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On Various 2-absorbing prime ideals in non commutative rings

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Abstract

In this paper we analyze strongly 2-absorbing prime ideals (shortly strongly 2-API), strongly 2-absorbing weak prime ideals (shortly strongly 2-AWPI) and 2-absorbing weak prime ideals (shortly 2-AWPI) in a non-commutative ring, which represent generalization of prime ideals (shortly PI) in a non-commutative ring. The relationship between the strongly 2-API and the 2-absorbing prime ideal (shortly 2-API) is examined. We provide examples to illustrate the new concept of strongly m_{a1}-system and strongly m_{a2}-system as well as the relationships beween them. Let I be an ideal of $\mathscr{R}$ and $\mathscr{M}$ be a strongly m_{a1}-system such that $I \cap \mathscr{M} = \phi$. Then there exists a strongly 2-API $\mathscr{P}$ of $\mathscr{R}$ containing I such that $\mathscr{P} \cap \mathscr{M} = \phi$. We prove that $\mathscr{P}$ is a strongly 2-API if and only if $\mathscr{A}_1\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$ implies that $\mathscr{A}_1 \subseteq \mathscr{P}$ or $\mathscr{A}_2 \subseteq \mathscr{P}$ or $\mathscr{A}_3 \subseteq \mathscr{P}$ for all ideals $\mathscr{A}_1, \mathscr{A}_2$ and $\mathscr{A}_3$ of $\mathscr{R}$.

1 Introduction

Numerous studies have described various forms of ideals in algebraic structures like semirings [1] and rings [2]. Noether expanded on the idea of ideals that Dedekind had developed for the theory of algebraic numbers in order to include associative rings. Good and Hughes first presented the idea of

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bi-ideals for semigroups in their 1952 [3]. Additionally, it is a specific instance of (m, n)-ideal introduced by Lajos. Many authors [4, 5, 6] have described different classes of semigroups using bi-ideals. Lajos and Szasz introduced bi-ideals for associative rings [7]. Quasi ideals are generalizations of both left and right ideals. The concept of quasi-ideals was introduced by Otto Steinfeld [8] for semigroups and rings in 1956. There are numerous ways to characterize PIs in semirings [1]. Although it has been used extensively in the theory of commutative rings, the study of non-commutative rings did not use the PI concept much. McCoy has analyzed several aspects of PIs in general rings [2], [9] . The role of PIs in the semiring theory and the ring theory is mentioned in [10, 1, 11]. Various ideals based on semigroups, semirings, ternary semirings are analyzed by Palanikumar et al. [12, 13, 14, 15, 16]. In 2021, Palanikumar et al. [17] introduced the novel idea of prime k-ideals in semirings. Also, Palanikumar et al. discussed several kinds of prime and semiprime bi-ideals of a ring [18] in 2021.
Walt [19] explored the prime and semiprime bi-ideals of unity-based associative rings in 1983. The results of prime and semiprime bi-ideals of associative rings with unity were extended to associative rings without unity by Roux [20] in 1995. In 2005, Flaska, Kepka, and Saroch provided some characterizations of bi-ideals in simple semirings [21]. The ideal theory of commutative semirings with non-zero identities was described by Atani [22] in 2012. In order to examine the factorization in a commutative ring with zero divisors, Anderson and Smith [23] established the concept of weakly PIs in a commutative ring. Badawi was the first to introduce the idea of 2-absorbing ideals, which is a generalization of PIs in commutative rings [24]. There are various methods to generalize Galovich's idea of a PI in 1978. An ideal p in $\mathscr{R}$ is a PI if and only if the complement of p in $\mathscr{R}$ is an m-system. The characterization of PIs plays an important role in the sequel. In 2013, the topic of weakly 2-absorbing ideals of commutative rings was studied by Badawi et al. [25]. The new idea of weakly 2-absorbing ideals in non-commutative rings was introduced by Malik Bataineh et al. [26] in 2018. For an ideal $\mathscr{P}$ of a commutative ring $\mathscr{R}$, the following propositions are equivalent: (i) $\mathscr{P}$ is 2-absorbing ideal, (ii) For ideals $\mathscr{A}_1, \mathscr{A}_2$ and $\mathscr{A}_3$ of $\mathscr{R}$ with $\mathscr{A}_1\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$, then either $\mathscr{A}_1\mathscr{A}_2 \subseteq \mathscr{P}$ or $\mathscr{A}_1\mathscr{A}_3 \subseteq \mathscr{P}$ or $\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$. For rings that are not always commutative, it is obvious that (ii) does not require (i).

