DOI: 10.2478/auom-2024-0011
An. Şt. Univ. Ovidius Constanţa **Vol. 32(1),2024, 201–219**

On nonnil-S-Noetherian and nonnil-u-S-Noetherian rings

Najib Mahdou, El Houssaine Oubouhou and Ece Yetkin Celikel

Abstract

Let R be a commutative ring with identity, and let S be a multiplicative subset of R. Then R is called a nonnil-S-Noetherian ring if every nonnil ideal of R is S-finite. Also, R is called a u-S-Noetherian ring if there exists an element $s \in S$ such that for each ideal I of R, $sI \subseteq K$ for some finitely generated sub-ideal K of I. In this paper, we examine some new characterization of nonnil-S-Noetherian rings. Then, as a generalization of nonnil-S-Noetherian rings and u-S-Noetherian rings, we introduce and investigate the nonnilu-S-Noetherian rings class.

1 Introduction

Throughout this paper, it is assumed that all rings are commutative with nonzero identity. If R is a ring, we denote by $Nil(R)$ the ideal of all nilpotent elements of R. Recall that an ideal I of R is said to be a nonnil ideal if $I \nsubseteq Nil(R)$. A nonempty subset S of R is said to be a multiplicative subset if $1 \in S$, and for each $a, b \in S$ we have $ab \in S$.

Badawi established the concept of nonnil-Noetherian rings in [2]. Remember that a commutative ring R is Nonnil-Noetherian if every nonnil ideal of R is finitely generated. Many of the features of Noetherian rings are analogously

Key Words: Nonnil-S-Noetherian rings, Nonnil-u-S-Noetherian rings, Nonnil-Noetherian rings.
2010 Mathematics Subject Classification: Primary 13A15; Secondary 13A99.
Received: 27.02.2023
Accepted: 31.05.2023

proved for the Nonnil-Noetherian rings. In [2], the trivial extension construction is provided to give examples of nonnil-Noetherian rings which are not Noetherian rings.

In [1], Anderson and Dumitrescu introduced the notion of S-Noetherian rings as a generalization of Noetherian rings. Let R be a ring, S be a multiplicative set of R, and M be an R-module. We say that M is S-finite if there exist a finitely generated submodule F of M and $s \in S$ such that $sM \subseteq F$. Also, we say that M is S-Noetherian if each submodule of M is S-finite. A ring R is said to be S-Noetherian if it is S-Noetherian as an R-module (i.e., if each ideal of R is S-finite). In addition, they gave various construction of the S-variants of the well-known results for Noetherian rings: S-versions of Cohens result, the Eakin-Nagata theorem, the Hilbert Basis theorem, and under certain supplementary hypothesis. In particular, they studied the transfer of the S-Noetherian property to the ring of polynomials and the ring of formal power series. In [10] a ring R is said to be a uniformly S-Noetherian (u-S-Noetherian for abbreviation) provided there exists an element $s \in S$ such that for any ideal I of R, $sI \subseteq K$ for some finitely generated sub-ideal K of I. Trivially, Noetherian rings are u-S-Noetherian, and u-S-Noetherian rings are S-Noetherian.

In [8], Known and Lim introduced the notion of nonnil-S-Noetherian rings as a generalization of both nonnil-Noetherian rings and S-Noetherian rings. Let R be a ring, S be a multiplicative set of R. Then R is said to be a nonnil-S-Noetherian ring if each nonnil ideal of R is S-finite. If S consists of units of R, then the concept of S-finite ideals is the same as that of finitely generated ideals; so if S consists of units of R, then the notion of nonnil-S-Noetherian rings is identical to that of nonnil-Noetherian ring. Moreover, if $Nil(R) = 0$, then the concept of nonnil-S-Noetherian rings is exactly the same as that of S-Noetherian rings. Obviously, if $S_1 \subseteq S_2$ are multiplicative subsets, then any nonnil-S_1-Noetherian ring is nonnil-S_2-Noetherian; and if S^* is the saturation of S in R, then R is a nonnil-S-Noetherian ring if and only if R is a nonnil-S^*-Noetherian ring. The nonnil-S-Noetherian rings was studied in [8] using the Cohen-type theorem, the flat extension, the faithfully flat extension, the polynomial ring extension and the power series ring extension.

Let A and B be two rings, J an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A \ \ and \ j \in J\}$$

is called the amalgamation of A and B along J with respect to f. This construction is a generalization of the amalgamated duplication of a ring along an ideal denoted by $A \bowtie I$ (introduced and studied by D'Anna and Fontana in [4]). The interest of amalgamation resides, partly, in its ability to cover several basic constructions in commutative algebra, pullbacks and trivial ring extensions. See for instance [5, 7].

This paper consists of three sections including introduction. In Section 2, we look at several new nonnil-S-Noetherian ring properties. First, we establish the Eakin-Nagata-Formanek Theorem for nonnil-S-Noetherian ring. After that we show that the polynomial ring $R[X]$ is a nonnil-S-Noetherian ring if and only if it is S-Noetherian, and we also give a characterisation when a ring is nonnil-S-Noetherian by using the polynomial ring. In the case when R is a ϕ-ring, R is a nonnil-S-Noetherian ring if and only if $R/Nil(R)$ is a $\overline{S}$-Noetherian domain with $\overline{S} = S + Nil(R)$. The characterize of the amalgamation $A \bowtie^f J$ to be nonnil-S-Noetherian provided that is a ϕ-ring which brings this section to a close.

However, in the definition of nonnil-S-Noetherian rings, the choice of $s \in S$ such that $sI \subseteq K \subseteq I$ with K finitely generated is dependent on the nonnil ideal I. This dependence sets many obstacles to the further study of nonnil-S-Noetherian rings. The main motivation of section 3 of this work is to introduce and study a uniform version of nonnil-S-Noetherian rings. In fact, if there exists an element $s \in S$ such that for any nonnil ideal I of R, $sI \subseteq K$ for some finitely generated sub-ideal K of I, we say that a ring R is nonnil uniformly S-Noetherian (nonnil-u-S-Noetherian for short). Trivially, nonnil-Noetherian and nonnil-u-S-Noetherian rings are nonnil-S-Noetherian.

2 On nonnil-S-Noetherian rings

Let R be a commutative ring and S be a multiplicative set of R. Then if there exists $s \in S \cap Nil(R)$, so there exists a positive integre n such that $0 = s^n \in S$. Hence in this paper we always assume that $S \cap Nil(R) = \emptyset$. If $Nil(R)$ is a prime ideal of R, Then a nonnil ideal I is S-finite if and only if there is $s \in S$ and a nonnil finitely generated ideal F such that $sI \subseteq F \subseteq I$.

Recall that a ring R is called a ϕ-von Neumann regular ring if $R/Nil(R)$ is a field by [12, Theorem 4.1]. We begin this section with the following theorem, which defines when each S-Noetherian (Resp., u-S-Noetherian) R-module is Noetherian, for each multiplicative subset S of R.

