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On the unrestricted virtual singular braid

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Abstract

Let $n, m \in \mathbb{N}, n \geq 3$ and $n \geq m \geq 2$. In this paper, we study some properties of the unrestricted virtual singular braid of braid monoid $\boldsymbol{UVSB}_{mn}$ and group $\boldsymbol{UVSG}_{mn}$. These properties emerged by studying the analogous properties of the unrestricted virtual singular pseudosymmetric braid of braid monoid and group. These mathematical objects are submonoids and subgroups of the virtual singular braid group VSG_n. For many of these quotients, we have obtained reduced presentations.

1 Introduction

The Artin braid groups or braid groups on n strands B_n are introduced by E. Artin as a tool for working with classical knots and links. B_n is the group defined [1] by generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and relations

(R0) $\quad \sigma_i\sigma_j = \sigma_j\sigma_i \qquad |j - i| > 1$
(R3) $\quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad i = 1, 2, \ldots, n\text{–}2.$

For work on classical braids and classical knots, see [16, 17] and references therein.

The term *Braid of Braids* identifies classes of algebraic groups of braids that each, in a certain sense, includes as a particular case a class of braids representing one of the classical braid groups, e.g. if m=1, the braid of braid group $\boldsymbol{BB}_{mn}$ coincides with the braid group of n strands B_n. The term Braid of Braids was first introduced by Moran at [12], where some of their properties

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referring to Artins braid groups were also studied. Lin in one of his works, [11], gives a note to this type of algebraic structures but without using the term *Braid-of-Braids*. The term indicates that each strand of this type of group that uses it is generally made up of several strands that we will call elementary and "borrowed" from classic braid groups.

The *Pseudosymmetric group* is the quotient of B_n by the relations $\sigma_i\sigma_{i+1}^{-1}\sigma_i = \sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}$. These groups appeared in 2010 in a paper of Panaite and Staic [15].

The *singular braid monoid* on n strands SB_n was introduced the same period by Baez 1992 [2] and Birman 1993 [4] independently and includes the B_n group as a submonoid. SB_n is generated by a set of n–1 classical generators: $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ and n–1 singular generators: $\{x_1, x_2, \ldots, x_{n-1}\}$ satisfying a sets of braid relations and singular relations.

The groups VB_n were introduced by Kauffman as a tool for working with virtual knots and links. The virtual braid group VB_n contains the braid group B_n in a natural way just as classical knots embed in virtual knots. This fact may be most easily deduced from [8, 10]. Hence, the virtual braid group on n strands, VB_n, is an extension of the classical braid group B_n by the symmetric group S_n. VB_n is generated by a set of n–1 classical generators: $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ and n–1 virtual generators: $\{\rho_1, \rho_2, \ldots, \rho_{n-1}\}$ satisfying a set of braid, virtual and mixed braid-virtual relations.

The *virtual singular monoids* denote VSB_n, was introduced by Caprau et all. in [5]. Virtual singular monoids VSB_n are similar to classical Artin braid monoids, with the difference that they contain singular crossings x_i and virtual ρ_i, besides classical crossings σ_i.

The group of *unrestricted virtual braids*, UVB_n, was introduced by Kauffman and Lambropoulou in [9]. Unrestricted virtual braid groups are used for working with fused links [14]. In [3] has given a description of the structure of UVB_n. Finally, the group of *unrestricted virtual singular braids*, $UVSG_n$, was introduced in [13].

The paper is organized as follows: in section 2 we recall some definitions and classical results for the *unrestricted virtual singular braid of braid*. A reduced presentation for the *unrestricted virtual singular pseudosymmetric braid of braid group* $\boldsymbol{UVSPSG}_{mn}$ has been built with generators $\{\underline{\sigma}_1, \underline{x}_1, \underline{\rho}_1, \underline{\rho}_2, \ldots, \underline{\rho}_{n-1}\}$ (see Theorem 2.10). Finally, the Theorem 2.12 states that the $\boldsymbol{UVSPSG}_{mn}$ group is isomorphic to the *unrestricted virtual singular pseudosymmetric group* $UVSPSG_n$. In section 3 we introduced the extended fusing b-strings $\underline{\mu}_i$, $\underline{\mu}_i^{-1}$, $\underline{\gamma}_i$, $\underline{\gamma}_i^{-1}$, $\underline{\delta}_1$, $\underline{\delta}_1^{-1}$, and then we tried a second reduced presentation of $\boldsymbol{UVSPSG}_{mn}$ with $\{\underline{\rho}_1, \underline{\rho}_2, \ldots, \underline{\rho}_{n-1}, \underline{\mu}_1, \underline{\mu}_1^{-1}, \underline{\gamma}_1, \underline{\gamma}_1^{-1}\}$ as a generator set (see Theorem 3.5), and a third presentation has been proposed with generators $\{\underline{\sigma}_1, \underline{\sigma}_2, \ldots, \underline{\sigma}_{n-1}, \underline{\mu}_1, \underline{\mu}_i^{-1}, \underline{\delta}_1, \underline{\delta}_1^{-1}\}$ (see Theorem 3.7).

2 Unrestricted virtual singular braid of braid

It has previously been indicated that Caprau et al. defined the *virtual singular braid* of n strands VSB_n [5] and in [6] they defined the *virtual singular braid group* of n strands VSG_n. In some of our proofs below, we'll underline the portion of the relation that we'll be working with on the row next.

Definition 2.1. (Moran [12]). Let the Artin braid group B_{mn} generated by $\sigma_1, \sigma_2, \ldots, \sigma_{mn-1}$. If $\varepsilon = \pm 1$ and $k = 1, 2, \ldots, n$–1, we define

$$\underline{\sigma}_k^{(m,\,\varepsilon)} = \prod_{i=1}^{m} \sigma^{\varepsilon}_{km-i+1}\sigma^{\varepsilon}_{km-i+2}\cdots\sigma^{\varepsilon}_{km-i+m}.$$

The symbol

$$\underline{\sigma}_k = \underline{\sigma}_k^{(m,\,1)} = \prod_{i=1}^{m} \sigma_{km-i+1}\sigma_{km-i+2}\cdots\sigma_{km-i+m}$$

represents the positions of each of the km, km–1, ..., km–m+1 strands on each of the km+1, km+2, ..., km+m strands. Each ordered (km–m+1, ..., km–1, km), $k = 1, 2, \ldots, n$–1, grouping of strands is called *m-strand*.

Definition 2.2. Let the virtual singular braid monoid VSB_{mn} generated by the classical σ_i, singular x_i, and virtual ρ_i braids, for $i = 1, 2, \ldots, mn$–1. If $k = 1, 2, \ldots, n$–1, we define

$$\underline{\mathrm{x}}_k^{(m)} = \prod_{i=1}^{m} x_{km-i+1}x_{km-i+2}\cdots x_{km-i+m},$$

and

$$\underline{\rho}_k^{(m)} = \prod_{i=1}^{m} \rho_{km-i+1}\rho_{km-i+2}\cdots\rho_{km-i+m}.$$

In cases of unambiguity the symbol $\underline{\mathrm{x}}_k$ is used to indicate $\underline{\mathrm{x}}_k^{(m)}$:

$$\underline{\mathrm{x}}_k = \prod_{i=1}^{m} x_{km-i+1}x_{km-i+2}\cdots x_{km-i+m},$$

and $\underline{\rho}_k$ is used to indicate $\underline{\rho}_k^{(m)}$:

$$\underline{\rho}_k = \prod_{i=1}^{m} \rho_{km-i+1}\rho_{km-i+2}\cdots\rho_{km-i+m}.$$

Definition 2.3. Let the virtual singular braid monoid VSB_{mn}. The *virtual singular braid of braid monoid*, denoted by $\boldsymbol{VSB}_{mn}$, is the following generated subgroup of VSB_{mn}:

$$\boldsymbol{VSB}_{mn} = \langle \underline{\sigma}_1, \underline{\sigma}_2, \ldots, \underline{\sigma}_{n-1}, \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_{n-1}, \underline{\rho}_1, \underline{\rho}_2, \ldots, \underline{\rho}_{n-1} \rangle.$$

Virtual singular braid of braid group $\boldsymbol{VSG}_{mn}$ is an abstract group that has a presentation that has the same set of generators and defining relations as the monoid-presentation for the virtual singular braid of braid monoid $\boldsymbol{VSB}_{mn}$, with the only additional properties that the elements are invertible in $\boldsymbol{VSG}_{mn}$ and the relations $(RS0)$, $(RRS0)$, $(RRS2)$, $(RRS3)$, $(RRS4)$, $(RSV0)$, and $(RSV3)$ remain valid by replacing $\underline{\mathrm{x}}_k$ with $\underline{\mathrm{x}}_k^{-1}$, for $k = 1, 2, \ldots, n$–1.

Each element of $\boldsymbol{VSB}_{mn}$ is called *virtual singular braid of braid.* A *virtual singular braid of braid diagram* is a diagram that represents any $\beta \in \boldsymbol{VSB}_{mn}$ element. The justaposition operation of the $\boldsymbol{VSB}_{mn}$ group is induced by the justaposition operation of VSB_{mn}. The *neutral element* $\underline{1}_n$ of $\boldsymbol{VSB}_{mn}$ group is induced by the neutral element 1_{mn} of VSB_{mn}.

Example 2.4. In Figure 1 a geometric representation of the braid $\underline{\mathrm{x}}_k$ (left) of VSB_{2n} monoid and its representation (right) as a generator of the monoid $\boldsymbol{VSB}_{2n}$.

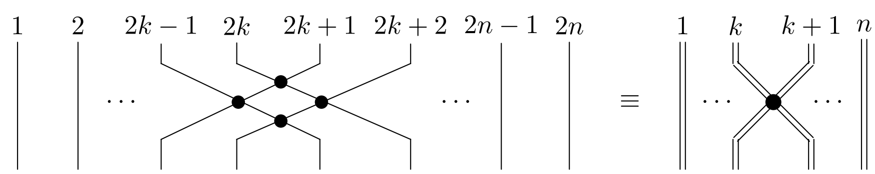

Figure 1: The braid $\underline{\mathrm{x}}_k$ (left) of VSB_{2n} monoid and its representation (right) as a generator of $\boldsymbol{VSB}_{2n}$.