In this paper, the PI concept in a non commutative ring approach is broadened to analyze a strongly 2-API in a non commutative ring. The article is divided into the following four sections. An introduction is the section 1, the section 2 contains some preliminaries. Also, the strongly 2-API in non commutative ring is presented in section 2. Strongly 2-AWPI in a non commutative ring is discussed in section 3 along with several examples. Section 4 provides a conclusion of the article. The purpose of this paper is,

1. To show that strongly 2-API implies 2-API, but the converse implication is not valid see Example 2.8.
2. To show that strongly m_{a1}-system implies strongly m_{a2}-system and its reverse implication is not valid see Example 2.13.
3. To establish that strongly 2-API implies strongly 2-AWPI, but opposite implication is not true see Example 3.4.
4. To prove that every strongly 2-AWPI is a 2-AWPI, but reverse implication fails see Example 3.6.

2 Preliminaries and Strongly 2-APIs

Throughout this paper, $\mathscr{R}$ denotes a non-commutative ring unless otherwise specified. We recall some basic notions that we use in what follows.

Definition 2.1. *[2] (i) A proper ideal $\mathscr{P}$ of $\mathscr{R}$ is called PI if $\mathscr{A}_1\mathscr{A}_2 \subseteq \mathscr{P}$, then either $\mathscr{A}_1 \subseteq \mathscr{P}$ or $\mathscr{A}_2 \subseteq \mathscr{P}$ for ideals $\mathscr{A}_1$ and $\mathscr{A}_2$ of $\mathscr{R}$.*
(ii) A proper ideal $\mathscr{P}$ of $\mathscr{R}$ is called weakly PI if $0 \neq \mathscr{A}_1\mathscr{A}_2 \subseteq \mathscr{P}$, then either $\mathscr{A}_1 \subseteq \mathscr{P}$ or $\mathscr{A}_2 \subseteq \mathscr{P}$ for ideals $\mathscr{A}_1$ and $\mathscr{A}_2$ of $\mathscr{R}$.

Definition 2.2. *[26] (i) A proper ideal $\mathscr{P}$ of $\mathscr{R}$ is called 2-absorbing ideal (shortly 2-AI) if $\mathscr{A}_1\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$, then either $\mathscr{A}_1\mathscr{A}_2 \subseteq \mathscr{P}$ or $\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$ or $\mathscr{A}_1\mathscr{A}_3 \subseteq \mathscr{P}$, for ideals $\mathscr{A}_1, \mathscr{A}_2$ and $\mathscr{A}_3$ of $\mathscr{R}$.*
(ii) A proper ideal $\mathscr{P}$ of $\mathscr{R}$ is called weakly 2-AI if $0 \neq \mathscr{A}_1\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$, then either $\mathscr{A}_1\mathscr{A}_2 \subseteq \mathscr{P}$ or $\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$ or $\mathscr{A}_1\mathscr{A}_3 \subseteq \mathscr{P}$, for ideals $\mathscr{A}_1, \mathscr{A}_2$ and $\mathscr{A}_3$ of $\mathscr{R}$.

Definition 2.3. *[24],[25] (i) A proper ideal $\mathscr{P}$ of a commutative ring R is called 2-AI if whenever $a, b, c \in R$ and $abc \in \mathscr{P}$, then either $ab \in \mathscr{P}$ or $bc \in \mathscr{P}$, or $ac \in \mathscr{P}$.*
(ii) A proper ideal $\mathscr{P}$ of a commutative ring R is called weakly 2-AI if whenever $a, b, c \in R$ and $0 \neq abc \in \mathscr{P}$, then either $ab \in \mathscr{P}$ or $bc \in \mathscr{P}$, or $ac \in \mathscr{P}$.

Lemma 2.4. *[2] For $x \in \mathscr{R}$, (i) The ideal generated by "x" is defined as $< x >= \{nx + sx + xt + \sum s_i x t_i | n \in N, s, t, s_i, t_i \in \mathscr{R}\}$.*
(ii) The right ideal generated by "x" is defined as $< x >_r = \{nx + \sum x t_i | n \in N, t_i \in \mathscr{R}\}$.
(iii) The left ideal generated by "x" is defined as $< x >_l = \{nx + \sum t_i x | n \in N, t_i \in \mathscr{R}\}$.

Now, we present various kinds of 2-APIs and m-systems.

Definition 2.5. *(i) A proper ideal $\mathscr{P}$ of $\mathscr{R}$ is called a strongly 2-API if $a\mathscr{R}b\mathscr{R}c \subseteq \mathscr{P}$ implies $a \in \mathscr{P}$ or $b \in \mathscr{P}$ or $c \in \mathscr{P}$ for $a, b, c \in \mathscr{R}$.*
(ii) A proper ideal $\mathscr{P}$ of $\mathscr{R}$ is called a 2-API if $\mathscr{A}_1\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$ implies $\mathscr{A}_1\mathscr{A}_2 \subseteq \mathscr{P}$ or $\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$ or $\mathscr{A}_1\mathscr{A}_3 \subseteq \mathscr{P}$ for ideals $\mathscr{A}_1, \mathscr{A}_2$ and $\mathscr{A}_3$ of $\mathscr{R}$.