Theorem 2.1. *Let R be a ring. Then the following conditions are equivalent:*

1. *For every multiplicative subset $S \subseteq R \setminus Nil(R)$, an $R-$module is S-Noetherian if and only if it is Noetherian,*
2. *For every multiplicative subset $S \subseteq R \setminus Nil(R)$, an R-module is u-S-Noetherian if and only if it is Noetherian,*
3. *R is a ϕ-von Neumann regular ring.*

Proof. (1) $\Rightarrow$ (2) Straightforward.

(2) $\Rightarrow$ (3) Let $a \in R \setminus Nil(R)$. Set $S = \{a^n \mid n \in \mathbb{N}\}$. Consider the following R-module $M = \bigoplus_{i\in\mathbb{N}} R/aR$. Since $aM = 0$, M is u-S-Noetherian. Then M is Noetherian and consequently $R/aR = 0$, so a is a unit, hence every non nilpotent element in R is a unit, thus $(R, Nil(R))$ is a local ring. Therefore, R is a ϕ-Von Neumann regular ring.

(3) $\Rightarrow$ (1) Let S be a multiplicative subset S of R. Then $S \subseteq R \setminus Nil(R) = U(R)$, so every element in S is a unit. Therefore an R-module M is S-Noetherian if and only if it is Noetherian. □

In order to generalize some known results on nonnil-S-Noetherian rings. We start with recalling the following definitions.

Definition 2.2. *Let R be a commutative ring, $S \subseteq R$ be a multiplicative set, and M an R-module.*

1. *An ascending chain $(N_n)_{n\in\mathbf{N}}$ of submodules of M is called S-stationary if there exists a positive integer k and $s \in S$ such that for each $n \geq k, sN_n \subseteq N_k$.*
2. *Let Ω be a family of submodules of M. An element $N \in \Omega$ is said to be S-maximal if there exists $s \in S$ such that for each $L \in \Omega$, if $N \subseteq L$ then $sL \subseteq N$.*

Now, we will give Eakin-Nagata-Formanek Theorem for nonnil-S-Noetherian rings for any multiplicative subset S of R.

Theorem 2.3. *Let R be a ring and let S be a multiplicative subset of R. Then the following conditions are equivalent:*

1. *Every nonempty family of nonnil-ideals has an $S-$maximal element,*
2. *R is nonnil-S-Noetherian,*
3. *Every ascending chain of nonnil-ideals of R is S-stationary,*

4. For every nonnil-ideal I of R, R/I is a $\overline{S}$-Noetherian ring with $\overline{S} = S+I$.

Proof. (1) $\Rightarrow$ (2) Let I be a nonnil ideal of R. Set Ω be the set of S-finite nonnil ideals of R wich are included in I. Since I is a nonnil ideal of R, there exists $a \in R \setminus Nil(R)$ such that $a \in I$. Hence $aR \in \Omega$, so Ω is nonempty. By assumption Ω has an S-maximal element L. Therefore, there exists $s_1 \in S$ such that if $J \in \Omega$ and $L \subseteq J$, then $s_1 J \subseteq L$. On the other hand L is S-finite, then there exists $s_2 \in S$, $x_1, .., x_n \in L$ such that $s_2 L \subseteq F = x_1 R + .. + x_n R$. Now, our aim is to prove that $s_1 s_2 I \subseteq F$. For this, let $\alpha \in I$. If $\alpha \in F$, then $s_1 s_2 \alpha \in F$. If $\alpha \notin F$, set $Q = L + \alpha R$, then $Q \subseteq I$ and Q is S-finite nonnil ideal of R. Hence $Q \in \Omega$. Since $L \subseteq Q$, then by S-maximality of L, $s_2 Q \subseteq L$. Therefore, $s_1 s_2 \alpha \in s_1 s_2 Q \subseteq s_1 L \subseteq F$. Hence $sI \subseteq F \subseteq I$ for $s = s_1 s_2 \in S$. Thus R is a nonnil-S-Noetherian ring.

(2) $\Rightarrow$ (3) Let $(I_n)_{n\in\mathbb{N}}$ be an ascending chain of nonnil ideals of R. Let $I = \bigsqcup_{n\in\mathbb{N}} I_n$ is a nonnil ideal of R. Since by hypothesis I is S-finite, then there exists $s \in S$ and $a_1, .., a_p \in I$ such that $sI \subseteq F = Ra_1 + .. + Ra_p$. Hence there exists $k \in \mathbb{N}$ such that $F \subseteq I_k$. So $sI_n \subseteq sI \subseteq F \subseteq I_k$ for any $n \geq k$. Thus, $(I_n)_{n\in\mathbb{N}}$ is S-stationary.

(3) $\Rightarrow$ (4) Let I be a nonnil ideal of R. Let $L_1/I \subseteq L_2/I \subseteq ...$ be an ascending chain of non zero ideal of R/I. Then $L_1 \subseteq L \subseteq L_2 \subseteq ...$ is an ascending chain of nonnil ideal of R. Hence by hypothesis there exists $s \in S$ and $k \in \mathbb{N}$ such that $sL_{n+1} \subseteq L_n$ for every $n > k$. So $\overline{s}L_{n+1}/I \subseteq L_n/I$ for every $n > k$. Hence $(L_n/I)_{n\in\mathbb{N}}$ is $\overline{S}$-stationary. Thus, R/I is $\overline{S}$-Noetherian.

(4) $\Rightarrow$ (1) Let Ω be a non empty set of nonnil-ideal of R which is not satisfying the property in (1). Then for every $I \in \Omega$ and every $s \in S$ there exists $J \in \Omega$ such that $I \subseteq J$ and $sJ \nsubseteq I$. Let $I \in \Omega$ and set $\Theta = \{J \in \Omega \mid I \subseteq J\}$. Then Θ is also does not have an S-maximmal element. Hence $\Lambda = \{J/I \mid J \in \Theta\}$ is a set of ideals of R/I wich is also does not have an $\overline{S}$-maximmal element, which contraduces the fact that R/N is $\overline{S}$-Neotherian. □

Let R be a ring, M an $R-$module and $R \propto M$ the set of pairs (r, m) with component-by-component addition and multiplication defined by: $(r, m)(b, f) = (rb, rf + bm)$, is a unitary commutative ring called the trivial extension (or idealization) of R by M.

The following example shows that the polynomial ring over a nonnil-S-Noetherian ring need not be nonnil-S-Noetherian.

