagger(AW)^k),\mathcal{N}([(WA)^k]^*(WA)^3)} \ \textit{and} \ \mathcal{R}(X^*) \subseteq \mathcal{R}((A^{\circledW,W}WAW)^*). \tag{14}$$

Proof. (i) By Theorem 2.5 and $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*)$, we conclude that (13) holds for $X = A^{\dagger,WG,W}$.

If two matrices X and X_1 satisfy (13), then

$$A(X-X_1)=P_{\mathcal{R}((AW)^k),\mathcal{N}([(WA)^k]^*(WA)^2W)}-P_{\mathcal{R}((AW)^k),\mathcal{N}([(WA)^k]^*(WA)^2W)}=0$$

gives $\mathcal{R}(X-X_1)\subseteq\mathcal{N}(A)$. Now $\mathcal{R}(X-X_1)\subseteq\mathcal{R}(A^*)\cap\mathcal{N}(A)=\mathcal{N}(A)^\perp\cap\mathcal{N}(A)=\{0\}$, that is, (13) posses the unique solution.

(ii) According to Theorem 2.5 and $A^{\dagger,WG,W}=A^\dagger A^{\circledW,W}WAW$, $A^{\dagger,WG,W}$ satisfies (14).

For X and X_1 such that (14) holds, we observe that $\mathcal{R}(X^*-X_1^*)\subseteq\mathcal{R}((A^{\circledW,W}WAW)^*)\cap\mathcal{N}(A^*)\subseteq\mathcal{R}((AA^{\dagger,WG,W})^*)\cap\mathcal{N}((AA^{\dagger,WG,W})^*)=\{0\}$. $\square$

3 Properties of the W-MPWG inverse

By the corresponding decompositions of A and W given in [7], we establish an expression for W-MPWG inverse.

Lemma 3.1. *For $\{A,W\}\in\mathbb{C}^{m,n,k}$ and $\operatorname{rank}((AW)^k)=p$, we have*

$$A=U\begin{bmatrix}A_1 & A_2\\ 0 & A_3\end{bmatrix}V^* \qquad \text{and} \qquad W=V\begin{bmatrix}W_1 & W_2\\ 0 & W_3\end{bmatrix}U^*, \tag{15}$$

where $U\in\mathbb{C}^{m\times m}$ and $V\in\mathbb{C}^{n\times n}$ are unitary, $A_1,W_1\in\mathbb{C}^{p\times p}$ are invertible, $A_3\in\mathbb{C}^{(m-p)\times(n-p)}$ and $W_3\in\mathbb{C}^{(n-p)\times(m-p)}$ such that A_3W_3 and W_3A_3 are nilpotent of indices $\operatorname{ind}(AW)$ and $\operatorname{ind}(WA)$, respectively. Furthermore,

$$A^{\dagger,WG,W}=V\begin{bmatrix}A_1^*\triangle & A_1^*\triangle T\\ (I-A_3^\dagger A_3)A_2^*\triangle & (I-A_3^\dagger A_3)A_2^*\triangle T\end{bmatrix}U^*, \tag{16}$$

where $\triangle=(A_1A_1^*+A_2(I-A_3^\dagger A_3)A_2^*)^{-1}$ and

$$\begin{aligned}T &= W_1^{-1}W_2+(A_1W_1)^{-1}A_2W_3\\ &+ (W_1A_1W_1)^{-1}[W_2+A_1^{-1}(A_2+W_1^{-1}W_2A_3)W_3]A_3W_3.\end{aligned}$$

Proof. Recall that (15) holds by [7]. Using [8, 28] and [4], we observe that

$$A^{\circledW,W}=U\begin{bmatrix}(W_1A_1W_1)^{-1} & (A_1W_1)^{-2}(A_2+W_1^{-1}W_2A_3)\\ 0 & 0\end{bmatrix}V^*$$

and

$$A^\dagger=V\begin{bmatrix}A_1^*\triangle & -A_1^*\triangle A_2A_3^\dagger\\ (I-A_3^\dagger A_3)A_2^*\triangle & A_3^\dagger-(I-A_3^\dagger A_3)A_2^*\triangle A_2A_3^\dagger\end{bmatrix}U^*.$$

The rest is clear by the definition of W-MPWG inverse. $\square$

Based on projections $G = I - AA^{\dagger,WG,W}$ and $H = I - A^{\dagger,WG,W}A$, we can represent the W-MPWG inverse as follows.