Definition 2.5. *Unrestricted virtual singular braid of braid monoid* on n m-strands, denoted by $\boldsymbol{UVSB}_{mn}$, is defined as the group generated by *b-classical generators* $\underline{\sigma}_i$, *b-singular generators* $\underline{\mathrm{x}}_i$, and *b-virtual generators* $\underline{\rho}_i$, $i = 1, 2, \ldots, n$–1, satisfying the following *b-classical relations*:

(RR0)	$\underline{\sigma}_i\underline{\sigma}_j = \underline{\sigma}_j\underline{\sigma}_i$	$\lvert j - i\rvert > 1$
(RR2)	$\underline{\sigma}_i^{-1}\underline{\sigma}_i = 1_n = \underline{\sigma}_i\underline{\sigma}_i^{-1}$	$i = 1, 2, \ldots, n$–1
(RR3)	$\underline{\sigma}_i\underline{\sigma}_{i+1}\underline{\sigma}_i = \underline{\sigma}_{i+1}\underline{\sigma}_i\underline{\sigma}_{i+1}$	$i = 1, 2, \ldots, n$–2,

b-singular relations:

(RS0)	$\underline{\mathrm{x}}_i\underline{\mathrm{x}}_j = \underline{\mathrm{x}}_j\underline{\mathrm{x}}_i$	$\lvert j - i\rvert > 1$,

b-virtual relations:

(RV0)	$\underline{\rho}_i\underline{\rho}_j = \underline{\rho}_j\underline{\rho}_i$	$\lvert j - i\rvert > 1$
(RV2)	$\underline{\rho}_i^2 = 1_n$	$i = 1, 2, \ldots, n$–1
(RV3)	$\underline{\rho}_i\underline{\rho}_{i+1}\underline{\rho}_i = \underline{\rho}_{i+1}\underline{\rho}_i\underline{\rho}_{i+1}$	$i = 1, 2, \ldots, n$–2,

mixed b-real-singular relations:

(RRS0) $\underline{x}_i\underline{\sigma}_j = \underline{\sigma}_j\underline{x}_i$ $\quad |j - i| > 1$
(RRS2) $\underline{x}_i\underline{\sigma}_i = \underline{\sigma}_i\underline{x}_i$ $\quad i = 1, 2, \ldots, n–1$
(RRS3) $\underline{x}_i\underline{\sigma}_{i+1}\underline{\sigma}_i = \underline{\sigma}_{i+1}\underline{\sigma}_i\underline{x}_{i+1}$ $\quad i = 1, 2, \ldots, n–2$
(RRS4) $\underline{\sigma}_i\underline{\sigma}_{i+1}\underline{x}_i = \underline{x}_{i+1}\underline{\sigma}_i\underline{\sigma}_{i+1}$ $\quad i = 1, 2, \ldots, n–2,$

mixed b-real-virtual relations:

(RRV0) $\underline{\sigma}_i\underline{\rho}_j = \underline{\rho}_j\underline{\sigma}_i$ $\quad |j - i| > 1$
(RRV3) $\underline{\sigma}_i\underline{\rho}_{i+1}\underline{\rho}_i = \underline{\rho}_{i+1}\underline{\rho}_i\underline{\sigma}_{i+1}$ $\quad i = 1, 2, \ldots, n–2,$

mixed b-real-virtual forbidden relations:

(RVF1) $\underline{\rho}_i\underline{\sigma}_{i+1}\underline{\sigma}_i = \underline{\sigma}_{i+1}\underline{\sigma}_i\underline{\rho}_{i+1}$ $\quad i = 1, 2, \ldots, n–2$
(RVF2) $\underline{\sigma}_i\underline{\sigma}_{i+1}\underline{\rho}_i = \underline{\rho}_{i+1}\underline{\sigma}_i\underline{\sigma}_{i+1}$ $\quad i = 1, 2, \ldots, n–2,$

and *mixed b-singular-virtual relations*:

(RSV0) $\underline{x}_i\underline{\rho}_j = \underline{\rho}_j\underline{x}_i$ $\quad |j - i| > 1$
(RSV3) $\underline{x}_i\underline{\rho}_{i+1}\underline{\rho}_i = \underline{\rho}_{i+1}\underline{\rho}_i\underline{x}_{i+1}$ $\quad i = 1, 2, \ldots, n–2.$

Unrestricted virtual singular braid of braid group, denoted by $\boldsymbol{UVSG}_{mn}$, is a group that has a presentation that has the same set of generators and defining relations as the monoid-presentation for the monoid $\boldsymbol{UVSB}_{mn}$, with the only additional properties that the elements are invertible in $\boldsymbol{UVSG}_{mn}$ and the relations ($RS0$), ($RRS0$), ($RRS2$), ($RRS3$), ($RRS4$), ($RSV0$), and ($RSV3$) remain valid by replacing $\underline{x}_k$ with $\underline{x}_k^{-1}$.

The $\boldsymbol{UVSPSB}_{mn}$, *Unrestricted virtual singular pseudosymmetric braid of braid monoid*, is a monoid that has a presentation that has the same set of generators and defining relations as the monoid-presentation for the monoid $\boldsymbol{UVSB}_{mn}$, with the only additional pseudosymmetric relation:

(RR4) $\underline{\sigma}_i\underline{\sigma}_{i+1}^{-1}\underline{\sigma}_i = \underline{\sigma}_{i+1}\underline{\sigma}_i^{-1}\underline{\sigma}_{i+1}$, $\quad i = 1, 2, \ldots, n–2.$

The $\boldsymbol{UVSPSG}_{mn}$, *unrestricted virtual singular pseudosymmetric braid of braid group*, is a group that has a presentation that has the same set of generators and defining relations as the monoid-presentation for the $\boldsymbol{UVSPSB}_{mn}$, with the additional properties that the elements are invertible in $\boldsymbol{UVSPSG}_{mn}$: *(RS2)* $\underline{x}_k^{-1}\underline{x}_k = 1_n = \underline{x}_k\underline{x}_k^{-1}$, and the relations ($RS0$), ($RRS0$), ($RRS2$), ($RRS3$), ($RRS4$), ($RSV0$), and ($RSV3$) remain valid by replacing $\underline{x}_k$ with $\underline{x}_k^{-1}$.

Definition 2.6. The $\boldsymbol{WSB}_{mn}$, *welded singular braid of braid monoid*, is a monoid that has a presentation with the same generator set as $\boldsymbol{UVSB}_{mn}$ while its relations are the same as $\boldsymbol{UVSB}_{mn}$ except for the ($RVF2$) relations.

The $\boldsymbol{WSG}_{mn}$, *welded singular braid of braid group*, is a group that has a presentation with the same sets of generators and relations as $\boldsymbol{WSB}_{mn}$ with only the added properties that each of its elements has an inverse element in $\boldsymbol{WSG}_{mn}$: *(RS2)*, and the relations $(RS0)$, $(RRS0)$, $(RRS2)$, $(RRS3)$, $(RRS4)$, $(RSV0)$, and $(RSV3)$ remain valid by replacing $\underline{\mathrm{x}}_k$ with $\underline{\mathrm{x}}_k^{-1}$.

The $\boldsymbol{WSPSB}_{mn}$, *welded singular pseudosymmetric braid of braid monoid*, is a monoid that has a presentation with the same generator set as $\boldsymbol{WSB}_{mn}$ while its relations are the same as $\boldsymbol{UVSB}_{mn}$ with the only additional pseudosymmetric relation *(RR4)*.

The $\boldsymbol{WSPSG}_{mn}$, *welded singular pseudosymmetric braid of braid group*, is a group that has a presentation with the same sets of generators and relations as $\boldsymbol{WSPSB}_{mn}$ with only the added properties that each of its elements has an inverse element in $\boldsymbol{WSPSG}_{mn}$: *(RS2)*, and the relations $(RS0)$, $(RRS0)$, $(RRS2)$, $(RRS3)$, $(RRS4)$, $(RSV0)$, and $(RSV3)$ remain valid by replacing $\underline{\mathrm{x}}_k$ with $\underline{\mathrm{x}}_k^{-1}$.

To prove the following equations we proceed by induction on the relations $(RRV3)$, $(RVF1)$, $(RRS4)$, $(RRS3)$, and $(RSV3)$.

Lemma 2.7. *For $1 \leq i \leq n{-}2$, the following statement holds in $\boldsymbol{UVSB}_{mn}$:*

(a) $\underline{\sigma}_{i+1} = (\varrho_i \cdots \varrho_1)(\varrho_{i+1} \cdots \varrho_2)\underline{\sigma}_1(\varrho_2 \cdots \varrho_{i+1})(\varrho_1 \cdots \varrho_i)$.
(b) $\varrho_{i+1} = (\underline{\sigma}_i^{-1} \cdots \underline{\sigma}_1^{-1})(\underline{\sigma}_{i+1}^{-1} \cdots \underline{\sigma}_2^{-1})\varrho_1(\underline{\sigma}_2 \cdots \underline{\sigma}_{i+1})(\underline{\sigma}_1 \cdots \underline{\sigma}_i)$.
(c) $\underline{x}_{i+1} = (\underline{\sigma}_i \cdots \underline{\sigma}_1)(\underline{\sigma}_{i+1} \cdots \underline{\sigma}_2)\underline{x}_1(\underline{\sigma}_2^{-1} \cdots \underline{\sigma}_{i+1}^{-1})(\underline{\sigma}_1^{-1} \cdots \underline{\sigma}_i^{-1})$.
(d) $\underline{x}_{i+1} = (\underline{\sigma}_i^{-1} \cdots \underline{\sigma}_1^{-1})(\underline{\sigma}_{i+1}^{-1} \cdots \underline{\sigma}_2^{-1})\underline{x}_1(\underline{\sigma}_2 \cdots \underline{\sigma}_{i+1})(\underline{\sigma}_1 \cdots \underline{\sigma}_i)$.
(e) $\underline{x}_{i+1} = (\varrho_i \cdots \varrho_1)(\varrho_{i+1} \cdots \varrho_2)\underline{x}_1(\varrho_2 \cdots \varrho_{i+1})(\varrho_1 \cdots \varrho_i)$.

Two Lemmas attributed to Kauffman and Lambropoulou follow, adapted for the $\boldsymbol{UVSG}_{mn}$ group. We prove only the Lemma 2.9.

Lemma 2.8 ([9])**.** *Let be the group $\boldsymbol{UVSG}_{mn}$. For $1 \leq i \leq n{-}3$ holds:*

$(\varrho_4\varrho_3\varrho_2\varrho_1) \cdots \varrho_{i+2}\varrho_{i+1}\varrho_i\varrho_{i-1}) = (\varrho_4 \cdots \varrho_{i+2})(\varrho_3 \cdots \varrho_{i+1})(\varrho_2 \cdots \varrho_i)(\varrho_1 \cdots \varrho_{i-1})$.