Example 2.4. *Let K be a field and E be a K-vector space of infinite dimensional and set $R = K \propto E$. Then R is a nonnil-S-Noetherian ring for every multiplicative subset S of R, and if $0 \notin S$ we have $R[X]$ is not a nonnil-S-Noetherian ring.*

Proof. We have $Nil(R) = 0 \propto E$ is a maximal ideal of R, so the unique nonnil ideal of R is R. Then R is a nonnil-S-Noetherian ring for every multiplicative subset S of R. If $S \cap Nil(R) = \emptyset$, then $S \subseteq U(R)$. Since E is a K-vector space infinite dimensional, $Nil(R) = 0 \propto E$ is not a finitely generated ideal of R by [2, Lemma 3.2]. By absurdity, assume that $R[X]$ is not a nonnil-S-Noetherian ring. Then, the nonnil ideal $\mathrm{Nil}(R) + XR[X]$ of $R[X]$ is finitely generated. Therefore:

$$\mathrm{Nil}(R) + X = P_1R[X] + \cdots + P_nR[X].$$

As a result, we get

$$\mathrm{Nil}(R) = P_1(0)R + \cdots + P_n(0)R.$$

Thus $\mathrm{Nil}(R)$ is a finitely generated ideal of R, which is absurd since $Nil(R)$ is not a finitely generated ideal of R. □

We next shows that the polynomial ring $R[X]$ is nonnil-S-Noetherian if and only if it is S-Noetherian.

Theorem 2.5. *Let R be a ring and S be a multiplicative subset of R. Then the following statements are equivalent:*

1. *$R[X]$ is a nonnil-S-Noetherian ring,*
2. *$R[X]$ is an S-Noetherian ring.*

Proof. (1) $\Rightarrow$ (2). Let P be a prime ideal of $R[X]$. If $Nil(R[X]) \nsubseteq P$, then P is a nonnil ideal of $R[X]$ so it is S-finite. If $P = Nil(R[X]) = Nil(R)[X]$. Since the nonnil ideal $\mathrm{Nil}(R) + XR[X]$ of $R[X]$ is S-finite, there exists $s \in S$ and $P_1, \cdots, P_n \in R[X]$ such that:

$$s(\mathrm{Nil}(R) + XR[X]) \subseteq P_1R[X] + \cdots + P_nR[X] \subseteq \mathrm{Nil}(R) + XR[X].$$

As a result, we get

$$s\,\mathrm{Nil}(R) \subseteq P_1(0)R + \cdots + P_n(0)R \subseteq \mathrm{Nil}(R).$$

Therefore,

$$s\,\mathrm{Nil}(R)[X] \subseteq P_1(0)R[X] + \cdots + P_n(0)R[X] \subseteq \mathrm{Nil}(R)[X].$$

Thus in all cases P is an S-finite ideal of $R[X]$. Then $R[X]$ is S-Noetherian by [1, Corollary 5].

(1) $\Rightarrow$ (2) Straightforward. □

Let R be a ring and S be a multiplicative subset of R. Recall that S is an anti-Archimedean subset of R if $\cap_{n\geq 1} s^n R \cap S \neq \emptyset$, for all $s \in S$. As a direct corollary of Theorem 2.5 and [1, Proposition 9], we deduce [8, Theorem 3].

Corollary 2.6. *Let R be a ring and $S \subseteq R$ an anti-Archimedean multiplicative subset of R. Then the following statements are equivalent:*

1. *$R[X_1, \cdots, X_n]$ is nonnil-S-Noetherian for every $n \in \mathbb{N}^*$,*
2. *$R[X_1, \cdots, X_n]$ is S-Noetherian for every $n \in \mathbb{N}^*$,*
3. *R is S-Noetherian.*

By using the polynomial ring, the following Theorem characterizes rings that are nonnil-S-Noetherian.

Theorem 2.7. *Let R be a ring and S be a multiplicative subset of R. Then the following conditions are equivalent:*

1. *R is a nonnil-S-Notherian ring,*
2. *$R[X]/X^{n+1}R[X]$ is a nonnil-$\overline{S}$-Notherian ring with $\overline{S} = S + X^{n+1}R[X]$ For every integer $n > 0$,*
3. *$R[X]/X^{n+1}R[X]$ is a nonnil-$\overline{S}$-Notherian ring with $\overline{S} = S + X^{n+1}R[X]$ For some integer $n > 0$.*

Proof. Let $n \in \mathbb{N}$ and set $U = X + X^{n+1}R[X]$. Then $R[X]/X^{n+1}R[X] = R[U] = R + RU + \cdots + RU^n$ since $U^{n+1} = 0$.

(1) $\Rightarrow$ (2) Let I be a nonnil prime ideal of $R[U]$. Then two cases are possibles:

Case 1: $U \in I$. Set $A_0 = \{f(0) \mid f(U) \in I\}$, then A_0 is an ideal of R. Assume that $A_0 \subseteq Nil(R)$. So for any $f(U) = a_0 + a_1U + \cdots + a_nU^n \in I$, there exists a positive integer m such that $a_0^m = 0$. Thus,

$$\begin{aligned}(f(U))^{m(n+1)} &= (a_0 + a_1U + \cdots + a_nU^n)^{m(n+1)}\\ &= (a_0^m + b_1U + \cdots + b_nU^n)^{n+1}\\ &= (U(b_1 + b_2U + \cdots + b_nU^{n-1})^{n+1}\\ &= 0\end{aligned}$$

which is impossible. So A_0 is a nonnil ideal of R. Hence there exists $s \in S$ and $x_1, \cdots, x_m \in A_0$ such that $sA_0 \subseteq F = x_1R + \cdots + x_mR$. On the other hand we have $I \subseteq A_0 + UR[U]$, for the converse. Let $a_0 \in A_0$, so $a_0 + a_1U + \cdots + a_nU^n \in I$ for some $a_1, \cdots, a_n \in R$. Then $a_0 \in I$ since $a_0 + a_1U + \cdots + a_nU^n = a_0 + U(a_1 + a_2U + \cdots + a_nU^{n-1}) \in I$ and $U \in I$. Hence $A_0 \subseteq I$, and consequently $I = A_0 + UR[U]$. Therefore

$$sI \subseteq sA_0 + sUR[U] \subseteq x_1R[U] + x_2R[U] + \cdots + x_mR[U] + UR[U] \subseteq I.$$

Thus, I is $\overline{S}$-finite.

Case 2: $U \notin I$**.** Set $A = \{$ the coefficient of $f(U) \mid f(U) \in I\}$. Then A is a nonnil ideal of R, so there exists $s \in S$ and $r_1, \cdots, r_m \in A$ such that $sA \subseteq r_1R + r_2R + \cdots + r_mR$, so for every $a_i \in A$, $sa_i = \sum_{j=1}^{m} r_j r_j^i$ for some $r_j^i \in R$. Hence for every $f(U) = \sum_{i=1}^{n} a_iU^i \in I$ we have:

$$\begin{aligned} sf(U) &= \sum_{i=1}^{n} sa_iU^i \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} r_j r_j^i U^i \\ &= \sum_{j=1}^{m} r_j \sum_{i=1}^{n} r_j^i U^i \\ &\in r_1R[U] + r_2R[U] + \cdots + r_mR[U]. \end{aligned}$$

Thus $sI \subseteq r_1R[U] + r_2R[U] + \cdots + r_mR[U]$. Now, let $f(U) = x_0 + x_1U + \cdots + x_nU^n \in I$. Then

$$U^n f(U) = U^n(x_0 + x_1U + \cdots + x_nU^n) = x_0U^n \in I.$$

Since $U \notin I$ and I is a prime ideal of $R[U]$, we get $x_0 \in I$. Therefore $x_1U + \cdots + x_nU^n \in I$, hence $(x_1U + \cdots + x_nU^n)U^{n-1} = x_1U^n \in I$. Since $U \notin I$ and I is a prime ideal of $R[U]$, $x_1 \in I$. Continuing this procedure yields that $x_i \in I$ for every $i \in \{0, 1, \cdots, n\}$. Hence $A \subseteq I$. Since $r_i \in A$ for all $i = 1, \cdots, m$, then all $r_i \in I$. Therefore

$$sI \subseteq r_1R[U] + r_2R[U] + \cdots + r_mR[U] \subseteq I.$$

Hence in both cases we have I is $\overline{S}$-finite. Thus, every nonnil prime ideal of $R[U]$ is $\overline{S}$-finite. Therefore $R[U]$ is a nonnil-$\overline{S}$-Noetherian ring by [8, Theorem 1].