Theorem 3.1. *Let* $\{A, W\} \in \mathbb{C}^{m,n,k}$. *For* $\alpha \in \mathbb{C}\backslash\{0\}$, $G = I - AA^{\dagger,WG,W}$ *and* $H = I - A^{\dagger,WG,W}A$, $AW \pm \alpha G$ *are invertible and*

$$A^{\dagger,WG,W} = (I - H)W(AW \pm \alpha G)^{-1}(I - G). \tag{17}$$

Proof. Let A, W and $A^{\dagger,WG,W}$ be expressed by (15) and (16). Notice that

$$\begin{aligned} I - G &= AA^{\dagger,WG,W} \\ &= U \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} V^*V \begin{bmatrix} A_1^*\triangle & A_1^*\triangle T \\ (I - A_3^\dagger A_3)A_2^*\triangle & (I - A_3^\dagger A_3)A_2^*\triangle T \end{bmatrix} U^* \\ &= U \begin{bmatrix} I & T \\ 0 & 0 \end{bmatrix} U^*, \end{aligned}$$

$$G = I - AA^{\dagger,WG,W} = U \begin{bmatrix} 0 & -T \\ 0 & I \end{bmatrix} U^*$$

and

$$\begin{aligned} I - H &= A^{\dagger,WG,W}A \\ &= V \begin{bmatrix} A_1^*\triangle & A_1^*\triangle T \\ (I - A_3^\dagger A_3)A_2^*\triangle & (I - A_3^\dagger A_3)A_2^*\triangle T \end{bmatrix} U^*U \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} V^* \\ &= V \begin{bmatrix} A_1^*\triangle A_1 & A_1^*\triangle(A_2 + TA_3) \\ (I - A_3^\dagger A_3)A_2^*\triangle A_1 & (I - A_3^\dagger A_3)A_2^*\triangle(A_2 + TA_3) \end{bmatrix} V^*. \end{aligned}$$

Since A_1, W_1 and $A_3W_3 \pm \alpha I$ are invertible, we deduce that

$$\begin{aligned} AW \pm \alpha G &= U \begin{bmatrix} A_1W_1 & A_1W_2 + A_2W_3 \\ 0 & A_3W_3 \end{bmatrix} U^* \pm \alpha U \begin{bmatrix} 0 & -T \\ 0 & I \end{bmatrix} U^* \\ &= U \begin{bmatrix} A_1W_1 & A_1W_2 + A_2W_3 \mp \alpha T \\ 0 & A_3W_3 \pm \alpha I \end{bmatrix} U^* \end{aligned}$$

is invertible too and, for $M = -(A_1W_1)^{-1}(A_1W_2 + A_2W_3 \mp \alpha T)(A_3W_3 \pm \alpha I)^{-1}$,

$$(AW \pm \alpha G)^{-1} = U \begin{bmatrix} (A_1W_1)^{-1} & M \\ 0 & (A_3W_3 \pm \alpha I)^{-1} \end{bmatrix} U^*.$$

Therefore,

$$\begin{aligned}(I-H)W &= V\begin{bmatrix} A_1^*\triangle A_1 & A_1^*\triangle(A_2+TA_3) \\ (I-A_3^\dagger A_3)A_2^*\triangle A_1 & (I-A_3^\dagger A_3)A_2^*\triangle(A_2+TA_3)\end{bmatrix}V^* \\ &\times\ V\begin{bmatrix} W_1 & W_2 \\ 0 & W_3\end{bmatrix}U^* \\ &= V\begin{bmatrix} A_1^*\triangle A_1W_1 & * \\ (I-A_3^\dagger A_3)A_2^*\triangle A_1W_1 & *\end{bmatrix}U^*\end{aligned}$$

and

$$\begin{aligned}(AW\pm\alpha G)^{-1}(I-G) &= U\begin{bmatrix} (A_1W_1)^{-1} & M\end{bmatrix}U^*U\begin{bmatrix} I & T \\ 0 & 0\end{bmatrix}U^* \\ &= U\begin{bmatrix} (A_1W_1)^{-1} & (A_1W_1)^{-1}T \\ 0 & 0\end{bmatrix}U^*\end{aligned}$$

give

$$\begin{aligned}&(I-H)W(AW\pm\alpha G)^{-1}(I-G) = \\ &= V\begin{bmatrix} A_1^*\triangle & A_1^*\triangle T \\ (I-A_3^\dagger A_3)A_2^*\triangle & (I-A_3^\dagger A_3)A_2^*\triangle T\end{bmatrix}U^* \\ &= A^{\dagger,WG,W}.\end{aligned}$$

□

The W-MPWG inverse can be correlated with an bordered invertible matrix as follows.