Definition 2.6. *(i) A subset $\mathscr{M}$ of $\mathscr{R}$ is called strongly m_{a1}-system if for any $a, b, c \in \mathscr{M}$, there exist $r_1, r_2 \in \mathscr{R}$ such that $ar_1br_2c \in \mathscr{M}$.*
(ii) A subset $\mathscr{M}$ of $\mathscr{R}$ is called strongly m_{a2}-system if for any three ideals $\mathscr{A}_1, \mathscr{A}_2$ and $\mathscr{A}_3$ of $\mathscr{R}$ with $\mathscr{A}_1\mathscr{A}_2 \cap \mathscr{M} \neq \phi$, $\mathscr{A}_2\mathscr{A}_3 \cap \mathscr{M} \neq \phi$ and $\mathscr{A}_1\mathscr{A}_3 \cap \mathscr{M} \neq \phi$, then there exists $a \in \mathscr{A}_1, b \in \mathscr{A}_2$ and $c \in \mathscr{A}_3$ such that $abc \in \mathscr{M}$.

Theorem 2.7. *If $\mathscr{P}$ is a strongly 2-API of $\mathscr{R}$, then $\mathscr{P}$ is a 2- API of $\mathscr{R}$.*

Proof. Suppose that $\mathscr{P}$ is a strongly 2-API of $\mathscr{R}$ and $\mathscr{A}_1\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$, for the ideals $\mathscr{A}_1, \mathscr{A}_2$ and $\mathscr{A}_3$ of $\mathscr{R}$. Let us shows that $\mathscr{A}_1\mathscr{A}_2 \subseteq \mathscr{P}$ or $\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$ or $\mathscr{A}_1\mathscr{A}_3 \subseteq \mathscr{P}$. Suppose that $\mathscr{A}_2\mathscr{A}_3 \nsubseteq \mathscr{P}$ and $\mathscr{A}_1\mathscr{A}_3 \nsubseteq \mathscr{P}$. Then there exist $a_1 \in \mathscr{A}_1, b_1 \in \mathscr{A}_2$ and $c, c_1 \in \mathscr{A}_3$ such that $b_1c_1 \in \mathscr{A}_2\mathscr{A}_3 \setminus \mathscr{P}$ and $a_1c \in \mathscr{A}_1\mathscr{A}_3 \setminus \mathscr{P}$. This implies that $b_1c_1 \notin \mathscr{P}$ and $a_1c \notin \mathscr{P}$. We show that $\mathscr{A}_1\mathscr{A}_2 \subseteq \mathscr{P}$. Let $ab \in \mathscr{A}_1\mathscr{A}_2$. Thus $(ab) \subseteq \mathscr{A}_1\mathscr{A}_2$. Now, $(ab)\mathscr{R}(b_1c_1)\mathscr{R}(a_1c) \subseteq \mathscr{A}_1\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$. This implies that $(ab) \subseteq \mathscr{P}$. Therefore $\mathscr{A}_1\mathscr{A}_2 \subseteq \mathscr{P}$. Thus $\mathscr{P}$ is a 2- API of $\mathscr{R}$.

The converse of the Theorem 2.7 is not true as we can see from the following Example.

Example 2.8. *Consider the ring* $\mathscr{R} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Z}_2 \right\}$.
Let $\mathscr{P} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ *is a 2- API, but not a strongly 2-API of $\mathscr{R}$. We have* $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \mathscr{R} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathscr{R} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \subseteq \mathscr{P}$, *but* $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \notin \mathscr{P}$.

Theorem 2.9. *If $\mathscr{P}$ is a proper ideal of $\mathscr{R}$, then $\mathscr{P}$ is a strongly 2-API of $\mathscr{R}$ with unity if and only if $\mathscr{A}_1\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$ implies $\mathscr{A}_1 \subseteq \mathscr{P}$ or $\mathscr{A}_2 \subseteq \mathscr{P}$ or $\mathscr{A}_3 \subseteq \mathscr{P}$ for all ideals $\mathscr{A}_1, \mathscr{A}_2$ and $\mathscr{A}_3$ of $\mathscr{R}$.*

Proof. Suppose that $\mathscr{P}$ is a strongly 2-API of $\mathscr{R}$ and $\mathscr{A}_1\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$, for the ideals $\mathscr{A}_1, \mathscr{A}_2$ and $\mathscr{A}_3$ of $\mathscr{R}$. Let us show that $\mathscr{A}_1 \subseteq \mathscr{P}$ or $\mathscr{A}_2 \subseteq \mathscr{P}$ or $\mathscr{A}_3 \subseteq \mathscr{P}$. Suppose that $\mathscr{A}_2 \nsubseteq \mathscr{P}$ and $\mathscr{A}_3 \nsubseteq \mathscr{P}$. Then there exist $b \in \mathscr{A}_2$ and $c \in \mathscr{A}_3$ such that $b \notin \mathscr{P}$ and $c \notin \mathscr{P}$. We show that $\mathscr{A}_1 \subseteq \mathscr{P}$. Let $a \in \mathscr{A}_1$. Now, $a\mathscr{R}b\mathscr{R}c \in \mathscr{A}_1\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$. This implies that $a \in \mathscr{P}$. Therefore $\mathscr{A}_1 \subseteq \mathscr{P}$.