(2) $\Rightarrow$ (3) Straightforward.

(3) $\Rightarrow$ (1) Let $I_1 \subseteq I_2 \subseteq \cdots$ be any ascending chain of nonnil ideals of R. then, $I_1R[U] \subseteq I_2R[U] \subseteq \cdots$ is an ascending chain of nonnil ideals of $R[U]$. So by Theorem 2.3 there exists $s \in S$ and a positive positive integer k such that $sI_{m+1}R[U] \subseteq I_mR[U]$ for every $m > k$. Hence $sI_{m+1} \subseteq I_m$. Thus R is a nonnil-S-Noetherian ring by Theorem 2.10. $\square$

As a consequence of the previous Theorem, we have the following two corollaries.

Corollary 2.8. *Let R be a ring, $X_1, X_2, \cdots, X_k$ a finite indeterminates over R, $n_1, n_2, \cdots, n_k \in \mathbb{N}$ and S be a multiplicative subset of R. Then R is a nonnil-S-Noetherian ring if and only if $R[X_1, \cdots, X_k]/(X_1^{n_1+1}, \cdots, X_k^{n_k+1})$ is a nonnil-$\bar{S}$-Noetherian ring with $\bar{S} = S + (X_1^{n_1+1}, \cdots, X_k^{n_k+1})$.*

Proof. It is easy to show that $R[X_1, \cdots, X_k]/(X_1^{n_1+1}, \cdots, X_k^{n_k+1}) \cong (R[X_1, \cdots, X_{k-1}]/(X_1^{n_1+1}, \cdots, X_{k-1}^{n_{k-1}+1}))[X_k]/(X_k^{n_k+1})$ via the isomorphism $\alpha : (R[X_1, \cdots, X_{k-1}]/(X_1^{n_1+1}, \cdots, X_{k-1}^{n_{k-1}+1})[X_k]/(X_k^{n_k+1}) \to R[X_1, \cdots, X_k]/(X_1^{n_1+1}, \cdots, X_k^{n_k+1})$, with $\alpha(\sum_{i=0}^n f_iX_k^i + (X_k^{n_k+1}) = \sum_{i=0}^n f_iX_k^i + \left(X_1^{n_1+1}, \cdots, X_k^{n_k+1}\right)$. Therefore, $R[X_1, \cdots, X_k]/(X_1^{n_1+1}, \cdots, X_k^{n_k+1})$ is a nonnil-$\bar{S}$-Noetherian ring if and only if R is nonnil-S-Noetherian. $\square$

Corollary 2.9. *Let R be a ring and S be a multiplicative subset of $R \propto R$. Set S' the trace of S in R. Then $R \propto R$ is a nonnil-S-Noetherian ring if and only if R is a nonnil-S'-Noetherian ring.*

Proof. Let S be a multiplicative subset of $R \propto R$ and S' its trace in R. Then S and $S' \propto 0$ have the same saturation. On the other hand we have $R \propto R \cong R[X]/(X^2)$ via the isomorphism $(a, b) \to a + bX$. Then by Theorem 2.7, we get $R \propto R$ is a nonnil-S-Noetherian ring if and only if R is a nonnil-S'-Noetherian ring. $\square$

Recall that a prime ideal P of R is called a divided prime if it is comparable to every ideal of R. Set $H = \{R \mid R$ is a commutative ring and $Nil(R)$ is a divided prime ideal of $R\}$. If $R \in H$, then R is called a ϕ-ring. For a ring $R \in H$, we have the following result.

Theorem 2.10. *Let R be a $\phi-$ring and S a multiplicative subset of R. Then R is a nonnil-S-Noetherian ring if and only if $R/Nil(R)$ is an $\overline{S}$-Noetherian domain with $\overline{S} = S + Nil(R)$.*

Proof. Assume that R is a is a nonnil-S-Noetherian ring. Set $A = R/Nil(R)$ and let Q be a non zero ideal of A. Then $Q = J/Nil(R)$ for some nonnil ideal J of R, hence there exists $s \in S$ and a finitely generated ideal F of R such that $sP \subseteq F \subseteq P$. Since F is a finitely generated ideal of R, $L = F/Nil(R)$ is a finitely generated ideal of A. Thus $\overline{s}Q \subseteq L \subseteq Q$, hence A is an $\overline{S}$-Noetherian domain.

Conversely, Assume that $A = R/Nil(R)$ is a is a $\overline{S}$-Noetherian ring. Let I be a nonnil ideal of R, since $Nil(R)$ is a divided ideal of R, $Nil(R) \subseteq I$. Then $J = I/Nil(R)$ is an ideal of A, so there exists $s \in S$ and $i_1, \cdots, i_n \in I$ such that $\overline{s}J \subseteq (i_1 + Nil(R), \cdots, i_n + Nil(R)) \subseteq J$. Let x be a non nilpotent element of I. Then $sx + \mathrm{Nil}(R) = c_1 i_1 + \cdots + c_n i_n + Nil(R)$ in A for some $c_1, \cdots, c_n \in R$. Hence there is $w \in \mathrm{Nil}(R)$ such that $sx + w = c_1 i_1 + \cdots + c_n i_n$ in R. Since $sx \in I \setminus \mathrm{Nil}(R)$, $Nil(R) \subseteq Rsx$, so $w = sxf$ for some $f \in \mathrm{Nil}(R)$. Hence $sx + w = sx + sxf = sx(1+f) = c_1 i_1 + \cdots + c_n i_n$ in R. Since $f \in Nil(R)$, $1+f$ is a unit of R. Thus $sx \in i_1 R + \cdots + i_n R$, Hence $sI \subseteq i_1 R + \cdots + i_n R \subseteq I$. Thus I is S-finite. Therefore R is a Nonnil-S-Noetherian ring. □

Assume that R is ϕ-ring, and set $\phi : R \to R_{Nil(R)}$ such that $\phi(r) = \frac{r}{1}$ for every $r \in R$. Then $R/Nil(R) \cong \phi(R)/Nil(\phi(R))$ by [2, Lemma 1.1]. We have the following corollary as a direct consequence of this result and the previous Theorem.