Theorem 3.2. *For $\{A,W\}\in\mathbb{C}^{m,n,k}$, let full column rank matrices M and N^* satisfy*

$$\mathcal{N}\left([(WA)^k]^*(WA)^2W\right)=\mathcal{R}(M) \qquad \textit{and} \qquad \mathcal{R}(A^\dagger(WA)^k)=\mathcal{N}(N).$$

Then

$$S=\begin{bmatrix} A & M \\ N & 0\end{bmatrix}$$

is nonsingular and

$$S^{-1}=\begin{bmatrix} A^{\dagger,WG,W} & (I-A^{\dagger,WG,W}A)N^\dagger \\ M^\dagger(I-AA^{\dagger,WG,W}) & -M^\dagger(A-AA^{\dagger,WG,W}A)N^\dagger\end{bmatrix}. \tag{18}$$

Proof. Theorem 2.5 gives $\mathcal{R}(A^{\dagger,WG,W})=\mathcal{R}(A^\dagger(WA)^k)=\mathcal{N}(N)$ and $\mathcal{R}(I-AA^{\dagger,WG,W})=\mathcal{N}(A^{\dagger,WG,W})=\mathcal{N}([(WA)^k]^*(WA)^2W)=\mathcal{R}(M)=\mathcal{N}(I-MM^\dagger)$. Therefore, $NA^{\dagger,WG,W}=0$ and $(I-MM^\dagger)(I-AA^{\dagger,WG,W})=0$.

Denote by R the right hand side of (18) and $E = MM^\dagger(I - AA^{\dagger,WG,W}) = I - AA^{\dagger,WG,W}$. From

$$\begin{aligned} SR &= \begin{bmatrix} AA^{\dagger,WG,W} + E & A(I - A^{\dagger,WG,W}A)N^\dagger - EAN^\dagger \\ NA^{\dagger,WG,W} & N(I - A^{\dagger,WG,W}A)N^\dagger \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \\ &= I \end{aligned}$$

and analogously $RS = I$, we conclude that $S^{-1} = R$. □

4 Integral and limit expressions for W-MPWG inverse

This section contains limit and integral formulae for the W-MPWG inverse.

Utilizing the limit representation for the MP inverse proved in [32], we develop the limit representations for the W-MPWG inverse.

Theorem 4.1. *For $\{A, W\} \in \mathbb{C}^{m,n,k}$ with $l \geq k$, it follows*

$$\begin{aligned} & A^{\dagger,WG,W} = \\ = \;& \lim_{u\to 0} A^*(AA^* + uI)^{-1} A \lim_{v\to 0} (WA)^l[(WA)^{l+3}]^* \\ \times \;& \left((WA)^{l+3}[(WA)^{l+3}]^* + vI\right)^{-1} (WA)^2W \\ = \;& \lim_{v\to 0} P_{\mathcal{R}(A^*)}(WA)^l[(WA)^{l+3}]^* \left((WA)^{l+3}[(WA)^{l+3}]^* + vI\right)^{-1} (WA)^2W \\ = \;& \lim_{u\to 0} (A^*A + uI)^{-1}A^*A \lim_{v\to 0} (WA)^l[(WA)^{l+3}]^* \\ \times \;& \left((WA)^{l+3}[(WA)^{l+3}]^* + vI\right)^{-1} (WA)^2W \\ = \;& \lim_{v\to 0} P_{\mathcal{R}(A^*)}(WA)^l \left([(WA)^{l+3}]^*(WA)^{l+3} + vI\right)^{-1} [(WA)^{l+3}]^*(WA)^2W. \end{aligned}$$

Proof. Recall that, by (6), $A^{\dagger,WG,W} = A^\dagger A(WA)^l[(WA)^{l+3}]^\dagger WAWAW$. According to [32], we have

$$A^\dagger = \lim_{u\to 0} A^*(AA^* + uI)^{-1} = \lim_{u\to 0} (A^*A + uI)^{-1}A^* \tag{19}$$

and thus

$$\begin{aligned} [(WA)^{l+3}]^\dagger &= \lim_{v\to 0} [(WA)^{l+3}]^* \left((WA)^{l+3}[(WA)^{l+3}]^* + vI\right)^{-1} \\ &= \lim_{v\to 0} \left([(WA)^{l+3}]^*(WA)^{l+3} + vI\right)^{-1} [(WA)^{l+3}]^*. \end{aligned}$$

The proof can be finished now. □

By Theorem 4.1, the following limit representations for MPWG inverse can be obtained.

Corollary 4.1. *For $A \in \mathbb{C}^{m\times n}$ with $l \geq k = \operatorname{ind}(A)$, it follows*

$$\begin{aligned} A^{\dagger,WG} &= \lim_{u\to 0} A^*(AA^*+uI)^{-1}A \lim_{v\to 0} A^l(A^{l+3})^* \left(A^{l+3}(A^{l+3})^*+vI\right)^{-1} A^2 \\ &= \lim_{v\to 0} P_{\mathcal{R}(A^*)} A^l(A^{l+3})^* \left(A^{l+3}(A^{l+3})^*+vI\right)^{-1} A^2 \\ &= \lim_{u\to 0} (A^*A+uI)^{-1}A^*A \lim_{v\to 0} A^l(A^{l+3})^* \left(A^{l+3}(A^{l+3})^*+vI\right)^{-1} A^2 \\ &= \lim_{v\to 0} P_{\mathcal{R}(A^*)} A^l \left((A^{l+3})^*A^{l+3}+vI\right)^{-1} (A^{l+3})^*A^2. \end{aligned}$$

According to the limit formula for outer inverse presented in [18], the next limit representation for W-MPWG inverse follows.