Conversely, suppose that $\mathscr{A}_1\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$ implies that $\mathscr{A}_1 \subseteq \mathscr{P}$ or $\mathscr{A}_2 \subseteq \mathscr{P}$

or $\mathscr{A}_3 \subseteq \mathscr{P}$ for the ideals $\mathscr{A}_1, \mathscr{A}_2$ and $\mathscr{A}_3$ of $\mathscr{R}$. Suppose that $a\mathscr{R}b\mathscr{R}c \subseteq \mathscr{P}$. Then $\mathscr{R}a\mathscr{R}b\mathscr{R}c\mathscr{R} \subseteq \mathscr{R}\mathscr{P}\mathscr{R} \subseteq \mathscr{P}$. Also $(\mathscr{R}a\mathscr{R})(\mathscr{R}b\mathscr{R})(\mathscr{R}c\mathscr{R}) = \mathscr{R}a\mathscr{R}^2b\mathscr{R}^2c\mathscr{R} \subseteq \mathscr{R}a\mathscr{R}b\mathscr{R}c\mathscr{R} \subseteq \mathscr{P}$. This implies that $\mathscr{R}a\mathscr{R} \subseteq \mathscr{P}$ or $\mathscr{R}b\mathscr{R} \subseteq \mathscr{P}$ or $\mathscr{R}c\mathscr{R} \subseteq \mathscr{P}$. Hence $a \in \mathscr{P}$ or $b \in \mathscr{P}$ or $c \in \mathscr{P}$. Hence $\mathscr{P}$ is a strongly 2-API of $\mathscr{R}$.

Theorem 2.10. *If $\mathscr{P}$ is a ideal of $\mathscr{R}$, then $\mathscr{P}$ is a strongly 2-API if and only if $\mathscr{R} \setminus \mathscr{P}$ is a strongly m_{a1}-system.*

Proof. Let $\mathscr{P}$ be a strongly 2-API of $\mathscr{R}$. Let us show that $\mathscr{M} = \mathscr{R} \setminus \mathscr{P}$ is a strongly m_{a1}-system. Let $a, b, c \in \mathscr{R} \setminus \mathscr{P}$. Thus $a \notin \mathscr{P}, b \notin \mathscr{P}$ and $c \notin \mathscr{P}$. Hence $a\mathscr{R}b\mathscr{R}c \nsubseteq \mathscr{P}$. Then there exist $r_1, r_2 \in \mathscr{R}$ such that $ar_1br_2c \notin \mathscr{P}$. Thus $ar_1br_2c \in \mathscr{M}$. Hence $\mathscr{R} \setminus \mathscr{P}$ is an strongly m_{a1}-system.

Conversely, Let $\mathscr{R} \setminus \mathscr{P}$ is a strongly m_{a1}-system. Suppose that $a\mathscr{R}b\mathscr{R}c \subseteq \mathscr{P}$. Let us shows that $a \in \mathscr{P}$ or $b \in \mathscr{P}$ or $c \in \mathscr{P}$. Suppose that $a \notin \mathscr{P}, b \notin \mathscr{P}$ and $c \notin \mathscr{P}$. Now, $a, b, c \in \mathscr{R} \setminus \mathscr{P}$. Since $\mathscr{R} \setminus \mathscr{P}$ is an strongly m_{a1}-system, then there exist $r_1, r_2 \in \mathscr{R}$ such that $ar_1br_2c \in \mathscr{R} \setminus \mathscr{P}$. Since $ar_1br_2c \in a\mathscr{R}b\mathscr{R}c \subseteq \mathscr{P}$. Thus $ar_1br_2c \in \mathscr{P}$, which is a contradiction. Hence $a \in \mathscr{P}$ or $b \in \mathscr{P}$ or $c \in \mathscr{P}$. Therefore $\mathscr{P}$ is a strongly 2-API of $\mathscr{R}$.