Corollary 2.11. *Let R be a ϕ-ring. Then the following statements are equivalent:*

1. *R is a Nonnil-S-Noetherian ring,*
2. *$R/\mathrm{Nil}(R)$ is an $\overline{S}$-Noetherian domain, with $\overline{S} = S + Nil(R)$,*
3. *$\phi(R)/\mathrm{Nil}(\phi(R))$ is an S'-Noetherian domain, with $S' = \phi(S) + Nil(\phi(R))$,*
4. *$\phi(R)$ is a Nonnil-$\phi(S)$-Noetherian ring.*

The next corollary studies when the amalgamated duplication $A \bowtie I$ is a nonnil-S-Noetherian ring, provided $A \bowtie I$ is a ϕ-ring.

Corollary 2.12. *Let A be a ring, I an ideal of A such that $A \bowtie I$ is a ϕ-ring. Let S be a multiplicative subset of $A \bowtie I$. Set S' the trace of S in A. Then $A \bowtie I$ is a nonnil-S-Noetherian ring if and only if A is a nonnil-S'-Noetherian ring.*

Proof. By [6, Theorem 2.1], it follows immediately that $I \subseteq Nil(A)$. Hence $Nil(A \bowtie I) = Nil(A) \bowtie I$, therefore $A \bowtie I/Nil(A \bowtie I) \cong A/Nil(A)$. Thus the conclusion is an easy consequence of Theorem 2.10. □

Recall from [6, Corollary 2.4], That the the trivial ring extension $A \propto E$ is a ϕ-ring if and only if A is a ϕ-ring and $E = aE$ for each $a \in A \setminus Nil(A)$. The following corollary is an immediate result of Corollary 2.12, which examines when the trivial ring extension is a nonnil-S-Noetherian ring.

Corollary 2.13. *Let A be a $\phi-$ring, E be an A-module such that $E = aE$ for every $a \in A \setminus Nil(A)$. Let S be a multiplicative subset of $A \propto E$. Set S' the trace of S in R. Then $A \propto E$ is a nonnil-S-Noetherian ring if and only if A is a nonnil-S'-Noetherian ring.*

Let A and B be two rings, J a nonzero ideal of B, and $f : A \to B$ be a ring homomorphism. Set $R := A \bowtie^f J$ and $N(J) := Nil(B) \cap J$. Recall from [6, Theorem 2.1] that (1) If J is a nonnil ideal of B, then R is a ϕ-ring if and only if $f^{-1}(J) = 0$, A is an integral domain, and $N(J)$ is a divided prime ideal of $f(A) + J$. (2) If $J \subseteq Nil(B)$, then R is a ϕ-ring if and only if A is a ϕ-ring, and for each $i, j \in J$ and each $a \in A \setminus Nil(A)$, there exists $x \in Nil(A)$ and $k \in J$ such that $xa = 0$ and $j = kf(a)+i(f(x)+k)$. Moreover let $\iota : A \to A \bowtie^f J$ be the natural embedding defined by $a \to (a, f(a))$ for each $a \in A$, and S a multiplicative subset of A, then $S' := \{(s, f(s)) \mid s \in S\}$ and $f(S)$ are multiplicative subsets of $A \bowtie^f J$ and B, respectively.

Theorem 2.14. *Let A and B be two rings, J a nonzero ideal of B, and let $f : A \to B$ be a ring homomorphism such that $A \bowtie^f J$ is a ϕ-ring and S a multiplicative subset of A. Then the following statements are equivalent:*

1. *$A \bowtie^f J$ is a nonnil-S'-Noetherian ring,*
2. *A is a nonnil-S-Noetherian ring and $f(A)+J$ is a nonnil-$f(S)$-Noetherian ring.*

Before proving Theorem 2.14, we need the following lemma of independent interest.

Lemma 2.15. *Let $\alpha : R \to R'$ be a surjective ring homomorphism and $S \subseteq R$ a multiplicative set of R. If R is nonnil-S-Noetherian, then R' is nonnil-$\alpha(S)$-Noetherian.*

Proof. let J be a nonnil ideal of R', then $J = f(I)$ for some nonnil ideal I of R. Since R is a nonnil-S-Northerian ring, there exist $x_1, \cdots, x_n \in I$ and $s \in S$ such that

$$sI \subseteq Rx_1 + ... + Rx_n \subseteq I.$$

Whence

$$f(s)J \subseteq R'f(x_1) + \cdots + R'f(x_n) \subseteq J.$$

So R' is nonnil-$\alpha(S)$-Noetherian. □

Proof of Theorem 2.14
(1) $\Rightarrow$ (2) Set $p_A : A \bowtie^f J \to A$ and $p_B : A \bowtie^f J \to f(A) + J$ the two canonical projections. Since $p_A(S') = S$ and $p_B(S') = f(S)$, we conclude that A is a nonnil-S-Noetherian ring and $f(A)+J$ is a nonnil-$f(S)$-Noetherian ring by lemma 2.15.

(2) $\Rightarrow$ (1) Set $\bar{A} = A/\operatorname{Nil}(A)$, $\bar{B} = B/\operatorname{Nil}(B)$, $\pi : B \to \bar{B}$ the canonical projection and $\bar{J} = \pi(J)$. Consider the ring homomorphism $\bar{f} : \bar{A} \to \bar{B}$ defined by setting $\bar{f}(\bar{a}) = \overline{f(a)}$. It is easy to see that $\bar{f}$ is well defined and it is clearly a ring homomorphism. The kernel of the restriction to $A \bowtie^f J$ of the canonical projection $A \times B \to \bar{A} \times \bar{B}$ is obviously $\operatorname{Nil}(A \bowtie^f J)$ and the image is $\bar{A} \bowtie^{\bar{f}} \bar{J}$ by the proof of [11, Theorem 2.7]. Hence, we have the following isomorphism of rings:

$$\begin{array}{ll} \varphi : (A \bowtie^f J)/\operatorname{Nilp}(A \bowtie^f J) & \longrightarrow \bar{A} \bowtie^{\bar{f}} \bar{J} \\ \overline{(a, f(a) + j)} & \longrightarrow (\bar{a}, \bar{f}(\bar{a}) + \bar{j}) \end{array}$$

on the other hand A and $f(A)+J$ are ϕ-rings by [6, Lemma 2.3]. Thus $\bar{A}$ is $\bar{S}$-Noetherian ring and $f(A) + J/Nil(f(A) + J) \cong \bar{f}(\bar{A}) + \bar{J}$ is $\bar{f}(\bar{S})$-Noetherian ring. So $\bar{A} \bowtie^{\bar{f}} \bar{J}$ is $\bar{S}'$-Noetherian domain by [9, Theorem 3.2]. Whence $A \bowtie^f J$ is a nonnil-S'-Noetherian ring by Theorem 2.10. □
In the case where $S = \{1\}$, we find the following result.