Theorem 4.2. *Let $\{A, W\} \in \mathbb{C}^{m,n,k}$ with $l \geq k$. For $E \in \mathbb{C}_s^{n\times s}$, $\mathcal{R}(E) = \mathcal{R}(A^\dagger(AW)^k)$, $E_1 \in \mathbb{C}_s^{s\times n}$ and $\mathcal{N}(E_1) = \mathcal{N}([(WA)^k]^*(WA)^2W)$, it follows*

$$\begin{aligned} A^{\dagger,WG,W} &= \lim_{v\to 0} E(vI+E_1AE)^{-1}E_1 \\ &= \lim_{v\to 0} (vI+EE_1A)^{-1}EE_1 = \lim_{v\to 0} EE_1(vI+AEE_1)^{-1}. \end{aligned}$$

Proof. By [18, Theorem 7], we obtain

$$A^{\dagger,WG,W} = A^{(2)}_{\mathcal{R}(A^\dagger(AW)^k),\mathcal{N}([(WA)^k]^*(WA)^2W)} = \lim_{u\to 0} E(uI+E_1AE)^{-1}E_1.$$

□

We propose some integral representations for the W-MPWG inverse.

Theorem 4.3. *For $\{A, W\} \in \mathbb{C}^{m,n,k}$ with $l \geq k$, we have*

$$\begin{aligned} A^{\dagger,WG,W} &= \int_0^\infty A^*\exp(-AA^*v)\mathrm{d}v \int_0^\infty A(WA)^l[(WA)^{l+3}]^*E_1\mathrm{d}u \\ &= P_{\mathcal{R}(A^*)} \int_0^\infty (WA)^l[(WA)^{l+3}]^*E_1\mathrm{d}u, \end{aligned}$$

where $E_1 = \exp(-(WA)^{l+3}[(WA)^{l+3}]^*u)(WA)^2W$.

Proof. By [12]

$$A^\dagger = \int_0^\infty A^*\exp(-AA^*v)\mathrm{d}v, \tag{20}$$

and therefore

$$[(WA)^{l+3}]^{\dagger} = \int_0^{\infty} [(WA)^{l+3}]^* \exp(-(WA)^{l+3}[(WA)^{l+3}]^* u)\mathrm{d}u.$$

The rest follows by (6). $\square$

Theorem 4.4. *Let $\{A, W\} \in \mathbb{C}^{m,n,k}$ and $E \in \mathbb{C}^{n\times n}$ satisfying $\mathcal{R}(E) = \mathcal{R}(A^{\dagger}(AW)^k)$ and $\mathcal{N}(E) = \mathcal{N}\left([(WA)^k]^*(WA)^2W\right)$. Then*

$$A^{\dagger,WG,W} = \int\limits_0^{\infty} \exp\left[-E(EAE)^*EAv\right]E(EAE)^*E\,\mathrm{d}v.$$

Proof. Since $A^{\dagger,WG,W} = A^{(2)}_{\mathcal{R}(A^{\dagger}(AW)^k),\mathcal{N}([(WA)^k]^*(WA)^2W)}$, the proof can be finished by [38, Theorem 2.2]. $\square$

Consequently, the following integral representations can be verified for the MPWG inverse.

Corollary 4.2. *Let $A \in \mathbb{C}^{m\times n}$ with $l \geq k = \mathrm{ind}(A)$.* (i) *Then*

$$\begin{aligned} A^{\dagger,WG} &= \int_0^{\infty} A^* \exp(-AA^*v)\mathrm{d}v \\ &\times \int_0^{\infty} A^{l+1}(A^{l+3})^* \exp(-A^{l+3}(A^{l+3})^*u)A^2\mathrm{d}u \\ &= P_{\mathcal{R}(A^*)} \int_0^{\infty} A^{l}(A^{l+3})^* \exp(-A^{l+3}(A^{l+3})^*u)A^2\mathrm{d}u. \end{aligned}$$

(ii) *For $E \in \mathbb{C}^{n\times n}$ satisfying $\mathcal{R}(E) = \mathcal{R}(A^{\dagger}A^k)$ and $\mathcal{N}(E) = \mathcal{N}\left((A^k)^*A^2\right)$, it follows*

$$A^{\dagger,WG} = \int\limits_0^{\infty} \exp\left[-E(EAE)^*EAv\right]E(EAE)^*E\,\mathrm{d}v.$$

5 Applications of W-MPWG inverse

Applications of the W-MPWG inverse in solving some systems of linear equations, are investigated in this section.

In the cases when ordinary inverses of matrices do not exist, we can use generalized inverses to solve some matrix equations. Generalized inverses of matrices play an important role in theoretical and numerical methods of linear algebra and have numerous applications in statistics, econometrics, logistics, electrical network theory, the theory of differential and difference equations. Remark that the equation (21) is a generalization of a equation which appeared in [34, Theorem 7.1].