Theorem 2.11. *If $\mathscr{P}$ is a ideal of $\mathscr{R}$, then $\mathscr{P}$ is a 2-API if and only if $\mathscr{R} \setminus \mathscr{P}$ is a strongly m_{a2}-system.*

Proof. Let $\mathscr{P}$ be a 2-API of $\mathscr{R}$. Let us show that $\mathscr{M} = \mathscr{R} \setminus \mathscr{P}$ is an strongly m_{a2}-system. Let $\mathscr{A}_1, \mathscr{A}_2$ and $\mathscr{A}_3$ be the ideals of $\mathscr{R}$ with $\mathscr{A}_1\mathscr{A}_2 \cap \mathscr{M} \neq \phi$, $\mathscr{A}_2\mathscr{A}_3 \cap \mathscr{M} \neq \phi$ and $\mathscr{A}_1\mathscr{A}_3 \cap \mathscr{M} \neq \phi$. Hence $\mathscr{A}_1\mathscr{A}_2\mathscr{A}_3 \nsubseteq \mathscr{P}$. Then there exist $x \in \mathscr{A}_1$, $y \in \mathscr{A}_2$ and $z \in \mathscr{A}_3$ such that $xyz \notin \mathscr{P}$. Thus $xyz \in \mathscr{R} \setminus \mathscr{P}$. Hence $\mathscr{R} \setminus \mathscr{P}$ is a strongly m_{a2}-system.

Conversely, Let $\mathscr{R} \setminus \mathscr{P}$ be a strongly m_{a2}-system. Let $\mathscr{A}_1, \mathscr{A}_2$ and $\mathscr{A}_3$ be the ideals of $\mathscr{R}$ and $\mathscr{A}_1\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$. Let us shows that $\mathscr{A}_1\mathscr{A}_2 \subseteq \mathscr{P}$ or $\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$ or $\mathscr{A}_1\mathscr{A}_3 \subseteq \mathscr{P}$. Suppose that $\mathscr{A}_1\mathscr{A}_2 \nsubseteq \mathscr{P}$ or $\mathscr{A}_2\mathscr{A}_3 \nsubseteq \mathscr{P}$ or $\mathscr{A}_1\mathscr{A}_3 \nsubseteq \mathscr{P}$. This implies that $\mathscr{A}_1\mathscr{A}_2 \cap \mathscr{M} \neq \phi$, $\mathscr{A}_2\mathscr{A}_3 \cap \mathscr{M} \neq \phi$ and $\mathscr{A}_1\mathscr{A}_3 \cap \mathscr{M} \neq \phi$. Since $\mathscr{M}$ is an strongly m_{a2}-system, then there exist $a \in \mathscr{A}_1$, $b \in \mathscr{A}_2$ and $c \in \mathscr{A}_3$ such that $abc \in \mathscr{M}$. Since $abc \in \mathscr{A}_1\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$, which is a contradiction. Thus $\mathscr{A}_1\mathscr{A}_2 \subseteq \mathscr{P}$ or $\mathscr{A}_2\mathscr{A}_3 \subseteq \mathscr{P}$ or $\mathscr{A}_1\mathscr{A}_3 \subseteq \mathscr{P}$. Hence $\mathscr{P}$ is a 2-API of $\mathscr{R}$.

Lemma 2.12. *Every strongly m_{a1}-system is a strongly m_{a2}-system.*

Proof. Suppose that $\mathscr{M}$ is a strongly m_{a1}-system. Let $x, y \in \mathscr{M}$, there exist $r_1, r_2 \in \mathscr{R}$ such that $xr_1yr_2z \in \mathscr{M}$. Let $x \in \mathscr{A}_1\mathscr{A}_2 \cap \mathscr{M}$, $y \in \mathscr{A}_2\mathscr{A}_3 \cap \mathscr{M}$ and $z \in \mathscr{A}_1\mathscr{A}_3 \cap \mathscr{M}$, for ideals $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ of $\mathscr{R}$. This implies that $x = ab_1$, $y = bc_1$ and $z = a_2c$. Since $xr_1yr_2z \in \mathscr{M}$ implies that $(ab_1r_1)(bc_1r_2)(a_2c) \in \mathscr{M}$ and hence $abc \in \mathscr{M}$. Therefore $\mathscr{M}$ is a strongly m_{a2}-system.

Converse of the Lemma 2.12 need not true by the Example.

Example 2.13. *Consider the ring* $\mathscr{R} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Z}_2 \right\}$.

Let $\mathscr{M} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ *is a* m_{a2}*-system, but not a* m_{a1}*- system. For* $x = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $z = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathscr{M}$*, but there are no* $r_1, r_2 \in \mathscr{R}$ *such that* $xr_1yr_2z \in \mathscr{M}$.

Theorem 2.14. *Let* I *be an ideal of* $\mathscr{R}$ *and* $\mathscr{M}$ *be a strongly* m_{a1}*-system with* $I \cap \mathscr{M} = \phi$. *Then there exists a strongly* 2*- API* $\mathscr{P}$ *of* $\mathscr{R}$ *containing* I *with* $\mathscr{P} \cap \mathscr{M} = \phi$.