Corollary 2.16. *Let A and B be two rings, J a nonzero ideal of B, and let $f : A \to B$ be a ring homomorphism such that $A \bowtie^f J$ is a ϕ-ring. Then the following statements are equivalent:*

1. *$A \bowtie^f J$ is a nonnil-Noetherian ring,*
2. *A and $f(A) + J$ are nonnil-Noetherian rings.*

It must be noted that the autours of [11] have been studied when $A \bowtie^f J$ is a nonnil-Noetherian ring, and it shows that if $A \bowtie^f J$ is a ϕ-ring. Then $A \bowtie^f J$ is a nonnil-Noetherian ring if and only if A and $f(A) + J$ are nonnil-Noetherian rings and $f^{-1}(J) \subseteq Nil(A)$.

Remark 2.17. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B, if $A \bowtie^f J$ is a ϕ-ring, then $f^{-1}(J) \subseteq Nil(A)$ by [6, Lemma 2.3]. Whence our corollary 2.16 and [11, Theorem 2.7] are identical.*

The following example shows that the condition R is a ϕ-ring is a necessary condition in Theorem 2.14.

Example 2.18. *([11, Example 2.10])*
Set $A = \mathbb{Z} \propto \mathbb{Q}$ and consider the surjective ring homomorphism $f : A \to \mathbb{Z}/6\mathbb{Z}$; $f((n, q)) = \bar{n}$. Consider $J = 3\mathbb{Z}/6\mathbb{Z}$ the ideal of $\mathbb{Z}/6\mathbb{Z}$. Then, R and $f(A)+J$ are nonnil-Noetherian rings. However, $A \bowtie^f J$ is not.

3 On nonnil-u-S-Noetherian rings

Recall from [10] that a ring R is said to be a u-S-Noetherian provided there exists an element $s \in S$ such that for any ideal I of R, $sI \subseteq K$ for some finitely generated sub-ideal K of I. Now we state our definition of nonnil-u-S-Noetherian rings.

Definition 3.1. *Let R be a ring and S be a multiplicative subset of R. Then :*

1. *R is called a nonnil uniformly S-Noetherian (nonnil-u-S-Noetherian for abbreviation) ring provided there exists an element $s \in S$ such that for any nonnil ideal I of R there exists a finitely generated ideal F of R, $sI \subseteq F \subseteq I$.*
2. *R is called a nonnil uniformly S-Principal ideal ring (nonnil-u-S-PIR for short) provided there exists an element $s \in S$ such that for any nonnil ideal I of R there exists $a \in I$, $sI \subseteq Ra$.*

If S consists of units of R, then the notion of nonnil-u-S-Noetherian rings coincides with that of nonnil-Noetherian ring. Furthermore, if $Nil(R) = (0)$, then the concept of nonnil-u-S-Noetherian rings is precisely the same as that of u-S-Noetherian rings. Clearly, if $S_1 \subseteq S_2$ are multiplicative subsets, then any nonnil-u-S_1-Noetherian ring is nonnil-u-S_2-Noetherian; and if S^* is the saturation of S in R, then R is a nonnil-u-S^*-Noetherian ring if and only if R is a nonnil-u-S-Noetherian ring. Also, every nonnil-Noetherian ring is nonnil-u-S-Noetherian. However, the converse does not hold general.

Example 3.2. *Let $R = \prod_{i=1}^{\infty} \mathbb{Z}/4\mathbb{Z}$ be the countable infinite direct product of $\mathbb{Z}/4\mathbb{Z}$, then R is not nonnil-Noetherian. Let e_i be the element in R with the i-th component 1 and others 0. Denote $S = \{1, e_i \mid i = 1, 2 \cdots\}$. Then R is a nonnil-u-S-PIR, let I be a nonnil ideal of R. Then if all elements in I have 1-th components equal to 0, we have $e_1 I = 0$. Otherwise $e_1 I = e_1 R$ or $e_1 I = 2e_1 R$. Thus $e_1 I$ is principally generated. Consequently R is a nonnil-u-S-PIR, and so is nonnil-u-S-Noetherian.*

Proposition 3.3. *Let R be a ring and S a multiplicative subset of R consisting of finite elements. Then R is a nonnil-u-S-Noetherian ring (resp., nonnil-u-S-PIR) if and only if R is is a nonnil-S-Noetherian ring (resp., nonnil-S-PIR).*

Assume that R is a nonnil-u-S-Noetherian ring (resp., nonnil-u-S-PIR). Then trivially R is a is a nonnil-S-Noetherian ring (resp., nonnil-S-PIR).

Conversely, assume that $S = \{s_1, s_2, \cdots, s_n\}$, R is is a nonnil-S-Notherian ring (resp., nonnil-u-S-PIR) and set $s = s_1 s_2 \cdots s_n$. Then for any nonnil ideal I of R, there exists a finitely generated ideal (resp., principal ideal) J of R such that $s_I I \subseteq J \subseteq I$. Hence $sI \subseteq J \subseteq I$. Thus, R is a nonnil-u-S-Noetherian ring (resp., nonnil-u-S-PIR). □

The following example shows that a nonnil-S-Noetherian ring is not a nonnil-u-S-Noetherian ring in general.

Example 3.4. *Let K be a field and $X = \{X_1, X_2, ...\}$ be an infinite set of indeterminates over K, let $R = K[X]$ and set $S = R \setminus 0$. Then R is an S-Noetherian ring so it is a nonnil-S-Notherian ring. However, R is not a nonnil-u-S-Noetherian by [10, Example 2.5].*

Next, we will give Eakin-Nagata-Formanek Theorem for nonnil-u-S-Noetherian rings for any multiplicative subset S of R. First, recall from [10] the notions of stationary ascending chains of R-modules with respective to $s \in S$ and maximal elements of a family of $R-$modules with respective to s. Let R be a ring, S a multiplicative subset of R and M an $R-$module. Denote by M^* an ascending chain $M_1 \subseteq M_2 \subseteq ...$ of submodules of M. An ascending chain M^* is called stationary with respective to s if there exists $k \geq 1$ such that $sM_n \subseteq M_k$ for any $n \geq k$. Let $\{M_i\}_{i \in \Gamma}$ be a family of sub-modules of M. We say an R-module $M_0 \in \{M_i\}_{i \in \Gamma}$ is maximal with respective to s provided that if $M_0 \subseteq M_j$ for some $M_j \in \{M_i\}_{i \in \Gamma}$, then $sM_j \subseteq M_0$.