Theorem 5.1. *For $\{A, W\} \in \mathbb{C}^{m,n,k}$, the general solution of equation*

$$[(WA)^{k+2}]^*(WA)^3x = [(WA)^{k+2}]^*(WA)^2Wb, \qquad b \in \mathbb{C}^m, \tag{21}$$

is expressed as

$$x = A^{\dagger,WG,W}b + (I - A^{\dagger,WG,W}A)y, \tag{22}$$

for arbitrary $y \in \mathbb{C}^n$.

Proof. Let x be given by (22). Using the equality $AA^{\dagger,WG,W} = A[(WA)^{\mathrm{D}}]^3(WA)^k[(WA)^k]^\dagger(WA)^2W$, note that

$$\begin{aligned} [(WA)^{k+2}]^*(WA)^3A^{\dagger,WG,W} &= [(WA)^{k+2}]^*(WA)^2WAA^{\dagger,WG,W} \\ &= [(WA)^{k+2}]^*(WA)^2WA[(WA)^{\mathrm{D}}]^3(WA)^k \\ &\times [(WA)^k]^\dagger(WA)^2W \\ &= [(WA)^{k+2}]^*(WA)^k[(WA)^k]^\dagger(WA)^2W \\ &= [(WA)^{k+2}]^*(WA)^2W. \end{aligned}$$

Thus,

$$\begin{aligned} [(WA)^{k+2}]^*(WA)^3x &= [(WA)^{k+2}]^*(WA)^3A^{\dagger,WG,W}b \\ &+ [(WA)^{k+2}]^*(WA)^3(I - A^{\dagger,WG,W}A)y \\ &= [(WA)^{k+2}]^*(WA)^2Wb + [(WA)^{k+2}]^*(WA)^3y \\ &- [(WA)^{k+2}]^*(WA)^2WAy \\ &= [(WA)^{k+2}]^*(WA)^2Wb, \end{aligned}$$

i.e. (21) is satisfied for x.

When x is a solution of (21), it follows

$$\begin{aligned} A^{\dagger,WG,W}b &= A^\dagger A[(WA)^{\mathrm{D}}]^3(WA)^k[(WA)^k]^\dagger(WA)^2Wb \\ &= A^\dagger A[(WA)^{\mathrm{D}}]^3P_{\mathcal{R}((WA)^k)}(WA)^2Wb \\ &= A^\dagger A[(WA)^{\mathrm{D}}]^3P_{\mathcal{R}((WA)^{k+2})}(WA)^2Wb \\ &= A^\dagger A[(WA)^{\mathrm{D}}]^3(WA)^{k+2}[(WA)^{k+2}]^\dagger(WA)^2Wb \\ &= A^\dagger A[(WA)^{\mathrm{D}}]^3\left([(WA)^{k+2}]^\dagger\right)^*[(WA)^{k+2}]^*(WA)^2Wb \\ &= A^\dagger A[(WA)^{\mathrm{D}}]^3\left([(WA)^{k+2}]^\dagger\right)^*[(WA)^{k+2}]^*(WA)^3x \\ &= A^{\dagger,WG,W}Ax. \end{aligned}$$

So,

$$x = A^{\dagger,WG,W}b + x - A^{\dagger,WG,W}Ax = A^{\dagger,WG,W}b + (I - A^{\dagger,WG,W}A)x,$$

i.e., x is presented as in (22). $\square$

Clearly, the equation (21) holds if and only if

$$[(WA)^k]^*(WA)^3x = [(WA)^k]^*(WA)^2Wb$$

is satisfied.

To find appropriate approximations to inconsistent linear equation $Ax = b$, $A \in \mathbb{C}^{m\times n}$, $x \in \mathbb{C}^n$ and $b \in \mathbb{C}^m$, one typical approach is to asks for, so called, generalized solutions, defined as solutions to $BAx = Bb$ with respect to an appropriate matrix $B \in \mathbb{C}^{n\times m}$ [20]. One particular choice is $B = A^*$, which leads to widely used least-squares solutions obtained as solutions to the normal equation $A^*Ax = A^*b$. When $m = n$, another important choice is $B = A^k$ with $k = ind(A)$, which leads to the so called Drazin normal equation $A^{k+1}x = A^kb$ and usage of the Drazin inverse solution $A^{\mathrm{D}}b$. Our equation (21) is one more case of $BAx = Bb$.

Theorem 5.1 presents an application to solve certain class of linear equation by means of the W-MPWG inverse. It is know, by Theorem 2.4, that $A^{\dagger,WG,W}$ is an outer inverse of A with prescribed range and null space. Since outer inverses with prescribed range and null space have a remarkable significance in matrix theory (i.e. application in defining iterative methods for solving nonlinear equations, in statistics in stable approximations of ill-posed problems and in linear and nonlinear problems involving rank-deficient generalized inverses [2]), the W-MPWG inverse can provide a starting point for future research [15, 16].