Proof. Let $X = \{J | J \text{ is an ideal with } I \subseteq J \text{ and } J \cap \mathscr{M} = \phi\}$. Clearly X is non-empty. By Zorn's lemma, there exist an maximal element $\mathscr{P}$ in $\mathscr{R}$ with $I \subseteq \mathscr{P}$. We claim that $\mathscr{P}$ is a strongly 2- API of $\mathscr{R}$. Suppose that $<a><b><c> \subseteq \mathscr{P}$. Let us show that $a \in \mathscr{P}$ or $b \in \mathscr{P}$ or $c \in \mathscr{P}$. Suppose $a, b, c \notin \mathscr{P}$ imply $\mathscr{P} \subset \mathscr{P} + <a>$, $\mathscr{P} \subset \mathscr{P} + <b>$ and $\mathscr{P} \subset \mathscr{P} + <c>$. By maximal property, then $(\mathscr{P} + <a>) \cap \mathscr{M} \neq \phi$, $(\mathscr{P} + <b>) \cap \mathscr{M} \neq \phi$ and $(\mathscr{P} + <c>) \cap \mathscr{M} \neq \phi$. The ideal $\mathscr{P} + <a>$, $\mathscr{P} + <b>$ and $\mathscr{P} + <c>$ contains an element m_1, m_2 and m_3 respectively of $\mathscr{M}$. We have $m_1 \in (\mathscr{P} + <a>) \cap \mathscr{M}$, $m_2 \in (\mathscr{P} + <b>) \cap \mathscr{M}$ and $m_3 \in (\mathscr{P} + <c>) \cap \mathscr{M}$. Since $\mathscr{M}$ is a strongly m_{a1}-system, then there exist $r_1, r_2 \in \mathscr{R}$ such that $m_1r_1m_2r_2m_3 \in \mathscr{M}$. But $m_1r_1m_2r_2m_3 \in (\mathscr{P} + <a>)(\mathscr{P} + <b>)(\mathscr{P} + <c>)$ is an ideal. But $(\mathscr{P} + <a>)(\mathscr{P} + <b>)(\mathscr{P} + <c>) = \mathscr{P} + <a><b><c> \subseteq \mathscr{P}$. Thus $\mathscr{P} \cap \mathscr{M} \neq \phi$, which is a contradiction to $\mathscr{P} \cap \mathscr{M} = \phi$. Thus, $<a><b><c> \nsubseteq \mathscr{P}$. Hence $\mathscr{P}$ is a strongly 2- API of $\mathscr{R}$.

Theorem 2.15. *Let* I *be an ideal of* $\mathscr{R}$ *and* $\mathscr{M}$ *be a strongly* m_{a2}*-system with* $I \cap \mathscr{M} = \phi$. *Then there exists a* 2*- API* $\mathscr{P}$ *of* $\mathscr{R}$ *containing* I *with* $\mathscr{P} \cap \mathscr{M} = \phi$.

Proof. Let $X = \{J | J \text{ is an ideal with } I \subseteq J \text{ and } J \cap \mathscr{M} = \phi\}$. Clearly X is non-empty. By Zorn's lemma, there exist an maximal element $\mathscr{P}$ in $\mathscr{R}$ with $I \subseteq \mathscr{P}$. We claim that $\mathscr{P}$ is a 2- API of $\mathscr{R}$. Suppose that $<a><b><c> \subseteq \mathscr{P}$. Let us show that $<a><b> \subseteq \mathscr{P}$ or $<b><c> \subseteq \mathscr{P}$ or $<a><c> \subseteq \mathscr{P}$. Suppose that $<a><b> \nsubseteq \mathscr{P}$, $<b><c> \nsubseteq \mathscr{P}$ and $<a><c> \nsubseteq \mathscr{P}$. This imply that $<a> \nsubseteq \mathscr{P}$, $<b> \nsubseteq \mathscr{P}$ and $<c> \nsubseteq \mathscr{P}$. Hence $\mathscr{P} \subset \mathscr{P} + <a>$, $\mathscr{P} \subset \mathscr{P} + <b>$ and $\mathscr{P} \subset \mathscr{P} + <c>$. By the maximal property, $(\mathscr{P} + <a>) \cap \mathscr{M} \neq \phi$, $(\mathscr{P} + <b>) \cap \mathscr{M} \neq \phi$ and $(\mathscr{P} + <c>) \cap \mathscr{M} \neq \phi$. Then $\mathscr{A}_1 = \mathscr{P} + <a>$, $\mathscr{A}_2 = \mathscr{P} + <b>$ and $\mathscr{A}_3 = \mathscr{P} + <c>$ are ideals of $\mathscr{R}$. Hence $\mathscr{A}_1\mathscr{A}_2 \cap \mathscr{M} = (\mathscr{P} + <a><b>) \cap \mathscr{M} \neq \phi$, $\mathscr{A}_2\mathscr{A}_3 \cap \mathscr{M} = (\mathscr{P} + <b><c>) \cap \mathscr{M} \neq \phi$ and $\mathscr{A}_1\mathscr{A}_3 \cap \mathscr{M} =$