Theorem 3.5. *Let R be a ring and let S be a multiplicative subset of R. Then the following conditions are equivalent:*

1. *There exists $s \in S$ such that any nonempty family of nonnil ideals of R has an maximal element with respective to s,*
2. *R is nonnil-u-S-Noetherian,*
3. *There exists $s \in S$ such that any ascending chain of nonnil ideals of R is stationary with respective to s,*
4. *For every nonnil ideal I of R, R/I is a u-$\overline{S}$-Noetherian ring with $\overline{S} = S + I$.*

Proof. (1) ⇒ (2) Let $s_0 \in S$ the element in (1) and set $s = s_0^2 \in S$. Let I be a nonnil ideal of R. Set Ω be the set of s_0-finite nonnil ideals of R which are included in I. Since I is a nonnil ideal of R, there exists $a \in R \setminus Nil(R)$ such that $a \in I$. Hence $aR \in \Omega$, so Ω is nonempty. By assumption Ω has an has an maximal element L with respective to s_0. Therefore, for each $J \in \Omega$ such that $L \subseteq J$, $s_0 J \subseteq L$. On the other hand L is s_0-finite, then there exists $x_1, \cdots, x_n \in L$ such that $s_0 L \subseteq F = x_1 R + \cdots + x_n R$. Now, our aim is to prove that $sI \subseteq F$. For this, let $\alpha \in I$. If $\alpha \in F$, then $s\alpha \in F$. If $\alpha \notin F$, set $Q = L + \alpha R$, then $Q \subseteq I$ and Q is s_0-finite nonnil ideal of R. Hence $Q \in \Omega$. Since $L \subseteq Q$, then by maximality of L with respective to s_0, $s_0 Q \subseteq L$. Therefore, $s\alpha \in s_0 Q \subseteq s_0 L \subseteq F$. Hence $sI \subseteq F \subseteq I$. Thus R is a nonnil-u-S-Noetherian ring.

The rest of the proof is analogous to the proof of Theorem 2.3. □

Let P be a prime ideal of R. We say R is nonnil-u-P-Noetherian provided that is nonnil-u-$(R \setminus P)$-Noetherian. The next result gives a local characterization of nonnil-Noetherian rings.

Proposition 3.6. *Let R be a ring. Then the following conditions are equivalent:*

1. *R is a nonnil-Noetherian ring,*
2. *R is a nonnil-u-P-Noetherian ring for all primes ideal P of R,*
3. *R is a nonnil-u-M-Noetherian ring for all maximal ideals M of R.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Straightforward.
(3) $\Rightarrow$ (1) Assume that R is a nonnil-u-M-Noetherian ring for all maximal ideals M of R. Let I be a nonnil ideal of R, so for every maximal ideal M of R, there exist an element $s_M \in R \backslash M$ and a finitely generated ideal F_M of R such that $s_M I \subseteq F_M \subseteq I$. Let $S = \{s_M \mid M \text{ is a maximal ideal of } R\}$. Since S generated R, there exists finite elements $s_{M_1}, \cdots, s_{M_n}$ of S such that

$$I = (s_{M_1} R + \cdots + s_{M_n} R) I \subseteq F_{M_1} + \cdots + F_{M_n} \subseteq I,$$

which means that $I = F_{M_1} + \cdots + F_{M_n}$, so I is finitely generated. Therefore, R is a nonnil-Noetherian ring. □

Corollary 3.7. *Let R be a local ring with maximal ideal M, then R is a nonnil-Noetherian ring if and only if R is a nonnil-u-M-Noetherian ring.*

Let R be a commutative ring with identity. Recall that R is decomposable if $R = R_1 \oplus R_2$ for some nonzero rings R_1 and R_2.

Theorem 3.8. *Let R be a decomposable commutative ring with identity, S a multiplicative subset of R and $\{\pi_i\}_{i \in \Lambda}$ the set of canonical epimorphisms from R to each component of decompositions of R. Then the following statements are equivalent:*

1. *R is an u-S-Noetherian ring,*
2. *R is a nonnil-u-S-Noetherian ring,*
3. *For each $i \in \Lambda, \pi_i(R)$ is a u-$\pi_i(S)$-Noetherian ring,*
4. *If e is a nonzero non unit idempotent element of R, there exists $s_e \in S$ such that every ideal of R contained in eR is s_e-finite.*

Proof. (1) $\Rightarrow$ (2) Straightforward.

(2) $\Rightarrow$ (3) Let $i \in \Lambda$. Then $R = \pi_i(R) \oplus \pi_j(R)$ for some $j \in \Lambda$. Let s the element in (2), and let I be an ideal of $\pi_i(R)$. So $I \oplus \pi_j(R)$ is a nonnil ideal of $\pi_i(R) \oplus \pi_j(R)$. Then there exist a finitely generated ideal F of R such that $s\left(I \oplus \pi_j(R)\right) \subseteq F \subseteq I \oplus \pi_j(R)$. Therefore $s\pi_i(I) \subseteq \pi_i(F) \subseteq I$. Since F is a finitely generated ideal of R, $\pi_i(F)$ is a finitely generated ideal of $\pi_i(R)$. Therefore $\pi_i(R)$ is a u-$\pi_i(S)$-Noetherian ring.

(3) $\Rightarrow$ (4) Let e be a nonzero non unit idempotent element of R,. Then $R = Re \oplus R(1-e)$. Then $Re = \pi_i(R)$ for some $i \in \lambda$. Hence by the assumption, Re is a u-$\pi_i(S)$-Noetherian ring. Then there exists $s \in S$ such that every ideal of eR is $\pi_i(s)$-finite. Let I be an ideal of R contained in eR. So there exists a finitely generated ideal F of eR such that $\pi_i(s)I \subseteq F \subseteq I$. Since F is a finitely generated ideal of R_i, $E = F \oplus 0$ is a finitely generated ideal of R, and $sI \subseteq E \subseteq I$. Thus I is s-finite.

(4) $\Rightarrow$ (1) Let e be a nonzero non unit idempotent element of R, Then $R = Re \oplus R(1-e)$, Hence $Re = \pi_i(R)$ and $R(1-e) = \pi_j(R)$ for some $i, j \in \lambda$. Then by assumption there exists $s_i \in S$ (resp., $s_j \in S$) such that every ideal of R contained in eR (resp., $(1-e)R$) is s_i-finite (resp., s_j-finite). Set $s = s_i s_j \in S$. Let I be an ideal of R. Then $I = \pi_i(I) \oplus \pi_j(I)$. By assumption there exists finitely generated ideals E and F such that $s_i\pi_i(I) \subseteq E \subseteq \pi_i(I)$ and $s_j\pi_j(I) \subseteq F \subseteq \pi_j(I)$. Set $L = E \oplus F$, then L is a finitely generated ideal of R and we have $sI \subseteq L \subseteq I$, witch implies that I is s-finite, Thus R is a u-S-Noetherian ring. □

Corollary 3.9. *Let $n \geq 2$ be an integer, $R_1, \cdots, R_n$ rings with identity, and let $S_1, \cdots, S_n$ be multiplicative subsets of $R_1, \cdots, R_n$, respectively. Then the following assertions are equivalent:*

1. *$\prod_{i=1}^n R_i$ is a nonnil-u-($\prod_{i=1}^n S_i$)-Noetherian ring,*
2. *$\prod_{i=1}^n R_i$ is a u-($\prod_{i=1}^n S_i$)-Noetherian ring,*
3. *For all $i = 1, \cdots, n$, R_i is an u-S_i-Noetherian ring.*

For a ϕ-ring, we have the following result.