Lemma 2.9 ([9])**.** *Let be the group $\boldsymbol{UVSG}_{mn}$. If $1 \leq i, j \leq n{-}2$ and $|i - j| > 1$, then*

$$\varrho_i\varrho_{i+1} \cdots \varrho_{j-1}\varrho_j\varrho_{j-1} \cdots \varrho_{i+1}\varrho_i = \varrho_j\varrho_{j-1} \cdots \varrho_{i+1}\varrho_i\varrho_{i+1} \cdots \varrho_{j-1}\varrho_j.$$

Proof. Without losing the generality it is considered $i{+}2 \leq j$. The $i{+}2 > j$ case is proved analogously. We have

$$\begin{aligned}
&\varrho_i\varrho_{i+1} \cdots \underline{\varrho_{j-1}\varrho_j\varrho_{j-1}} \cdots \varrho_{i+1}\varrho_i \\
&= \varrho_i\varrho_{i+1} \cdots \underline{\varrho_j}\varrho_{j-1}\underline{\varrho_j} \cdots \varrho_{i+1}\varrho_i \\
&= \varrho_j\varrho_i\varrho_{i+1} \cdots \underline{\varrho_{j-2}\varrho_{j-1}\varrho_{j-2}} \cdots \varrho_{i+1}\varrho_i\varrho_j \\
&= \varrho_j\varrho_i\varrho_{i+1} \cdots \underline{\varrho_{j-1}}\varrho_{j-2}\underline{\varrho_{j-1}} \cdots \varrho_{i+1}\varrho_i\varrho_j \\
&= \varrho_j\varrho_{j-1}\varrho_i\varrho_{i+1} \cdots \underline{\varrho_{j-3}\varrho_{j-2}\varrho_{j-3}} \cdots \varrho_{i+1}\varrho_i\varrho_{j-1}\varrho_j \\
&= \varrho_j\varrho_{j-1}\varrho_i\varrho_{i+1} \cdots \underline{\varrho_{j-2}}\varrho_{j-3}\underline{\varrho_{j-2}} \cdots \varrho_{i+1}\varrho_i\varrho_{j-1}\varrho_j
\end{aligned}$$

$= \dots$
$= \varrho_j\varrho_{j-1}\cdots\varrho_{i+2}\underline{\varrho_i\varrho_{i+1}\varrho_i}\varrho_{i+2}\cdots\varrho_{j-1}\varrho_j$
$= \varrho_j\varrho_{j-1}\cdots\varrho_{i+2}\varrho_{i+1}\varrho_i\varrho_{i+1}\varrho_{i+2}\cdots\varrho_{j-1}\varrho_j.$ □

2.1 Reduced presentation for $\boldsymbol{UVSPSG}_{mn}$ with $\underline{\sigma}_1$, $\underline{\mathrm{x}}_1$, $\varrho_1, \dots, \varrho_{n-1}$ generators

Theorem 2.10. *The $\boldsymbol{UVSPSG}_{mn}$ group has a reduced presentation with generators $\{\underline{\sigma}_1, \underline{\mathrm{x}}_1, \varrho_1, \varrho_2, \dots, \varrho_{n-1}\}$ and the following relations:*

(i)	$\underline{\sigma}_1(\varrho_2\varrho_3\varrho_1\varrho_2\underline{\sigma}_1\varrho_2\varrho_1\varrho_3\varrho_2) = (\varrho_2\varrho_3\varrho_1\varrho_2\underline{\sigma}_1\varrho_2\varrho_1\varrho_3\varrho_2)\underline{\sigma}_1,$	
(ii)	$\underline{\sigma}_1^{-1}\underline{\sigma}_1 = 1_n = \underline{\sigma}_1\underline{\sigma}_1^{-1},$	
(iii)	$\underline{\sigma}_1(\varrho_1\varrho_2\underline{\sigma}_1\varrho_2\varrho_1)\underline{\sigma}_1 = (\varrho_1\varrho_2\underline{\sigma}_1\varrho_2\varrho_1)\underline{\sigma}_1(\varrho_1\varrho_2\underline{\sigma}_1\varrho_2\varrho_1),$	
(iv)	$\underline{\sigma}_1(\varrho_1\varrho_2\underline{\sigma}_1^{-1}\varrho_2\varrho_1)\underline{\sigma}_1 = (\varrho_1\varrho_2\underline{\sigma}_1\varrho_2\varrho_1)\underline{\sigma}_1^{-1}(\varrho_1\varrho_2\underline{\sigma}_1\varrho_2\varrho_1),$	
(v)	$\varrho_i\varrho_j = \varrho_j\varrho_i$	$\lvert j - i\rvert > 1,$
(vi)	$\varrho_i^2 = 1_n$	$i = 1, 2, \dots, n{-}1,$
(vii)	$\varrho_i\varrho_{i+1}\varrho_i = \varrho_{i+1}\varrho_i\varrho_{i+1}$	$i = 1, 2, \dots, n{-}2,$
(viii)	$\underline{\mathrm{x}}_1(\varrho_2\varrho_3\varrho_1\varrho_2\underline{\mathrm{x}}_1\varrho_2\varrho_1\varrho_3\varrho_2) = (\varrho_2\varrho_3\varrho_1\varrho_2\underline{\mathrm{x}}_1\varrho_2\varrho_1\varrho_3\varrho_2)\underline{\mathrm{x}}_1,$	
(ix)	$\underline{\mathrm{x}}_1^{-1}\underline{\mathrm{x}}_1 = 1_n = \underline{\mathrm{x}}_1\underline{\mathrm{x}}_1^{-1},$	
(x)	$\underline{\mathrm{x}}_1(\varrho_2\varrho_3\varrho_1\varrho_2\underline{\sigma}_1\varrho_2\varrho_1\varrho_3\varrho_2) = (\varrho_2\varrho_3\varrho_1\varrho_2\underline{\sigma}_1\varrho_2\varrho_1\varrho_3\varrho_2)\underline{\mathrm{x}}_1,$	
(xi)	$\underline{\mathrm{x}}_1\underline{\sigma}_1 = \underline{\sigma}_1\underline{\mathrm{x}}_1,$	
(xii)	$\underline{\mathrm{x}}_1(\varrho_1\varrho_2\underline{\sigma}_1\varrho_2\varrho_1)\underline{\sigma}_1 = (\varrho_1\varrho_2\underline{\sigma}_1\varrho_2\varrho_1)\underline{\sigma}_1(\varrho_1\varrho_2\underline{\mathrm{x}}_1\varrho_2\varrho_1),$	
(xiii)	$\underline{\sigma}_1\varrho_i = \varrho_i\underline{\sigma}_1$	$i = 3, 4, \dots, n{-}2,$
(xiv)	$\varrho_1(\varrho_2\underline{\sigma}_1\varrho_2\varrho_1\underline{\sigma}_1) = (\varrho_2\underline{\sigma}_1\varrho_2\varrho_1\underline{\sigma}_1)\varrho_2,$	
(xv)	$(\underline{\sigma}_1\varrho_1\varrho_2\underline{\sigma}_1\varrho_2)\varrho_1 = \varrho_2(\underline{\sigma}_1\varrho_1\varrho_2\underline{\sigma}_1\varrho_2),$	
(xvi)	$\underline{\mathrm{x}}_1\varrho_i = \varrho_i\underline{\mathrm{x}}_1$	$i = 3, 4, \dots, n{-}2.$

Proof. The presentation assumes the relations Lemma 2.7-*(a)* and Lemma 2.7-*(e)*, which we refer to as the *special defining relations*. The relations ($RRV3$) and ($RSV3$) are not needed in the reduced presentation for $\boldsymbol{UVSPSG}_{mn}$, since they were implicitly used in the relations Lemma 2.7-*(a)* and Lemma 2.7-*(e)* respectively. The relation ($RRS4$) is also not included in the reduced presentation because it can be easily reconstructed from the relation ($RRS3$) which is present at the position *(xii)*.
The relations *(RV0)*: $\varrho_i\varrho_j = \varrho_j\varrho_i$, *(RV2)*: $\varrho_i^2 = 1_n$, *(RV3)*: $\varrho_i\varrho_{i+1}\varrho_i = \varrho_{i+1}\varrho_i\varrho_{i+1}$ of the definition of $\boldsymbol{UVSPSG}_{mn}$ group are identical, respectively, to the relations *(v)*, *(vi)* and *(vii)*.
The relations *(RR0)*: $\underline{\sigma}_i\underline{\sigma}_j = \underline{\sigma}_j\underline{\sigma}_i$ were proved in ([9], Lemma 3) for VB_n group. The base case of the relations, $\underline{\sigma}_1\underline{\sigma}_3 = \underline{\sigma}_3\underline{\sigma}_1$, corresponds to the relation *(i)*.
The relations *(RRS0)*: $\underline{\mathrm{x}}_i\underline{\sigma}_j = \underline{\sigma}_j\underline{\mathrm{x}}_i$ were proved in ([5], Lemma 3) for VSB_n group. The initial case of the relations, $\underline{\mathrm{x}}_1\underline{\sigma}_3 = \underline{\sigma}_3\underline{\mathrm{x}}_1$, corresponds to the relation *(x)*.

The relations *(RRS2)*: $\underline{x}_i\underline{\sigma}_i = \underline{\sigma}_i\underline{x}_i$ were proved in ([5], Lemma 7) for VSB_n group. The initial case of the relations, $\underline{x}_1\underline{\sigma}_1 = \underline{\sigma}_1\underline{x}_1$, corresponds to the relation *(xi)*.
The relations *(RSV0)*: $\underline{x}_i\underline{\rho}_j = \underline{\rho}_j\underline{x}_i$ were proved in ([5], Lemma 2) for VSB_n monoid. The initial case of the relations, $\underline{x}_1\underline{\rho}_3 = \underline{\rho}_3\underline{x}_1$, corresponds to the relation *(xvi)*.
The relations *(RRV0)*: $\underline{\sigma}_i\underline{\rho}_j = \underline{\rho}_j\underline{\sigma}_i$ were proved in ([9], Lemma 1) for VB_n group. The initial case of the relations, $\underline{\sigma}_1\underline{\rho}_3 = \underline{\rho}_3\underline{\sigma}_1$, corresponds to the relation *(xiii)*.
The relations *(RR3)*: $\underline{\sigma}_i\underline{\sigma}_{i+1}\underline{\sigma}_i = \underline{\sigma}_{i+1}\underline{\sigma}_i\underline{\sigma}_{i+1}$ were proved in ([9], Lemma 2) for VB_n group. The initial case of the relations, $\underline{\sigma}_1\underline{\sigma}_2\underline{\sigma}_1 = \underline{\sigma}_2\underline{\sigma}_1\underline{\sigma}_2$, corresponds to the relation *(iii)*.
The relations *(RRS3)*: $\underline{x}_i\underline{\sigma}_{i+1}\underline{\sigma}_i = \underline{\sigma}_{i+1}\underline{\sigma}_i\underline{x}_{i+1}$ were proved in ([5], Lemma 5) for VSB_n monoid. The initial case of the relations, $\underline{x}_1\underline{\sigma}_2\underline{\sigma}_1 = \underline{\sigma}_2\underline{\sigma}_1\underline{x}_2$, corresponds to the relation *(xii)*.
For $i = 1, \ldots, n–2$, the forbidden relations *(RVF2)*: $\underline{\sigma}_i\underline{\sigma}_{i+1}\underline{\rho}_i = \underline{\rho}_{i+1}\underline{\sigma}_i\underline{\sigma}_{i+1}$ follow from the basic relations (xv) of Theorem 2.10, the virtual relations, and Lemma 2.7.
The singular commuting relations *(RS0)*: $\underline{x}_i\underline{x}_j = \underline{x}_j\underline{x}_i$ for $i, j = 1, \ldots, n–1$ and $|i - j| \geq 2$ follow from the basic relations $(viii)$ of Theorem 2.10, the virtual relations, and Lemma 2.7.
For $i = 1, 2, \ldots, n–2$, the relations *(RVF1)*: $\underline{\rho}_i\underline{\sigma}_{i+1}\underline{\sigma}_i = \underline{\sigma}_{i+1}\underline{\sigma}_i\underline{\rho}_{i+1}$ follow from the basic relations (xiv) of Theorem 2.10, the virtual relations, and Lemmas 2.7, 2.8.
For $i = 1, 2, \ldots, n–1$, the relations *(RR2)*: $\underline{\sigma}_i^{-1}\underline{\sigma}_i = 1_n = \underline{\sigma}_i\underline{\sigma}_i^{-1}$ follow from the basic relations (ii) of Theorem 2.10, the virtual relations, and Lemma 2.7.
The real relations *(RR4)*: $\underline{\sigma}_{i+1}\underline{\sigma}_i^{-1}\underline{\sigma}_{i+1} = \underline{\sigma}_i\underline{\sigma}_{i+1}^{-1}\underline{\sigma}_i$, for $1 \leq i \leq n–2$, follow from the basic relations (iv), $(xiii)$ of Theorem 2.10, the virtual relations, and Lemmas 2.7, 2.9.
The singular relations *(RS2)*: $\underline{x}_i^{-1}\underline{x}_i = 1_n = \underline{x}_i\underline{x}_i^{-1}$, for $1 \leq i \leq n–1$, follow from the basic relations (ix) of Theorem 2.10, the virtual relations, and Lemma 2.7.
The proof is complete. □

Corollary 2.11. *From Theorem 2.10, if we remove the relations:*

- *(ix) we have a reduced presentation of* $\boldsymbol{UVSPSB}_{mn}$.
- *(iv) we have a reduced presentation of the* $\boldsymbol{UVSG}_{mn}$.
- *(iv), and (ix) we have a reduced presentation of* $\boldsymbol{UVSB}_{mn}$.
- *(iv), (ix), and (xviii) we have a reduced presentation of the* $\boldsymbol{WSB}_{mn}$.
- *(iv), and (xviii) we have a reduced presentation of the* $\boldsymbol{WSG}_{mn}$.
- *(ix), and (xviii) we have a reduced presentation of the* $\boldsymbol{WSPSB}_{mn}$.