$(\mathscr{P}+ < a >< c >) \cap \mathscr{M} \neq \phi$. Since $\mathscr{M}$ is a strongly m_{a2}-system, then there exist $a_1 \in (\mathscr{P}+ < a >)$, $b_1 \in (\mathscr{P}+ < b >)$ and $c_1 \in (\mathscr{P}+ < c >)$ such that $a_1b_1c_1 \in \mathscr{M}$. But $a_1b_1c_1 \in (\mathscr{P}+ < a >)(\mathscr{P}+ < b >)(\mathscr{P}+ < c >) = \mathscr{P}+ < a >< b >< c > \subseteq \mathscr{P}$. Thus $\mathscr{P} \cap \mathscr{M} \neq \phi$, which is a contradiction to $\mathscr{P} \cap \mathscr{M} = \phi$. Thus, $< a >< b >< c > \nsubseteq \mathscr{P}$. Hence $\mathscr{P}$ is a 2- API of $\mathscr{R}$.

3 Strongly 2-AWPIs

Here, we present various kinds of 2-AWPIs and weak m-systems.

Definition 3.1. *(i) A proper ideal $\mathscr{P}$ of $\mathscr{R}$ is called a strongly 2-AWPI if $0 \neq x\mathscr{R}y\mathscr{R}z \subseteq \mathscr{P}$, then $x \in \mathscr{P}$ or $y \in \mathscr{P}$ or $z \in \mathscr{P}$ for $x,\ y,\ z \in \mathscr{R}$.*
(ii) A proper ideal $\mathscr{P}$ of $\mathscr{R}$ is called a 2-AWPI if $0 \neq \mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \subseteq \mathscr{P}$, then $\mathscr{X}_1\mathscr{X}_2 \subseteq \mathscr{P}$ or $\mathscr{X}_2\mathscr{X}_3 \subseteq \mathscr{P}$ or $\mathscr{X}_1\mathscr{X}_3 \subseteq \mathscr{P}$ for ideals $\mathscr{X}_1$, $\mathscr{X}_2$ and $\mathscr{X}_3$ of $\mathscr{R}$.

Definition 3.2. *(i) A subset $\mathscr{M}$ of $\mathscr{R}$ is called weak m_{a1}-system if for any ideals $\mathscr{X}_1, \mathscr{X}_2$ and $\mathscr{X}_3$ of $\mathscr{R}$ with $\mathscr{X}_1 \cap \mathscr{M} \neq \phi, \mathscr{X}_2 \cap \mathscr{M} \neq \phi$ and $\mathscr{X}_3 \cap \mathscr{M} \neq \phi$, then either $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 = 0$ or $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \cap \mathscr{M} \neq \phi$.*
(ii) A subset $\mathscr{M}$ of $\mathscr{R}$ is called weak m_{a2}-system if for any three ideals $\mathscr{X}_1, \mathscr{X}_2$ and $\mathscr{X}_3$ of $\mathscr{R}$ with $\mathscr{X}_1\mathscr{X}_2 \cap \mathscr{M} \neq \phi$, $\mathscr{X}_2\mathscr{X}_3 \cap \mathscr{M} \neq \phi$ and $\mathscr{X}_1\mathscr{X}_3 \cap \mathscr{M} \neq \phi$, then either $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 = 0$ or $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \cap \mathscr{M} \neq \phi$.

Theorem 3.3. *If $\mathscr{P}$ is a strongly 2-API of $\mathscr{R}$, then $\mathscr{P}$ is a strongly 2- AWPI of $\mathscr{R}$.*

Proof. Straightforward.

The converse of the Theorem 3.3 is not true as we can see by the following Example.

Example 3.4. *Consider the ring* $\mathscr{R} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Z}_2 \right\}$.
Let $\mathscr{P} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ *is a strongly 2- AWPI, but $\mathscr{P}$ is not a strongly 2-API of $\mathscr{R}$. Since* $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \mathscr{R} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathscr{R} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \subseteq \mathscr{P}$, *but* $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \notin \mathscr{P}$.

Theorem 3.5. *If $\mathscr{P}$ is a strongly 2-AWPI of $\mathscr{R}$, then $\mathscr{P}$ is a 2- AWPI of $\mathscr{R}$.*

Proof. Suppose that $\mathscr{P}$ is a strongly 2-AWPI of $\mathscr{R}$ and $0 \neq \mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \subseteq \mathscr{P}$, for the ideals $\mathscr{X}_1, \mathscr{X}_2$ and $\mathscr{X}_3$ of $\mathscr{R}$. Let us shows that $\mathscr{X}_1\mathscr{X}_2 \subseteq \mathscr{P}$ or