As Theorem 5.1, we can verify the following for $A^{\dagger,D,W} = A^\dagger A^{\mathrm{D},W}WAW$ and $A^\dagger A^{\oplus,W}WAW$.

Remark 5.1. *For $\{A, W\} \in \mathbb{C}^{m,n,k}$, the general solution of equation*

(i) $(AW)^kAx = (AW)^kb,\ b \in \mathbb{C}^m$ *is expressed as*

$$x = A^{\dagger,D,W}b + (I - A^\dagger A)y;$$

(ii) $[(WA)^k]^*(WA)^2x = [(WA)^k]^*WAWb,\ b \in \mathbb{C}^m$ *is expressed as*

$$x = A^\dagger A^{\oplus,W}WAWb + (I - A^\dagger A)y;$$

for arbitrary $y \in \mathbb{C}^n$.

Under the additional hypothesis, the solution of equation (21) can be unique.

Theorem 5.2. *For* $\{A, W\} \in \mathbb{C}^{m,n,k}$, $x = A^{\dagger,WG,W}b$ *is the unique solution in* $\mathcal{R}(A^{\dagger}(AW)^k)$ *of equation* (21).

Proof. Applying Theorem 5.1, $x = A^{\dagger,WG,W}b$ is a solution to (21). Since $\mathcal{R}(A^{\dagger,WG,W}) = \mathcal{R}(A^{\dagger}(AW)^k)$ by Theorem 2.5, $x = A^{\dagger,WG,W}b \in \mathcal{R}(A^{\dagger}(AW)^k)$.

If $x, x_1 \in \mathcal{R}(A^{\dagger}(AW)^k)$ are two solutions of (21), we have

$$\begin{aligned} x - x_1 &\in \mathcal{R}(A^{\dagger}(AW)^k) \cap \mathcal{N}([(WA)^{k+2}]^*(WA)^3) \\ &= \mathcal{R}(A^{\dagger,WG,W}A) \cap \mathcal{N}(A^{\dagger,WG,W}A) = \{0\}. \end{aligned}$$

Hence, $x = A^{\dagger,WG,W}b$ is the uniquely determined solution in $\mathcal{R}(A^{\dagger}(AW)^k)$ of equation (21). $\square$

Theorem 5.3. *For* $\{A, W\} \in \mathbb{C}^{m,n,k}$, $x = A^{\dagger,WG,W}b$ *is the unique solution of restricted equation*

$$Ax = b, \qquad b \in \mathcal{R}((AW)^k), \quad x \in \mathcal{R}(A^{\dagger}(AW)^k). \tag{23}$$

Proof. Because $b \in \mathcal{R}((AW)^k) = \mathcal{R}(AA^{\dagger,WG,W})$, note that $b = AA^{\dagger,WG,W}b$. Thus, $x = A^{\dagger,WG,W}b \in \mathcal{R}(A^{\dagger}(AW)^k)$ is a solution to (23).

For two solutions $x = A^{\dagger,WG,W}b, x_1 \in \mathcal{R}(A^{\dagger}(AW)^k) = \mathcal{R}(A^{\dagger,WG,W}A)$ of (23), we see that $x = A^{\dagger,WG,W}b = A^{\dagger,WG,W}Ax_1 = x_1$. $\square$

Recall that $x = A^{-1}b$ is unique solution of equation $Ax = b$, for the invertible matrix $A \in \mathbb{C}^{n\times n}$ and $b \in \mathbb{C}^n$. Substituting l-th column of A by b, we obtain a matrix $A(l \to b)$. According to [1, 30], the Cramer's rule for the solution $x = (x_1, x_2, \ldots, x_n)^\top$ of nonsingular equation $Ax = b$ is:

$$x_l = \frac{\det(A(l \to b))}{\det(A)}, \quad l = \overline{1, n}. \tag{24}$$

The Cramer's rule for finding the unique solution of (23) in $\mathcal{R}(A^{\dagger}(AW)^k))$, is provided now.

Theorem 5.4. *Under assumptions of Theorem 3.2, the unique solution* $x = (x_1, x_2, \ldots, x_n)^\top$ *of* (23) *in* $\mathcal{R}(A^{\dagger}(AW)^k)$ *can be expressed componentwise by*

$$x_l = \det\left(\begin{bmatrix} A(l \to b) & M \\ N(l \to 0) & 0 \end{bmatrix}\right) / \det\left(\begin{bmatrix} A & M \\ N & 0 \end{bmatrix}\right), \quad l = \overline{1, n}.$$

Proof. Applying Theorem 5.3, $x = A^{\dagger,WG,W}b$ is the unique solution in $\mathcal{R}(A^\dagger(AW)^k)) = \mathcal{N}(N)$ of (23). So, $Nx = 0$ and

$$\begin{bmatrix} A & M \\ N & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

The components of $x = A^{\dagger,WG,W}b$ can be obtained using Theorem 3.2 and the Cramer rule (24). □

6 Conclusion

To generalize the concept of the MPWG inverse for square matrices to rectangular matrices, we define the weighted MP weak group inverse. Thus, a new and wider class of generalized inverses is introduced. We develop a number of characterizations and representations of W-MPWG inverse. The W-MPWG inverse for block matrices is considered. We apply the W-MPWG inverse in solving some linear equations and present their solutions. In this way, we extend the results for the MPWG inverse and give some new results for the MPWG inverse.