- *(xviii) we have a reduced presentation of the* $\boldsymbol{WSPSG}_{mn}$.

Theorem 2.12. *The braid of braid group* $\boldsymbol{UVSPSG}_{mn}$ *is isomorphic to the braid group* $UVSPSG_n$.

Proof. Let the groups $UVSPSG_{mn}$ and $\boldsymbol{UVSPSG}_{mn}$. For $k = 1, \ldots, n$–1, correspondence

$$\underline{\sigma}_k \mapsto \prod_{i=1}^{m} \sigma_{km-i+1}\sigma_{km-i+2}...\sigma_{km-i+m},$$

$$\underline{x}_k \mapsto \prod_{i=1}^{m} x_{km-i+1}x_{km-i+2}\cdots x_{km-i+m},$$

and

$$\underline{\rho}_k \mapsto \prod_{i=1}^{m} \rho_{km-i+1}\rho_{km-i+2}\cdots\rho_{km-i+m}.$$

defines an injective homomorphism ϑ: $\boldsymbol{UVSPSG}_{mn} \longrightarrow UVSPSG_{mn}$. According to the Definitions 2.1 and 2.2, the inverse map ϑ^{-1} of the ϑ is a surjective homomorphism and the proof is complete. So, $\boldsymbol{UVSPSG}_{mn} \cong UVSPSG_n$. □

3 Reduced presentations for $\boldsymbol{UVSPSG}_{mn}$ using real-fusing b-strings

In this section, we introduce a new presentation for the n m-stranded unrestricted virtual singular pseudosymmetric braid of braid group, $\boldsymbol{UVSPSG}_{mn}$. This reduced presentation uses as generators a particular type of unrestricted virtual singular braid of braids, which we now define.

Definition 3.1. In $\boldsymbol{VSG}_{mn}$, the *elementary real-fusing b-strings* $\underline{\mu}_i$, $\underline{\mu}_i^{-1}$, $\underline{\gamma}_i$, and $\underline{\gamma}_i^{-1}$ are n m-stranded virtual singular braid of braids defined as follows:

$$\underline{\mu}_i = \underline{\sigma}_i\underline{\rho}_i,\ \underline{\mu}_i^{-1} = \underline{\rho}_i\underline{\sigma}_i^{-1},\ \underline{\gamma}_i = \underline{x}_i\underline{\rho}_i,\ \underline{\gamma}_i^{-1} = \underline{\rho}_i\underline{x}_i^{-1},\ \underline{\delta}_i = \underline{x}_i\underline{\sigma}_i,\ \underline{\delta}_i^{-1} = \underline{\sigma}_i^{-1}\underline{x}_i^{-1},$$

where $i = 1, 2, \ldots, n$–1.

Remark 3.2. By Definition 3.1, we can describe the:

- real generators $\underline{\sigma}_i$, $\underline{\sigma}_i^{-1}$ and the singular generators $\underline{x}_i$, $\underline{x}_i^{-1}$ of $\boldsymbol{VSG}_{mn}$ in terms of the elementary real-fusing b-strings $\underline{\mu}_i$, $\underline{\mu}_i^{-1}$, $\underline{\gamma}_i$ and $\underline{\gamma}_i^{-1}$:

$$\underline{\sigma}_i = \underline{\mu}_i\underline{\rho}_i,\ \ \underline{\sigma}_i^{-1} = \underline{\rho}_i\underline{\mu}_i^{-1},\ \ \underline{x}_i = \underline{\gamma}_i\underline{\rho}_i,\ \ \underline{x}_i^{-1} = \underline{\rho}_i\underline{\gamma}_i^{-1},$$

- virtual generators $\underline{\rho}_i$ and the singular generators $\underline{x}_i$, $\underline{x}_i^{-1}$ of $\boldsymbol{VSG}_{mn}$ in terms of $\underline{\mu}_i$, $\underline{\delta}_i$ and $\underline{\delta}_i^{-1}$:

$$\underline{\rho}_i = \underline{\sigma}_i^{-1}\underline{\mu}_i, \quad \underline{x}_i = \underline{\delta}_i\underline{\sigma}_i^{-1}, \quad \underline{x}_i^{-1} = \underline{\sigma}_i\underline{\delta}_i^{-1},$$

where $i = 1, 2, \ldots, n$–1.

Translating pages 8-10 of [10] and Definition 7 of [6] to the algebraically language of the braid of braid we can state the following Lemma.

Lemma 3.3. *For 1 ≤ i ≤ n–2, the following relation holds in* $\boldsymbol{UVSG}_{mn}$*:*

(a) $\underline{\rho}_i\underline{\mu}_{i+1}\underline{\rho}_i = \underline{\rho}_{i+1}\underline{\mu}_i\underline{\rho}_{i+1}$ *and* $\underline{\rho}_i\underline{\mu}_{i+1}^{-1}\underline{\rho}_i = \underline{\rho}_{i+1}\underline{\mu}_i^{-1}\underline{\rho}_{i+1}$.
(b) $\underline{\rho}_i\underline{\Upsilon}_{i+1}\underline{\rho}_i = \underline{\rho}_{i+1}\underline{\Upsilon}_i\underline{\rho}_{i+1}$ *and* $\underline{\rho}_i\underline{\Upsilon}_{i+1}^{-1}\underline{\rho}_i = \underline{\rho}_{i+1}\underline{\Upsilon}_i^{-1}\underline{\rho}_{i+1}$.
(c) $\underline{\mu}_{i+1} = (\underline{\rho}_i \ldots \underline{\rho}_1)(\underline{\rho}_{i+1} \ldots \underline{\rho}_2)\underline{\mu}_1(\underline{\rho}_2 \ldots \underline{\rho}_{i+1})(\underline{\rho}_1 \ldots \underline{\rho}_i)$.
(d) $\underline{\mu}_{i+1}^{-1} = (\underline{\rho}_i \ldots \underline{\rho}_1)(\underline{\rho}_{i+1} \ldots \underline{\rho}_2)\underline{\mu}_1^{-1}(\underline{\rho}_2 \ldots \underline{\rho}_{i+1})(\underline{\rho}_1 \ldots \underline{\rho}_i)$.
(e) $\underline{\Upsilon}_{i+1} = (\underline{\rho}_i \ldots \underline{\rho}_1)(\underline{\rho}_{i+1} \ldots \underline{\rho}_2)\underline{\Upsilon}_1(\underline{\rho}_2 \ldots \underline{\rho}_{i+1})(\underline{\rho}_1 \ldots \underline{\rho}_i)$.
(f) $\underline{\Upsilon}_{i+1}^{-1} = (\underline{\rho}_i \ldots \underline{\rho}_1)(\underline{\rho}_{i+1} \ldots \underline{\rho}_2)\underline{\Upsilon}_1^{-1}(\underline{\rho}_2 \ldots \underline{\rho}_{i+1})(\underline{\rho}_1 \ldots \underline{\rho}_i)$.

Lemma 3.4. *For 1 ≤ i ≤ n–2, the following relation holds in* $\boldsymbol{UVSG}_{mn}$*:*

(a) $\underline{\mu}_{i+1} = (\underline{\sigma}_i^{-1} \ldots \underline{\sigma}_1^{-1})(\underline{\sigma}_{i+1}^{-1} \ldots \underline{\sigma}_2^{-1})\underline{\mu}_1(\underline{\sigma}_2 \ldots \underline{\sigma}_{i+1})(\underline{\sigma}_1 \ldots \underline{\sigma}_i)$.
(b) $\underline{\mu}_{i+1}^{-1} = (\underline{\sigma}_i^{-1} \ldots \underline{\sigma}_1^{-1})(\underline{\sigma}_{i+1}^{-1} \ldots \underline{\sigma}_2^{-1})\underline{\mu}_1^{-1}(\underline{\sigma}_2 \ldots \underline{\sigma}_{i+1})(\underline{\sigma}_1 \ldots \underline{\sigma}_i)$.
(c) $\underline{\Upsilon}_{i+1} = (\underline{\sigma}_2 \ldots \underline{\sigma}_{i+1})(\underline{\sigma}_1 \ldots \underline{\sigma}_i)\underline{\Upsilon}_1(\underline{\sigma}_i^{-1} \ldots \underline{\sigma}_1^{-1})(\underline{\sigma}_{i+1}^{-1} \ldots \underline{\sigma}_2^{-1})$.
(d) $\underline{\Upsilon}_{i+1}^{-1} = (\underline{\sigma}_2 \ldots \underline{\sigma}_{i+1})(\underline{\sigma}_1 \ldots \underline{\sigma}_i)\underline{\Upsilon}_1^{-1}(\underline{\sigma}_i^{-1} \ldots \underline{\sigma}_1^{-1})(\underline{\sigma}_{i+1}^{-1} \ldots \underline{\sigma}_2^{-1})$.
(e) $\underline{\delta}_{i+1} = (\underline{\sigma}_2 \ldots \underline{\sigma}_{i+1})(\underline{\sigma}_1 \ldots \underline{\sigma}_i)\underline{\delta}_1(\underline{\sigma}_i^{-1} \ldots \underline{\sigma}_1^{-1})(\underline{\sigma}_{i+1}^{-1} \ldots \underline{\sigma}_2^{-1})$.
(f) $\underline{\delta}_{i+1}^{-1} = (\underline{\sigma}_2 \ldots \underline{\sigma}_{i+1})(\underline{\sigma}_1 \ldots \underline{\sigma}_i)\underline{\delta}_1^{-1}(\underline{\sigma}_i^{-1} \ldots \underline{\sigma}_1^{-1})(\underline{\sigma}_{i+1}^{-1} \ldots \underline{\sigma}_2^{-1})$.