Theorem 3.10. *Let R be a ϕ-ring and S a multiplicative subset of R. Then R is a nonnil-u-S-Noetherian ring if and only if $R/Nil(R)$ is a u-$\overline{S}$-Noetherian domain with $\overline{S} = S + Nil(R)$.*

Proof. Analogue to Theorem 2.10. □

Corollary 3.11. *Let R be a ϕ-ring and S a multiplicative subset of R. Then the following statements are equivalent:*

1. *R is a Nonnil-u-S-Noetherian ring,*
2. *$R/Nil(R)$ is a u-$\overline{S}$-Noetherian domain with $\overline{S} = S + Nil(R)$,*
3. *$\phi(R)/\mathrm{Nil}(\phi(R))$ is a u-S'-Noetherian domain, with $S' = Nil(\phi(R)) + \phi(S)$,*
4. *$\phi(R)$ is a nonnil-u-$\phi(S)$-Noetherian ring.*

Let A and B be two rings, J a nonzero ideal of B, and let $f : A \to B$ be a ring homomorphism. Let $i : A \to A \bowtie^f J$ be the natural embedding defined by $a \to (a, f(a))$ for all $a \in A$. For a multiplicative subset S of A, put $S' := \{(s, f(s)) \mid s \in S\}$. Clearly, S' and $f(S)$ are multiplicative subsets of $A \bowtie^f J$ and B, respectively.

Theorem 3.12. *Let A and B be two rings, J a nonzero ideal of B, and let $f : A \to B$ be a ring homomorphism such that $A \bowtie^f J$ is a ϕ-ring, let S a multiplicative subset of A. Then the following statements are equivalent:*

1. *$A \bowtie^f J$ is a nonnil-u-S'-Noetherian ring.*
2. *A is a nonnil-u-S-Noetherian ring and $f(A) + J$ is a nonnil-u-$f(S)$-Noetherian ring.*

Before proving Theorem 2.14, we need the following lemma of independent interest.

Lemma 3.13. *Let $\alpha : R \to R'$ be a surjective ring homomorphism and $S \subseteq R$ a multiplicative set of R. If R is nonnil-u-S-Noetherian, then R' is nonnil-u-$\alpha(S)$-Noetherian.*

Proof. Let $s \in S$ the element such that every nonnil ideal of R is s-finite. Let J be a nonnil ideal of R', then $J = f(I)$ for some nonnil ideal I of R. Since R is a nonnil-u-S-Noetherian ring, there exist $x_1, \cdots, x_n \in I$ such that

$$sI \subseteq Rx_1 + \cdots + Rx_n \subseteq I.$$

Whence
$$f(s)J \subseteq R'f(x_1) + \cdots + R'f(x_n) \subseteq J.$$
So R' is nonnil-u-$\alpha(S)$-Noetherian. □

Proof of Theorem 3.12
(1) ⇒ (2) Set $p_A : A \bowtie^f J \to A$ and $p_A : A \bowtie^f J \to f(A) + J$ the two canonical projections. Since $p_A(S') = S$ and $p_B(S') = f(S)$, we conclude that A is a nonnil-u-S-Noetherian ring and $f(A)+J$ is a nonnil-u-$f(S)$-Noetherian ring.

(2) ⇒ (1) With the same notation in theorem 2.14, we have the following isomorphism of rings:

$$\begin{aligned} \varphi : (A \bowtie^f J)/\operatorname{Nilp}(A \bowtie^f J) &\longrightarrow \bar{A} \bowtie^{\bar{f}} \bar{J} \\ \overline{(a, f(a) + j)} &\longrightarrow (\bar{a}, \bar{f}(\bar{a}) + \bar{j}) \end{aligned}$$

on the other hand A and $f(A) + J$ are ϕ-rings by [6, Lemma 2.3]. Thus $\bar{A}$ is u-$\bar{S}$-Noetherian ring and $f(A) + J/Nil(f(A) + J) \cong \bar{f}(\bar{S}) + \bar{J}$ is u-$\bar{f}(\bar{S})$-Noetherian ring by [10, Lemma 3.3]. So $\bar{A} \bowtie^{\bar{f}} \bar{J}$ is $\bar{S}'$-Noetherian domain by [10, Proposition 3.4]. Whence $A \bowtie^f J$ is a nonnil-u-S'-Noetherian ring by Theorem 3.10. □

4 Declarations

There are no Funding and/or Conflicts of interests/Competing interests.

References

[1] D. D. Anderson and T. Dumitrescu, *S-Noetherian rings*, Commun. Algebra, 30(9) (2002), 4407-4416.

[2] A. Badawi, *On nonnil-Noetherian rings*, Commun. Algebra, 31(4) (2003), 1669–1677.

[3] Z. Bilgin, M. L. Reyes and U. Tekir, *On right S-Noetherian rings and S-Noetherian modules*, Commun. Algebra, 46(2) (2018), 863–869.

[4] M. D'Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl, 6(3) (2007), 443-459.

[5] M. D'Anna, C. A. Finacchiaro, and M. Fontana, *Amalgamated algebras along an ideal*, Comm. Algebra and Applications, Walter De Gruyter, (2009), 241-252.

[6] A. El Khalfi, H. Kim and N. Mahdou, *Amalgamated Algebras Issued from ϕ-Chained Rings and ϕ-Pseudo-Valuation Rings*, Bull. Iranian Math. Soc, 47(5) (2021), 1599-1609.

[7] A. El Khalfi, H. Kim and N. Mahdou, *Amalgamation extension in commutative ring theory, a survey*, Moroccan Journal of algebra and Geometry with applications, 1(1), (2022), 139–182.

[8] M. J. Kwon and J. W. Lim *On nonnil-S-Noetherian rings* Mathematics, 8(9) (2020), 1428.

[9] J. W. Lim and D. Y. Oh, *S-Noetherian properties on amalgamated algebras along an ideal*, J. Pure Appl. Algebra, 218(6) (2014), 1075–1080.

[10] W. Qi, H. Kim, F. G. Wang, M. Z. Chen and W. Zhao, *Uniformly S-Noetherian rings*, arXiv preprint arXiv, 2201.07913 (2022).

[11] M. Tamekkante, K. Louartiti, and M. Chhiti, *Chain conditions in amalgamated algebras along an ideal*, Arab. J. Math. 2(4) (2013), 403-408.

[12] W. Zhao, F. Wang and G. Tang, *On ϕ-von Neumann regular rings*, J. Korean Math. Soc, 50(1) (2013), 219–229.

Najib MAHDOU,
Laboratory of Modelling and Mathematical Structures,
Department of Mathematics,
Faculty of Science and Technology of Fez,
University S.M. Ben Abdellah Fez, Box 2202, Morocco.
Email: mahdou@hotmail.com

El Houssaine OUBOUHOU,
Laboratory of Modelling and Mathematical Structures,
Department of Mathematics,
Faculty of Science and Technology of Fez,
University S.M. Ben Abdellah Fez, Box 2202, Morocco.
Email: hossineoubouhou@gmail.com

Ece YETKIN CELIKEL
Department of Basic Sciences,
Faculty of Engineering,
Hasan Kalyoncu University,
Gaziantep, Turkey.
Email: ece.celikel@hku.edu.tr, yetkinece@gmail.com