Following increasing interest for the combination of WG and MP inverses, we expect further intervention on this topic and for next research can be considered:

1. extensions to tensor;
2. iterative methods for their approximation;
3. Recurrent neural network (RNN) models for their computation.

Acknowledgements. The first author is supported by the Ministry of Education, Science and Technological Development, Republic of Serbia, grant No. 451-03-47/2023-01/200124. The second author acknowledges the financial support from the Slovene Research Agency, ARRS (research core funding No. P1-0288). The authors also acknowledge the bilateral project between Serbia and Slovenia (Generalized inverses, operator equations and applications) was financially supported by the Ministry of Education, Science and Technological Development, Republic of Serbia (grant No. 337-00-21/2020-09/32) and by the Slovene Research Agency, ARRS (grant BI-RS/20-21-039).

References

[1] A. Ben-Israel, *A Cramer rule for least-norm solutions of consistent linear equations*, Linear Algebra Appl. 43 (1982), 223–226.

[2] A. Ben-Israel, T.N.E. Grevile, *Generalized inverses, theory and applications, Second edition*, Canadian Mathematical Society, Springer, New York, Beflin, Heidelberg, Hong Kong, London, Milan, Paris, Tokyo, 2003.

[3] R. Behera, G. Maharana, J.K. Sahoo, *Further results on weighted core-EP inverse of matrices*, Results Math 75, 174 (2020), https://doi.org/10.1007/s00025-020-01296-z.

[4] C.Y. Deng, H.K. Du, *Representations of the Moore-Penrose inverse of* 2×2 *block operator valued matrices*, J. Korean Math. Soc. 46(6) (2009), 1139–1150.

[5] G. Dolinar, B. Kuzma, J. Marovt, B. Ungor, *Properties of core-EP order in rings with involution*, Front. Math. China 14 (2019), 715–736.

[6] D.E. Ferreyra, F.E. Levis, A.N. Priori, N. Thome, *The weak core inverse*, Aequat. Math. 95 (2021), 351–373.

[7] D.E. Ferreyra, F.E. Levis, N. Thome, *Revisiting the core EP inverse and its extension to rectangular matrices*, Quaest. Math. 41(2) (2018), 265–281.

[8] D.E. Ferreyra, V. Orquera, N. Thome, *A weak group inverse for rectangular matrices*, Rev. R. Acad. Cienc. Exactas Fí́s. Nat. Ser. A Mat. RACSAM 113 (2019), 3727–3740.

[9] Z. Fu, K. Zuo, Y. Chen, *Further characterizations of the weak core inverse of matrices and the weak core matrix*, AIMS Mathematics, 7(3) (2021), 3630–3647.

[10] Y. Gao, J. Chen, *Pseudo core inverses in rings with involution*, Comm. Algebra 46(1) (2018), 38–50.

[11] Y. Gao, J. Li, *Perturbation bounds of core inverse under the Frobenius norm*, Bull. Malays. Math. Sci. Soc. 45 (2022), 2361–2371.

[12] C.W. Groetsch, *Generalized inverses of linear operators: representation and approximation*, in: Monographs and Textbooks in Pure and Applied Mathematics, Vol. 37, Marcel Dekker, Inc., New York, Basel, 1977.

[13] Y. Ke, L. Wang, J. Chen, *The core inverse of a product and* 2×2 *matrice*, Bull. Malays. Math. Sci. Soc. 42 (2019), 51–66.

[14] I.I. Kyrchei, *Determinantal representations of the core inverse and its generalizations with applications*, Journal of Mathematics 2019 (2019), Article ID 1631979, 13 pages.

[15] I.I. Kyrchei, *Weighted quaternion core-EP, DMP, MPD, and CMP inverses and their determinantal representations*, Rev. R. Acad. Cienc. Exactas Fı́s. Nat. Ser. A Mat. RACSAM (2020), 114:198, https://doi.org/10.1007/s13398-020-00930-3.

[16] I.I. Kyrchei, D. Mosić, P.S. Stanimirović, *Weighted minimization problems for quaternion matrices*, Adv. Appl. Clifford Algebras 31, 48 (2021), https://doi.org/10.1007/s00006-021-01153-4.

[17] N. Liu, H. Wang, *The characterizations of WG matrix and its generalized CayleyHamilton theorem*, Journal of Mathematics 2021 (2021), Article ID 4952943, 10 pages, https://doi.org/10.1155/2021/4952943.