Proof. Then, the definition relations $(RR0)$, $(RR2)$, $(RR3)$, $(RRS0)$, $(RRV0)$, $(RRS4$, $(RVF1)$, Definition 3.1, and Lemma 2.7 are used. We proceed with the proof of equations (a).

$$\underline{\rho}_{i+1} = (\underline{\sigma}_i^{-1} \ldots \underline{\sigma}_1^{-1})(\underline{\sigma}_{i+1}^{-1} \ldots \underline{\sigma}_2^{-1})\underline{\rho}_1(\underline{\sigma}_2 \ldots \underline{\sigma}_{i+1})(\underline{\sigma}_1 \ldots \underline{\sigma}_i)$$
$$\underline{\mu}_{i+1} = \underline{\sigma}_{i+1}\underline{\rho}_{i+1} = \underline{\sigma}_{i+1}(\underline{\sigma}_i^{-1} \ldots \underline{\sigma}_1^{-1})(\underline{\sigma}_{i+1}^{-1} \ldots \underline{\sigma}_2^{-1})\underline{\rho}_1(\underline{\sigma}_2 \ldots \underline{\sigma}_{i+1})(\underline{\sigma}_1 \ldots \underline{\sigma}_i)$$
$$= \underline{\sigma}_{i+1}(\underline{\sigma}_i^{-1}\underline{\sigma}_{i+1}^{-1} \ldots \underline{\sigma}_1^{-1})(\underline{\sigma}_i^{-1} \ldots \underline{\sigma}_2^{-1})\underline{\rho}_1(\underline{\sigma}_2 \ldots \underline{\sigma}_{i+1})(\underline{\sigma}_1 \ldots \underline{\sigma}_i)$$
$$= (\underline{\sigma}_i^{-1}\underline{\sigma}_{i+1}^{-1}\underline{\sigma}_i \ldots \underline{\sigma}_1^{-1})(\underline{\sigma}_i^{-1} \ldots \underline{\sigma}_2^{-1})\underline{\rho}_1(\underline{\sigma}_2 \ldots \underline{\sigma}_{i+1})(\underline{\sigma}_1 \ldots \underline{\sigma}_i)$$
$$= \ldots$$
$$= (\underline{\sigma}_i^{-1} \ldots \underline{\sigma}_1^{-1})(\underline{\sigma}_{i+1}^{-1} \ldots \underline{\sigma}_2^{-1})\underline{\sigma}_1\underline{\rho}_1(\underline{\sigma}_2 \ldots \underline{\sigma}_{i+1})(\underline{\sigma}_1 \ldots \underline{\sigma}_i)$$
$$= (\underline{\sigma}_i^{-1} \ldots \underline{\sigma}_1^{-1})(\underline{\sigma}_{i+1}^{-1} \ldots \underline{\sigma}_2^{-1})\underline{\mu}_1(\underline{\sigma}_2 \ldots \underline{\sigma}_{i+1})(\underline{\sigma}_1 \ldots \underline{\sigma}_i).$$

Equations (b), (c), (d), (e) and (f) are proved analogously. Note that the equations $\underline{x}_i = \underline{\Upsilon}_i\underline{\rho}_i = \underline{\Upsilon}_i\underline{\mu}_i^{-1}\underline{\sigma}_i$ hold. □

3.1 Reduced presentation for $\boldsymbol{UVSPSG}_{mn}$ with $\{\underline{\varrho}_1,\ \underline{\varrho}_2,\ \ldots,\ \underline{\varrho}_{n-1},\ \underline{\mu}_1,\ \underline{\Upsilon}_1\}$ generators

Theorem 3.5. *The $\boldsymbol{UVSPSG}_{mn}$ group has a reduced presentation with generators $\{\underline{\varrho}_1,\ \underline{\varrho}_2,\ \ldots,\ \underline{\varrho}_{n-1},\ \underline{\mu}_1,\ \underline{\mu}_1^{-1},\ \underline{\Upsilon}_1,\ \underline{\Upsilon}_1^{-1}\}$ and the following relations:*

(i)	$\underline{\varrho}_i\underline{\varrho}_j = \underline{\varrho}_j\underline{\varrho}_i$	$\lvert j-i\rvert > 1,$
(ii)	$\underline{\varrho}_i^2 = 1_n$	$i = 1,\ 2,\ \ldots,\ n{-}1,$
(iii)	$\underline{\varrho}_i\underline{\varrho}_{i+1}\underline{\varrho}_i = \underline{\varrho}_{i+1}\underline{\varrho}_i\underline{\varrho}_{i+1}$	$i = 1,\ 2,\ \ldots,\ n{-}2,$
(iv)	$\underline{\mu}_1^{-1}\underline{\mu}_1 = 1_n = \underline{\mu}_1\underline{\mu}_1^{-1},$	
(v)	$\underline{\Upsilon}_1^{-1}\underline{\Upsilon}_1 = 1_n = \underline{\Upsilon}_1\underline{\Upsilon}_1^{-1},$	
(vi)	$\underline{\mu}_1\underline{\varrho}_i = \underline{\varrho}_i\underline{\mu}_1,$	$i = 3,\ 4,\ \ldots,\ n{-}1,$
(vii)	$\underline{\Upsilon}_1\underline{\varrho}_i = \underline{\varrho}_i\underline{\Upsilon}_1$	$i = 3,\ 4,\ \ldots,\ n{-}1,$
(viii)	$\underline{\mu}_1\underline{\varrho}_1\underline{\Upsilon}_1 = \underline{\Upsilon}_1\underline{\varrho}_1\underline{\mu}_1,$	
(ix)	$\underline{\mu}_1\underline{\varrho}_2\underline{\varrho}_1\underline{\mu}_1^{-1}\underline{\varrho}_2\underline{\varrho}_1\underline{\mu}_1\underline{\varrho}_2\underline{\varrho}_1 = \underline{\varrho}_1\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_1\underline{\varrho}_2\underline{\mu}_1^{-1}\underline{\varrho}_1\underline{\varrho}_2\underline{\mu}_1,$	
(x)	$(\underline{\varrho}_1\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_2\underline{\varrho}_1)(\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_2) = (\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_2)(\underline{\varrho}_1\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_2\underline{\varrho}_1),$	
(xi)	$(\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_2)\underline{\mu}_1 = \underline{\mu}_1(\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_2),$	
(xii)	$(\underline{\varrho}_1\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_2\underline{\varrho}_1)(\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_2)\underline{\mu}_1 = \underline{\mu}_1(\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_2)(\underline{\varrho}_1\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_2\underline{\varrho}_1),$	
(xiii)	$(\underline{\varrho}_1\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_2\underline{\varrho}_1)(\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_2)\underline{\Upsilon}_1 = \underline{\Upsilon}_1(\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_2)(\underline{\varrho}_1\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_2\underline{\varrho}_1),$	
(xiv)	$(\underline{\varrho}_2\underline{\varrho}_1\underline{\varrho}_3\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_2\underline{\varrho}_3\underline{\varrho}_1\underline{\varrho}_2)\underline{\mu}_1 = \underline{\mu}_1(\underline{\varrho}_2\underline{\varrho}_1\underline{\varrho}_3\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_2\underline{\varrho}_3\underline{\varrho}_1\underline{\varrho}_2),$	
(xv)	$(\underline{\varrho}_2\underline{\varrho}_1\underline{\varrho}_3\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_2\underline{\varrho}_3\underline{\varrho}_1\underline{\varrho}_2)\underline{\Upsilon}_1 = \underline{\Upsilon}_1(\underline{\varrho}_2\underline{\varrho}_1\underline{\varrho}_3\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_2\underline{\varrho}_3\underline{\varrho}_1\underline{\varrho}_2),$	
(xvi)	$(\underline{\varrho}_2\underline{\varrho}_1\underline{\varrho}_3\underline{\varrho}_2\underline{\Upsilon}_1\underline{\varrho}_2\underline{\varrho}_3\underline{\varrho}_1\underline{\varrho}_2)\underline{\Upsilon}_1 = \underline{\Upsilon}_1(\underline{\varrho}_2\underline{\varrho}_1\underline{\varrho}_3\underline{\varrho}_2\underline{\Upsilon}_1\underline{\varrho}_2\underline{\varrho}_3\underline{\varrho}_1\underline{\varrho}_2).$	

Proof. The equations *(i)*-*(iv)*, *(vi)*-*(viii)*, and *(xii)*-*(xvi)* were discussed in Proposition 14 of [7] for the virtual singular monoid VSB_n.

We need to examine why the *(v)*, *(ix)*, *(x)* and *(xi)* relations are present. The four relations correspond respectively to the defining relations, of $\boldsymbol{UVSPSG}_{mn}$, $(RS2)$, $(RR4)$, $(RVF1)$ and $(RVF2)$ of Definition 2.5. To continue, it is sufficient to consider the basic cases of the four relations. In fact, with the use of Lemma 3.3 we can describe the cases for all $2 \le i \le n{-}2$.

The initial case of the relations *(RS2)*: $\underline{x}_i\underline{x}_i^{-1} = 1_n = \underline{x}_i\underline{x}_i^{-1}$ is $\underline{x}_1\underline{x}_1^{-1} = 1_n = \underline{x}_1\underline{x}_1^{-1}$. We prove *(v)* implies $(RS2)$. By *(ii)*, *(v)* and Remark 3.2:

$\underline{\Upsilon}_1\underline{\Upsilon}_1^{-1} = 1_n;\ (\underline{x}_1\underline{\varrho}_1)(\underline{\varrho}_1\underline{x}_1^{-1}) = 1_n;\ \underline{x}_1(\underline{\varrho}_1\underline{\varrho}_1)\underline{x}_1^{-1} = 1_n;\ \underline{x}_1(1_n)\underline{x}_1^{-1} = 1_n;\ \underline{x}_1\underline{x}_1^{-1} = 1_n$.

On the other side

$\underline{\varrho}_1\underline{\varrho}_1 = 1_n;\ \underline{\varrho}_1(1_n)\underline{\varrho}_1 = 1_n;\ \underline{\varrho}_1(\underline{\Upsilon}_1^{-1}\underline{\Upsilon}_1)\underline{\varrho}_1 = 1_n;\ (\underline{\varrho}_1\underline{\Upsilon}_1^{-1})(\underline{\Upsilon}_1\underline{\varrho}_1) = 1_n;\ \underline{x}_1^{-1}\underline{x}_1 = 1_n$.