[18] X. Liu, Y. Yu, J. Zhong, Y. Wei, *Integral and limit representations of the outer inverse in Banach space*, Linear Multilinear Algebra 60 (2012), 333–347.

[19] H. Ma, P.S. Stanimirović, *Characterizations, approximation and perturbations of the core-EP inverse*, Appl. Math. Comput. 359 (2019), 404–417.

[20] G. Maess, *Projection methods solving rectangular systems of linear equations*, J. Comput. Appl. Math. 24 (1988), 107–119.

[21] D. Mosić, *Maximal classes of operators determining some weighted generalized inverses*, Linear Multilinear Algebra 68(11) (2020), 2201–2220.

[22] D. Mosić, *Perturbation of the weighted core-EP inverse*, Ann. Funct. Anal. 11 (2020), 75–86.

[23] D. Mosić, *Weighted core–EP inverse of an operator between Hilbert spaces*, Linear Multilinear Algebra 67(2) (2019), 278–298.

[24] D. Mosić, *Weighted core-EP inverse and weighted core-EP pre-orders in a C^*-algebra*, J. Aust. Math. Soc. 111(1) (2021), 76–110.

[25] D. Mosić, J. Marovt, *Weighted weak core inverse of operators*, Linear Multilinear Algebra 70(20) (2022), 4991–5013.

[26] D. Mosić, P.S. Stanimirović, *Expressions and properties of weak core inverse*, Appl. Math. Comput. 415 (2022), 126704.

[27] D. Mosić, P.S. Stanimirović, *Representations for the weak group inverse*, Appl. Math. Comput. 397 (2021), 125957.

[28] D. Mosić, D. Zhang, *Weighted weak group inverse for Hilbert space operators*, Front. Math. China 15 (2020), 709–726.

[29] K.M. Prasad, K.S. Mohana, *Core-EP inverse*, Linear Multilinear Algebra 62(6) (2014), 792–802.

[30] S.M. Robinson, *A short proof of Cramer's rule*, Math. Mug. 43 (1970), 94–95.

[31] J.K. Sahoo, R. Behera, P.S. Stanimirović, V.N. Katsikis, H. Ma, *Core and core-EP inverses of tensors*, Comput. Appl. Math. 39, 9 (2020), doi:10.1007/s40314-019-0983-5.

[32] P.S. Stanimirović, *Limit representations of generalized inverses and related methods*, Appl. Math. Comput. 103 (1999), 51–68.

[33] K.S. Stojanović, D. Mosić, *Weighted MPCEP inverse of an operator between Hilbert spaces*, Bull. Iran. Math. Soc. 48 (2022), 53–71.

[34] C. Wang, X. Liu, H. Jin, *The MP weak group inverse and its application*, Filomat 36(18) (2022), 6085–6102.

[35] H. Wang, *Core-EP decomposition and its applications*, Linear Algebra Appl. 508 (2016), 289–300.

[36] H. Wang, J. Chen, *Weak group inverse*, Open Mathematics 16 (2018), 1218–1232.

[37] H. Wang, X. Liu, *The weak group matrix*, Aequat. Math. 93 (2019), 1261–1273.

[38] Y. Wei, D.S. Djordjević, *On integral representation of the generalized inverse $A^{(2)}_{T,S}$*, Appl. Math. Comput. 142 (2003), 189–194.

[39] H. Yan, H. Wang, K. Zuo, Y. Chen, *Further characterizations of the weak group inverse of matrices and the weak group matrix*, AIMS Mathematics, 6(9) (2021), 9322–9341.

[40] M. Zhou, J. Chen, Y. Zhou, *Weak group inverses in proper $*$-rings*, J. Algebra Appl. 19(12) (2020), 2050238.

[41] M. Zhou, J. Chen, Y. Zhou, N. Thome, *Weak group inverses and partial isometries in proper $*$-rings*, Linear Multilinear Algebra (2021), https://doi.org/10.1080/03081087.2021.1884639.

[42] Y. Zhou, J. Chen, *Weak core inverses and pseudo core inverses in a ring with involution*, Linear Multilinear Algebra (2021), https://doi.org/10.1080/03081087.2021.1971151.

[43] H. Zhu, Q.W. Wang, *Weighted pseudo core inverses in rings*, Linear Multilinear Algebra 68 (2020), 2434–2447.

Dijana MOSIĆ,
Faculty of Sciences and Mathematics,
University of Niš,
P.O. Box 224, 18000 Niš, Serbia.
Email: dijana@pmf.ni.ac.rs

Janko MAROVT,
University of Maribor,
Faculty of Economics and Business,
Razlagova 14, SI-2000 Maribor, Slovenia,
and IMFM,
Jadranska 19, SI-1000 Ljubljana, Slovenia.
Email: janko.marovt@um.si