The basic case of the relations

(RR4): $\underline{\sigma}_i\underline{\sigma}_{i+1}^{-1}\underline{\sigma}_i = \underline{\sigma}_{i+1}\underline{\sigma}_i^{-1}\underline{\sigma}_{i+1}$ is $\underline{\sigma}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1 = \underline{\sigma}_2\underline{\sigma}_1^{-1}\underline{\sigma}_2$. We prove *(ix)* implies $(RR4)$. By *(ii)*, *(iii)* and *(ix)*:

$\underline{\mu}_1\underline{\varrho}_2\underline{\varrho}_1\underline{\mu}_1^{-1}\underline{\varrho}_2\underline{\varrho}_1\underline{\mu}_1\underline{\underline{\varrho}_2\underline{\varrho}_1} = \underline{\varrho}_1\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_1\underline{\varrho}_2\underline{\mu}_1^{-1}\underline{\varrho}_1\underline{\varrho}_2\underline{\mu}_1$

$\underline{\mu}_1\underline{\varrho}_2\underline{\varrho}_1\underline{\mu}_1^{-1}\underline{\varrho}_2\underline{\varrho}_1\underline{\mu}_1 = \underline{\varrho}_1\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_1\underline{\varrho}_2\underline{\mu}_1^{-1}\underline{\varrho}_1\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_1\underline{\varrho}_2$

$\underline{\mu}_1\underline{\varrho}_2\underline{\varrho}_1\underline{\mu}_1^{-1}\underline{\varrho}_2\underline{\varrho}_1\underline{\mu}_1\underline{\varrho}_1 = \underline{\varrho}_1\underline{\varrho}_2\underline{\mu}_1\underline{\varrho}_1\underline{\varrho}_2\underline{\mu}_1^{-1}\underline{\varrho}_1\underline{\varrho}_2\underline{\mu}_1\underline{\underline{\varrho}_1\underline{\varrho}_2\underline{\varrho}_1}$

$$\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1\underline{(1_n)}\underline{\mu}_1^{-1}\underline{\rho}_2\underline{\rho}_1\underline{\mu}_1\underline{\rho}_1 = \underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1^{-1}\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2$$
$$\underline{\mu}_1\underline{\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2}\underline{\rho}_2\underline{\mu}_1^{-1}\underline{\rho}_2\underline{\rho}_1\underline{\mu}_1\underline{\rho}_1 = \underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1^{-1}\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2$$
$$\underline{\mu}_1\underline{\rho}_1\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1^{-1}\underline{\rho}_2\underline{\rho}_1\underline{\mu}_1\underline{\rho}_1 = \underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_1\underline{\rho}_2\underline{(1_n)}\underline{\mu}_1^{-1}\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2$$
$$\underline{\mu}_1\underline{\rho}_1\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1^{-1}\underline{\rho}_2\underline{\rho}_1\underline{\mu}_1\underline{\rho}_1 = \underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\underline{\rho}_1\underline{\rho}_2\underline{\rho}_1}\underline{\rho}_1\underline{\mu}_1^{-1}\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2$$
$$(\underline{\mu}_1\underline{\rho}_1)(\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1^{-1}\underline{\rho}_2\underline{\rho}_1)(\underline{\mu}_1\underline{\rho}_1) = (\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2)(\underline{\rho}_1\underline{\mu}_1^{-1})(\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2),$$

and by Remark 3.2 and the relations *(c)*, *(d)* of Lemma 3.3 with $i = 1$

$$(\underline{\sigma}_1)(\underline{\rho}_2\underline{\mu}_2^{-1})(\underline{\sigma}_1) = (\underline{\mu}_2\underline{\rho}_2)(\underline{\rho}_1\underline{\mu}_1^{-1})(\underline{\mu}_2\underline{\rho}_2)$$
$$\underline{\sigma}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1 = \underline{\sigma}_2\underline{\sigma}_1^{-1}\underline{\sigma}_2.$$

The initial case of the relations *(RVF1)*: $\underline{\rho}_i\underline{\sigma}_{i+1}\underline{\sigma}_i = \underline{\sigma}_{i+1}\underline{\sigma}_i\underline{\rho}_{i+1}$ is $\underline{\rho}_1\underline{\sigma}_2\underline{\sigma}_1 = \underline{\sigma}_2\underline{\sigma}_1\underline{\rho}_2$. We prove *(x)* implies $(RVF1)$. Using relations *(ii)*, *(iii)*, *(x)*, Remark 3.2 and Lemma 3.3:

$$(\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1)(\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2) = (\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2)(\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\underline{\rho}_2\underline{\rho}_1})$$
$$(\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1)\underline{\rho}_2\underline{\mu}_1\underline{\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2} = (\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2)\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1$$
$$(\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1)\underline{\rho}_2\underline{\mu}_1\underline{\rho}_1\underline{\rho}_2\underline{\underline{\rho}_1} = (\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2)\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1$$
$$(\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1)\underline{\rho}_2\underline{\mu}_1\underline{\rho}_1\underline{\rho}_2 = \underline{(1_n)}(\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2)\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_1$$
$$(\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1)\underline{\rho}_2\underline{\mu}_1\underline{\rho}_1\underline{\rho}_2 = \underline{\rho}_1\underline{\rho}_1(\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2)\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_1$$
$$(\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2)(\underline{\mu}_1\underline{\rho}_1)\underline{\rho}_2 = \underline{\rho}_1(\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2)(\underline{\mu}_1\underline{\rho}_1)$$
$$(\underline{\mu}_2\underline{\rho}_2)(\underline{\mu}_1\underline{\rho}_1)\underline{\rho}_2 = \underline{\rho}_1(\underline{\mu}_2\underline{\rho}_2)(\underline{\mu}_1\underline{\rho}_1)$$
$$\underline{\sigma}_2\underline{\sigma}_1\underline{\rho}_2 = \underline{\rho}_1\underline{\sigma}_2\underline{\sigma}_1.$$

The basic case of the relations *(RVF2)* $\underline{\sigma}_i\underline{\sigma}_{i+1}\underline{\rho}_i = \underline{\rho}_{i+1}\underline{\sigma}_i\underline{\sigma}_{i+1}$ is $\underline{\sigma}_1\underline{\sigma}_2\underline{\rho}_1 = \underline{\rho}_2\underline{\sigma}_1\underline{\sigma}_2$ We prove *(xi)* implies $(RVF2)$. Using the relations *(ii)*, *(iii)*, *(xi)*, Remark 3.2 and Lemma 3.3:

$$(\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2)\underline{\mu}_1 = \underline{\mu}_1(\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2)$$
$$(\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2)\underline{\mu}_1\underline{\underline{\rho}_1\underline{\rho}_2\underline{\rho}_1} = \underline{\mu}_1(\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2)\underline{\rho}_1\underline{\rho}_2\underline{\rho}_1$$
$$\underline{\rho}_2\underline{\mu}_1\underline{(1_n)}\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2 = \underline{\mu}_1(\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2)\underline{\rho}_1\underline{\rho}_2\underline{\rho}_1$$
$$\underline{\rho}_2\underline{\mu}_1(\underline{\rho}_1\underline{\rho}_1)\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2 = \underline{\mu}_1\underline{(1_n)}\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2\underline{\rho}_1$$
$$\underline{\rho}_2(\underline{\mu}_1\underline{\rho}_1)(\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2) = \underline{\mu}_1(\underline{\rho}_1\underline{\rho}_1)\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2\underline{\rho}_1$$
$$\underline{\rho}_2(\underline{\mu}_1\underline{\rho}_1)(\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2) = (\underline{\mu}_1\underline{\rho}_1)(\underline{\rho}_1\underline{\rho}_2\underline{\mu}_1\underline{\rho}_2\underline{\rho}_1\underline{\rho}_2)\underline{\rho}_1$$
$$\underline{\rho}_2(\underline{\mu}_1\underline{\rho}_1)(\underline{\mu}_2\underline{\rho}_2) = (\underline{\mu}_1\underline{\rho}_1)(\underline{\mu}_2\underline{\rho}_2)\underline{\rho}_1$$
$$\underline{\rho}_2\underline{\sigma}_1\underline{\sigma}_2 = \underline{\sigma}_1\underline{\sigma}_2\underline{\rho}_1.$$

It is easy to transfer the reasoning used to work with VSB_n generators to $\boldsymbol{UVSPSG}_{mn}$ generators. □

Corollary 3.6. *Let be Theorem 3.5. If we remove the relations :*

- *(v) we have a reduced presentation of* $\boldsymbol{UVSPSB}_{mn}$.
- *(ix) we have a reduced presentation of group* $\boldsymbol{UVSG}_{mn}$.
- *(xi) we have a reduced presentation of* $\boldsymbol{WSPSG}_{mn}$.
- *(v), and (ix) we have a reduced presentation of* $\boldsymbol{UVSB}_{mn}$.
- *(v), and (xi) we have a presentation of* $\boldsymbol{WSPSB}_{mn}$.
- *(x) and (xi) we have a reduced presentation of* $\boldsymbol{VSPSG}_{mn}$.

- *(ix), (x) and (xi) we have a reduced presentation of* $\boldsymbol{VSG}_{mn}$.
- *(v), (ix), and (xi) we have a presentation of* $\boldsymbol{WSB}_{mn}$.
- *(v), (x), and (xi) we have a presentation of* $\boldsymbol{VSPSB}_{mn}$.
- *(v), (ix), (x), and (xi) we have a reduced presentation of* $\boldsymbol{VSB}_{mn}$.

3.2 Reduced presentation for $\boldsymbol{UVSPSG}_{mn}$ with $\{\underline{\sigma}_1, \underline{\sigma}_2, \ldots, \underline{\sigma}_{n-1}, \underline{\mu}_1, \underline{\delta}_1\}$ generators

Theorem 3.7. *The* $\boldsymbol{UVSPSG}_{mn}$ *group has a reduced presentation with generators* $\{\underline{\sigma}_1, \underline{\sigma}_2, \ldots, \underline{\sigma}_{n-1}, \underline{\mu}_1, \underline{\delta}_1, \underline{\delta}_1^{-1}\}$ *and the following relations:*

(i) $\underline{\sigma}_i\underline{\sigma}_j = \underline{\sigma}_j\underline{\sigma}_i$ $\quad |j-i| > 1,$

(ii) $\underline{\sigma}_1^{-1}\underline{\sigma}_1 = 1_n = \underline{\sigma}_1\underline{\sigma}_1^{-1}$ $\quad i = 1, 2, \ldots, n-1,$

(iii) $\underline{\sigma}_i\underline{\sigma}_{i+1}\underline{\sigma}_i = \underline{\sigma}_{i+1}\underline{\sigma}_i\underline{\sigma}_{i+1}$ $\quad i = 1, 2, \ldots, n-2,$

(iv) $\underline{\sigma}_i\underline{\sigma}_{i+1}^{-1}\underline{\sigma}_i = \underline{\sigma}_{i+1}\underline{\sigma}_i^{-1}\underline{\sigma}_{i+1}$ $\quad i = 1, 2, \ldots, n-2,$

(v) $\underline{\delta}_1(\underline{\sigma}_2\underline{\sigma}_3\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}) = (\underline{\sigma}_2\underline{\sigma}_3\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1})\underline{\delta}_1,$

(vi) $\underline{\delta}_1^{-1}\underline{\delta}_1 = 1_n = \underline{\delta}_1\underline{\delta}_1^{-1}$ $\quad i = 1, 2, \ldots, n-1,$

(vii) $\underline{\mu}_1(\underline{\sigma}_2^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_1\underline{\sigma}_3\underline{\sigma}_2) = (\underline{\sigma}_2^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_1\underline{\sigma}_3\underline{\sigma}_2)\underline{\mu}_1,$

(viii) $\underline{\mu}_1\underline{\mu}_1^{-1} = 1_n = \underline{\mu}_1^{-1}\underline{\mu}_1,$

(ix) $\underline{\mu}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1} = \underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1,$

(x) $\underline{\delta}_1\underline{\sigma}_i = \underline{\sigma}_i\underline{\delta}_1$ $\quad i = 3, 4, \ldots, n-1,$

(xi) $\underline{\delta}_1\underline{\sigma}_1 = \underline{\sigma}_1\underline{\delta}_1,$

(xii) $\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\delta}_1\underline{\sigma}_2\underline{\sigma}_1 = \underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1},$

(xiii) $\underline{\sigma}_1^{-1}\underline{\mu}_1\underline{\sigma}_i = \underline{\sigma}_i\underline{\mu}_1^{-1}\underline{\sigma}_1$ $\quad i = 3, 4, \ldots, n-1,$

(xiv) $\underline{\sigma}_1^2\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1 = \underline{\sigma}_2^{-2}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1\underline{\sigma}_2,$

(xv) $(\underline{\sigma}_2\underline{\sigma}_1^2\underline{\sigma}_2)\underline{\mu}_1 = \underline{\mu}_1(\underline{\sigma}_2\underline{\sigma}_1^2\underline{\sigma}_2),$

(xvi) $\underline{\delta}_1\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_3\underline{\sigma}_1\underline{\sigma}_2\underline{\sigma}_1$
$= \underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_3\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1,$

(xvii) $\underline{\sigma}_2\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1(\underline{\sigma}_2\underline{\sigma}_1)^{-2}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_1 = \underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}.$

Proof. To proceed with the proof it is sufficient to consider the basic cases of the defining relations. In fact, with the use of Lemmas 3.3 and 3.4, we can describe the cases for all i.

The presentation assumes the relations Lemma 3.4-*(a)* and 3.4-*(e)*, which we refer to as the *special defining relations*. The relations ($RRS4$) and ($RVF1$) are not needed in the reduced presentation for $\boldsymbol{UVSPSG}_{mn}$, since they were implicitly used in the relations Lemma 3.4-*(a)* and 3.4-*(e)* respectively.

The relations *(RR0)*: $\underline{\sigma}_i\underline{\sigma}_j = \underline{\sigma}_j\underline{\sigma}_i$, *(RR2)*: $\underline{\sigma}_i\underline{\sigma}_i^{-1} = 1_n = \underline{\sigma}_i^{-1}\underline{\sigma}_i$, *(RR3)*: $\underline{\sigma}_i\underline{\sigma}_{i+1}\underline{\sigma}_i = \underline{\sigma}_{i+1}\underline{\sigma}_i\underline{\sigma}_{i+1}$, and *(RR4)*: $\underline{\sigma}_i\underline{\sigma}_{i+1}^{-1}\underline{\sigma}_i = \underline{\sigma}_{i+1}\underline{\sigma}_i^{-1}\underline{\sigma}_{i+1}$ of the definition of $\boldsymbol{UVSPSG}_{mn}$ group are identical, respectively, to the relations *(i)*, *(ii)*, *(iii)* and *(iv)*. Definition 3.1, is present in all of the proof procedures that follow.

The basic case of the relations *(RS0)*: $\underline{\mathrm{x}}_i\underline{\mathrm{x}}_j = \underline{\mathrm{x}}_j\underline{\mathrm{x}}_i$ is $\underline{\mathrm{x}}_1\underline{\mathrm{x}}_3 = \underline{\mathrm{x}}_3\underline{\mathrm{x}}_1$. We prove *(v)* implies the basic case of $(RS0)$. By Lemma 3.4-*(e)*, *(i)*, and *(ix)*:

$\underline{\delta}_1(\underline{\sigma}_2\underline{\sigma}_3\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}) = (\underline{\sigma}_2\underline{\sigma}_3\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1})\underline{\delta}_1$

$\underline{\delta}_1\underline{\delta}_3 = \underline{\delta}_3\underline{\delta}_1$; $\underline{\delta}_1\underline{\mathrm{x}}_3\underline{\sigma}_3 = \underline{\mathrm{x}}_3\underline{\sigma}_3\underline{\delta}_1$;

$\underline{\mathrm{x}}_1\underline{\sigma}_1\underline{\mathrm{x}}_3\underline{\sigma}_3 = \underline{\mathrm{x}}_3\underline{\sigma}_3\underline{\mathrm{x}}_1\underline{\sigma}_1$; $\underline{\mathrm{x}}_1\underline{\mathrm{x}}_3\underline{\sigma}_3\underline{\sigma}_1 = \underline{\mathrm{x}}_3\underline{\mathrm{x}}_1\underline{\sigma}_3\underline{\sigma}_1$; $\underline{\mathrm{x}}_1\underline{\mathrm{x}}_3 = \underline{\mathrm{x}}_3\underline{\mathrm{x}}_1$.

The basic case of the relations *(RS2)*: $\underline{\mathrm{x}}_i\underline{\mathrm{x}}_i^{-1} = 1_n = \underline{\mathrm{x}}_i^{-1}\underline{\mathrm{x}}_i$ is $\underline{\mathrm{x}}_1\underline{\mathrm{x}}_1^{-1} = 1_n = \underline{\mathrm{x}}_1^{-1}\underline{\mathrm{x}}_1$. We prove *(vi)* implies the basic case of $(RS2)$. By *(i)*, *(ii)*, and *(x)*:

$\underline{\delta}_1\underline{\delta}_1^{-1} = 1_n = \underline{\delta}_1^{-1}\underline{\delta}_1$; $\underline{\mathrm{x}}_1\underline{\sigma}_1\underline{\sigma}_1^{-1}\underline{\mathrm{x}}_1^{-1} = 1_n = \underline{\sigma}_1^{-1}\underline{\mathrm{x}}_1^{-1}\underline{\mathrm{x}}_1\underline{\sigma}_1$;

$\underline{\mathrm{x}}_1(1_n)\underline{\mathrm{x}}_1^{-1} = 1_n = \underline{\sigma}_1^{-1}\underline{\mathrm{x}}_1^{-1}\underline{\mathrm{x}}_1\underline{\sigma}_1$; $\underline{\mathrm{x}}_1(1_n)\underline{\mathrm{x}}_1^{-1} = 1_n = \underline{\mathrm{x}}_1^{-1}\underline{\sigma}_1^{-1}\underline{\mathrm{x}}_1\underline{\sigma}_1$;

$\underline{\mathrm{x}}_1\underline{\mathrm{x}}_1^{-1} = 1_n = \underline{\mathrm{x}}_1^{-1}\underline{\mathrm{x}}_1\underline{\sigma}_1^{-1}\underline{\sigma}_1$; $\underline{\mathrm{x}}_1\underline{\mathrm{x}}_1^{-1} = 1_n = \underline{\mathrm{x}}_1^{-1}\underline{\mathrm{x}}_1$.

The basic case of the relations *(RV0)*: $\underline{\rho}_i\underline{\rho}_j = \underline{\rho}_j\underline{\rho}_i$ is $\underline{\rho}_1\underline{\rho}_3 = \underline{\rho}_3\underline{\rho}_1$. We prove *(vii)* implies the basic case of $(RV0)$. By *(i)*, *(ii)*, and *(xii)*:

$\underline{\mu}_1(\underline{\sigma}_2^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_1\underline{\sigma}_3\underline{\sigma}_2) = (\underline{\sigma}_2^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_1\underline{\sigma}_3\underline{\sigma}_2)\underline{\mu}_1$

$\underline{\mu}_1\underline{\mu}_3 = \underline{\mu}_3\underline{\mu}_1$; $\underline{\mu}_1\underline{\sigma}_3\underline{\rho}_3 = \underline{\sigma}_3\underline{\rho}_3\underline{\mu}_1$;

$\underline{\sigma}_1\underline{\rho}_1\underline{\sigma}_3\underline{\rho}_3 = \underline{\sigma}_3\underline{\rho}_3\underline{\sigma}_1\underline{\rho}_1$; $\underline{\sigma}_1\underline{\sigma}_3\underline{\rho}_1\underline{\rho}_3 = \underline{\sigma}_3\underline{\sigma}_1\underline{\rho}_3\underline{\rho}_1$; $\underline{\rho}_1\underline{\rho}_3 = \underline{\rho}_3\underline{\rho}_1$.

The initial case of the relations *(RV2)*: $\underline{\rho}_i^2 = 1_n$ is $\underline{\rho}_1^2 = 1_n$. We prove *(viii)* implies the basic case of $(RV2)$. By *(i)*, *(ii)*, and *(x)*:

$\underline{\mu}_1\underline{\mu}_1^{-1} = 1_n = \underline{\mu}_1^{-1}\underline{\mu}_1$

$\underline{\sigma}_1\underline{\rho}_1\underline{\rho}_1\underline{\sigma}_1^{-1} = 1_n = \underline{\rho}_1\underline{\sigma}_1^{-1}\underline{\sigma}_1\underline{\rho}_1$; $\underline{\sigma}_1\underline{\rho}_1\underline{\rho}_1\underline{\sigma}_1^{-1} = 1_n = \underline{\rho}_1^2$;

$\underline{\sigma}_1^{-1}\underline{\sigma}_1\underline{\rho}_1^2\underline{\sigma}_1^{-1}\underline{\sigma}_1 = \underline{\sigma}_1^{-1}1_n\underline{\sigma}_1 = \underline{\rho}_1^2$; $\underline{\rho}_1^2 = 1_n$.

The initial case of the relations *(RV3)*: $\underline{\rho}_i\underline{\rho}_{i+1}\underline{\rho}_i = \underline{\rho}_{i+1}\underline{\rho}_i\underline{\rho}_{i+1}$ is $\underline{\rho}_1\underline{\rho}_2\underline{\rho}_1 = \underline{\rho}_2\underline{\rho}_1\underline{\rho}_2$. We prove *(ix)* implies the basic case of $(RV3)$. By *(i)*, *(ii)*, and *(iii)*:

$\underline{\mu}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1} = \underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1$

$\underline{\mu}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1 = (1_n)\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_1$

$\underline{\mu}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1 = \underline{\sigma}_1\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_1$

$\underline{\mu}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1 = \underline{\sigma}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_1$

$\underline{\sigma}_1^{-1}\underline{\mu}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2(1_n)\underline{\mu}_1 = \underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2(1_n)\underline{\mu}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_1$

$\underline{\sigma}_1^{-1}\underline{\mu}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_1\underline{\sigma}_1^{-1}\underline{\mu}_1 = \underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_1\underline{\sigma}_1^{-1}\underline{\mu}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_1$

$\underline{\sigma}_1^{-1}\underline{\mu}_1\underline{\rho}_2\underline{\sigma}_1^{-1}\underline{\mu}_1 = \underline{\rho}_2\underline{\sigma}_1^{-1}\underline{\mu}_1\underline{\rho}_2$; $\underline{\rho}_1\underline{\rho}_2\underline{\rho}_1 = \underline{\rho}_2\underline{\rho}_1\underline{\rho}_2$.

The initial case of the relations *(RRS0)*: $\underline{\mathrm{x}}_j\underline{\sigma}_i = \underline{\sigma}_i\underline{\mathrm{x}}_j$ is $\underline{\mathrm{x}}_1\underline{\sigma}_i = \underline{\sigma}_i\underline{\mathrm{x}}_1$, for $i = 3, 4, ..., n-1$. We prove *(x)* and *(i)* implies the basic case of $(RRS0)$:

$\underline{\delta}_1\underline{\sigma}_i = \underline{\sigma}_i\underline{\delta}_1$; $\underline{\delta}_1\underline{\sigma}_i\underline{\sigma}_1^{-1} = \underline{\sigma}_i\underline{\delta}_1\underline{\sigma}_1^{-1}$; $\underline{\delta}_1\underline{\sigma}_1^{-1}\underline{\sigma}_i = \underline{\sigma}_i\underline{\delta}_1\underline{\sigma}_1^{-1}$; $\underline{\mathrm{x}}_1\underline{\sigma}_i = \underline{\sigma}_i\underline{\mathrm{x}}_1$.

The initial case of the relations *(RRS2)*: $\underline{\mathrm{x}}_i\underline{\sigma}_i = \underline{\sigma}_i\underline{\mathrm{x}}_i$ is $\underline{\mathrm{x}}_1\underline{\sigma}_1 = \underline{\sigma}_1\underline{\mathrm{x}}_1$. We prove *(xi)* and *(ii)* implies the basic case of $(RRS2)$:

$\underline{\delta}_1\underline{\sigma}_1 = \underline{\sigma}_1\underline{\delta}_1$; $\underline{\delta}_1\underline{\sigma}_1^{-1}\underline{\sigma}_i = \underline{\sigma}_i\underline{\delta}_1\underline{\sigma}_1^{-1}$; $\underline{\mathrm{x}}_1\underline{\sigma}_i = \underline{\sigma}_i\underline{\mathrm{x}}_1$.

The initial case of the *(RRS3)*: $\underline{\mathrm{x}}_i\underline{\sigma}_{i+1}\underline{\sigma}_i = \underline{\sigma}_{i+1}\underline{\sigma}_i\underline{\mathrm{x}}_{i+1}$ is $\underline{\mathrm{x}}_1\underline{\sigma}_2\underline{\sigma}_1 = \underline{\sigma}_2\underline{\sigma}_1\underline{\mathrm{x}}_2$. We prove *(xii)*, *(i)*, *(ii)* and *(iii)* implies the basic case of $(RRS3)$:

$\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\delta}_1\underline{\sigma}_2\underline{\sigma}_1 = \underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}$

$\underline{\delta}_1\underline{\sigma}_2\underline{\sigma}_1 = \underline{\sigma}_2\underline{\sigma}_1\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}$
$\underline{\delta}_1\underline{\sigma}_2\underline{\sigma}_1\underline{\sigma}_2^{-1} = \underline{\sigma}_2\underline{\sigma}_1\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}$
$\underline{\delta}_1\underline{\sigma}_1^{-1}\underline{\sigma}_2\underline{\sigma}_1 = \underline{\sigma}_2\underline{\sigma}_1\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}$
$\underline{\mathrm{x}}_1\underline{\sigma}_2\underline{\sigma}_1 = \underline{\sigma}_2\underline{\sigma}_1\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}$
$\underline{\mathrm{x}}_1\underline{\sigma}_2\underline{\sigma}_1 = \underline{\sigma}_2\underline{\sigma}_1\underline{\delta}_2\underline{\sigma}_2^{-1}, \quad \underline{\mathrm{x}}_1\underline{\sigma}_2\underline{\sigma}_1 = \underline{\sigma}_2\underline{\sigma}_1\underline{\mathrm{x}}_2.$

The initial case of the relations *(RRV0)*: $\underline{\rho}_j\underline{\sigma}_i = \underline{\sigma}_i\underline{\rho}_j$ is $\underline{\rho}_1\underline{\sigma}_i = \underline{\sigma}_i\underline{\rho}_1$. We prove *(xiii)* implies the basic case of $(RRV0)$:
$\underline{\sigma}_1^{-1}\underline{\mu}_1\underline{\sigma}_i = \underline{\sigma}_i\underline{\mu}_1^{-1}\underline{\sigma}_1; \quad \underline{\rho}_1\underline{\sigma}_i = \underline{\sigma}_i\underline{\rho}_1.$

The initial case of the relations
(RRV3): $\underline{\sigma}_i\underline{\rho}_{i+1}\underline{\rho}_i = \underline{\rho}_{i+1}\underline{\rho}_i\underline{\sigma}_{i+1}$ is $\underline{\sigma}_1\underline{\rho}_2\underline{\rho}_1 = \underline{\rho}_2\underline{\rho}_1\underline{\sigma}_2$.
We prove *(xiv)*, *(ii)*, and *(iii)* implies the basic case of $(RRV3)$:
$\underline{\sigma}_1^2\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1 = \underline{\sigma}_2^{-2}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1\underline{\sigma}_2$
$\underline{\sigma}_2\underline{\sigma}_1^2\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1 = \underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1\underline{\sigma}_2$
$\underline{\sigma}_1^{-1}\underline{\sigma}_2\underline{\sigma}_1^2\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1 = \underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1\underline{\sigma}_2$
$\underline{\sigma}_2\underline{\sigma}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1 = \underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1\underline{\sigma}_2$
$\underline{\sigma}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2(1_n)\underline{\mu}_1 = \underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2(1_n)\underline{\mu}_1\underline{\sigma}_2$
$\underline{\sigma}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2(\underline{\sigma}_1\underline{\sigma}_1^{-1})\underline{\mu}_1 = \underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2(\underline{\sigma}_1\underline{\sigma}_1^{-1})\underline{\mu}_1\underline{\sigma}_2$
$\underline{\sigma}_1\underline{\sigma}_2^{-1}\underline{\mu}_2\underline{\sigma}_1^{-1}\underline{\mu}_1 = \underline{\sigma}_2^{-1}\underline{\mu}_2\underline{\sigma}_1^{-1}\underline{\mu}_1\underline{\sigma}_2, \quad \underline{\sigma}_1\underline{\rho}_2\underline{\rho}_1 = \underline{\rho}_2\underline{\rho}_1\underline{\sigma}_2.$

The initial case of the relations
(RVF2): $\underline{\sigma}_i\underline{\sigma}_{i+1}\underline{\rho}_i = \underline{\rho}_{i+1}\underline{\sigma}_i\underline{\sigma}_{i+1}$ is $\underline{\sigma}_1\underline{\sigma}_2\underline{\rho}_1 = \underline{\rho}_2\underline{\sigma}_1\underline{\sigma}_2$.
We prove *(xv)*, *(ii)*, and *(iii)* implies the basic case of $(RVF2)$:
$(\underline{\sigma}_2\underline{\sigma}_1^2\underline{\sigma}_2)\underline{\mu}_1 = \underline{\mu}_1(\underline{\sigma}_2\underline{\sigma}_1^2\underline{\sigma}_2)$
$\underline{\sigma}_1\underline{\sigma}_2\underline{\mu}_1 = \underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1(\underline{\sigma}_2\underline{\sigma}_1^2\underline{\sigma}_2)$
$\underline{\sigma}_2^{-1}\underline{\sigma}_1\underline{\sigma}_2\underline{\mu}_1 = \underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1(\underline{\sigma}_2\underline{\sigma}_1^2\underline{\sigma}_2)$
$\underline{\sigma}_1\underline{\sigma}_2\underline{\sigma}_1^{-1}\underline{\mu}_1 = \underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1(\underline{\sigma}_2\underline{\sigma}_1^2\underline{\sigma}_2$
$\underline{\sigma}_1\underline{\sigma}_2\underline{\rho}_1 = \underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1(\underline{\sigma}_2\underline{\sigma}_1^2\underline{\sigma}_2)$
$\underline{\sigma}_1\underline{\sigma}_2\underline{\rho}_1 = \underline{\sigma}_2^{-1}\underline{\mu}_2\underline{\sigma}_1\underline{\sigma}_2), \quad \underline{\sigma}_1\underline{\sigma}_2\underline{\rho}_1 = \underline{\rho}_2\underline{\sigma}_1\underline{\sigma}_2.$

The initial case of the relations *(RSV0)*: $\underline{\mathrm{x}}_i\underline{\rho}_j = \underline{\rho}_j\underline{\mathrm{x}}_i$ is $\underline{\mathrm{x}}_1\underline{\rho}_3 = \underline{\rho}_3\underline{\mathrm{x}}_1$. We prove *(xvi)*, *(i)*, *(ii)*, and *(iii)* implies the basic case of $(RSV0)$:
$\underline{\delta}_1\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_3\underline{\sigma}_1\underline{\sigma}_2\underline{\sigma}_1 = \underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_3\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1$
$\underline{\delta}_1\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_3\underline{\sigma}_1\underline{\sigma}_2 = \underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_3\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_1^{-1}$
$\underline{\delta}_1\underline{\sigma}_3^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_3\underline{\sigma}_1\underline{\sigma}_2 = \underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_3\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_1^{-1}$
$\underline{\delta}_1\underline{\sigma}_1^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_3\underline{\sigma}_1\underline{\sigma}_2 = \underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_3\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_1^{-1}$
$\underline{\delta}_1\underline{\sigma}_1^{-1}\underline{\sigma}_3^{-1}\underline{\mu}_3 = \underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_3\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_1^{-1}$
$\underline{\mathrm{x}}_1\underline{\rho}_3 = \underline{\sigma}_3^{-1}\underline{\sigma}_2^{-1}\underline{\sigma}_3^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2\underline{\sigma}_3\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_1^{-1}$
$\underline{\mathrm{x}}_1\underline{\rho}_3 = \underline{\sigma}_3^{-1}\underline{\mu}_3\underline{\delta}_1\underline{\sigma}_1^{-1}; \quad \underline{\mathrm{x}}_1\underline{\rho}_3 = \underline{\rho}_3\underline{\mathrm{x}}_1.$

The initial case of the relations *(RSV3)*: $\underline{\mathrm{x}}_i\underline{\rho}_j = \underline{\rho}_j\underline{\mathrm{x}}_i$ is $\underline{\mathrm{x}}_1\underline{\rho}_3 = \underline{\rho}_3\underline{\mathrm{x}}_1$. We prove *(xvii)*, and *(ii)* implies the basic case of $(RSV3)$:
$\underline{\sigma}_2\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1(\underline{\sigma}_2\underline{\sigma}_1)^{-2}\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1 = \underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}$
$\underline{\sigma}_2\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1(\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1})^2\underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1 = \underline{\mu}_1\underline{\sigma}_2\underline{\mu}_1\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}$
$\underline{\delta}_1(\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1})^2\underline{\mu}_1\underline{\sigma}_2(1_n)\underline{\mu}_1 = \underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2(1_n)\underline{\mu}_1\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}$

$$\begin{aligned}
&\underline{\delta}_1(\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1})^2\underline{\mu}_1\underline{\sigma}_2(\underline{\sigma}_1\underline{\sigma}_1^{-1})\underline{\mu}_1 = \underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2(1_n)\underline{\mu}_1\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\\
&\underline{\delta}_1\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_2\underline{\sigma}_1^{-1}\underline{\mu}_1 = \underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_1\underline{\sigma}_2(\underline{\sigma}_1\underline{\sigma}_1^{-1})\underline{\mu}_1\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\\
&\underline{\delta}_1\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_2\underline{\sigma}_1^{-1}\underline{\mu}_1 = \underline{\sigma}_2^{-1}\underline{\mu}_2\underline{\sigma}_1^{-1}\underline{\mu}_1\underline{\sigma}_1\underline{\sigma}_2\underline{\delta}_1\underline{\sigma}_2^{-1}\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\\
&\underline{\delta}_1\underline{\sigma}_1^{-1}\underline{\sigma}_2^{-1}\underline{\mu}_2\underline{\sigma}_1^{-1}\underline{\mu}_1 = \underline{\sigma}_2^{-1}\underline{\mu}_2\underline{\sigma}_1^{-1}\underline{\mu}_1\underline{\delta}_2\underline{\sigma}_2^{-1}, \quad \underline{\mathrm{x}}_1\underline{\rho}_2\underline{\rho}_1 = \underline{\rho}_2\underline{\rho}_1\underline{\mathrm{x}}_2. \qquad \square
\end{aligned}$$

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