$\mathscr{X}_2\mathscr{X}_3 \subseteq \mathscr{P}$ or $\mathscr{X}_1\mathscr{X}_3 \subseteq \mathscr{P}$. Suppose that $\mathscr{X}_2\mathscr{X}_3 \nsubseteq \mathscr{P}$ and $\mathscr{X}_1\mathscr{X}_3 \nsubseteq \mathscr{P}$. Then there exist $x_1 \in \mathscr{X}_1$, $y_1 \in \mathscr{X}_2$ and z, $z_1 \in \mathscr{X}_3$ such that $y_1z_1 \in \mathscr{X}_2\mathscr{X}_3 \setminus \mathscr{P}$ and $x_1z \in \mathscr{X}_1\mathscr{X}_3 \setminus \mathscr{P}$. This implies that $y_1z_1 \notin \mathscr{P}$ and $x_1z \notin \mathscr{P}$. To Show that $\mathscr{X}_1\mathscr{X}_2 \subseteq \mathscr{P}$. For all $xy \in \mathscr{X}_1\mathscr{X}_2$. Thus $(xy) \subseteq \mathscr{X}_1\mathscr{X}_2$. Now, $0 \neq (xy)\mathscr{R}(y_1z_1)\mathscr{R}(x_1z) \subseteq \mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \subseteq \mathscr{P}$. This implies $(xy) \subseteq \mathscr{P}$. Therefore $\mathscr{X}_1\mathscr{X}_2 \subseteq \mathscr{P}$. Thus $\mathscr{P}$ is a 2- AWPI of $\mathscr{R}$.

The converse of the Theorem 3.5 is not true as we can see by the next Example.

Example 3.6. *Consider the ring* $\mathscr{R} = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \middle| a, b, c \in \mathbb{Z}_2 \right\}$.
Let $\mathscr{P} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ *is a* 2- *AWPI, but not strongly* 2-*AWPI of* $\mathscr{R}$. *Since* $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathscr{R} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \mathscr{R} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \subseteq \mathscr{P}$, *but* $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \notin \mathscr{P}$.

Theorem 3.7. *If* $\mathscr{P}$ *be a proper ideal of* $\mathscr{R}$, *then* $\mathscr{P}$ *is a strongly* 2-*AWPI of* $\mathscr{R}$ *if and only if* $0 \neq \mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \subseteq \mathscr{P}$ *implies that* $\mathscr{X}_1 \subseteq \mathscr{P}$ *or* $\mathscr{X}_2 \subseteq \mathscr{P}$ *or* $\mathscr{X}_3 \subseteq \mathscr{P}$ *for all ideals* $\mathscr{X}_1, \mathscr{X}_2$ *and* $\mathscr{X}_3$ *of* $\mathscr{R}$.

Proof. Suppose that $0 \neq \mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \subseteq \mathscr{P}$ implies that $\mathscr{X}_1 \subseteq \mathscr{P}$ or $\mathscr{X}_2 \subseteq \mathscr{P}$ or $\mathscr{X}_3 \subseteq \mathscr{P}$ for the ideals $\mathscr{X}_1, \mathscr{X}_2$ and $\mathscr{X}_3$ of $\mathscr{R}$. Suppose that $0 \neq x\mathscr{R}y\mathscr{R}z \subseteq \mathscr{P}$. Then $0 \neq \mathscr{R}x\mathscr{R}y\mathscr{R}z\mathscr{R} \subseteq \mathscr{R}\mathscr{P}\mathscr{R} \subseteq \mathscr{P}$. Also $0 \neq (\mathscr{R}x\mathscr{R})(\mathscr{R}y\mathscr{R})(\mathscr{R}z\mathscr{R}) = \mathscr{R}x\mathscr{R}^2y\mathscr{R}^2z\mathscr{R} \subseteq \mathscr{R}x\mathscr{R}y\mathscr{R}z\mathscr{R} \subseteq \mathscr{P}$. This implies that $\mathscr{R}x\mathscr{R} \subseteq \mathscr{P}$ or $\mathscr{R}y\mathscr{R} \subseteq \mathscr{P}$ or $\mathscr{R}z\mathscr{R} \subseteq \mathscr{P}$. Since $\mathscr{R}$ is a ring with unity, hence $x \in \mathscr{P}$ or $y \in \mathscr{P}$ or $z \in \mathscr{P}$. Hence $\mathscr{P}$ is a strongly 2-AWPI of $\mathscr{R}$.

Conversely, suppose that $\mathscr{P}$ is a strongly 2-AWPI of $\mathscr{R}$ and $0 \neq \mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \subseteq \mathscr{P}$, for the ideals $\mathscr{X}_1$, $\mathscr{X}_2$ and $\mathscr{X}_3$ of $\mathscr{R}$. Let us shows that $\mathscr{X}_1 \subseteq \mathscr{P}$ or $\mathscr{X}_2 \subseteq \mathscr{P}$ or $\mathscr{X}_3 \subseteq \mathscr{P}$. Suppose that $\mathscr{X}_2 \nsubseteq \mathscr{P}$ and $\mathscr{X}_3 \nsubseteq \mathscr{P}$. Then there exist $y \in \mathscr{X}_2$ and $z \in \mathscr{X}_3$ such that $y \notin \mathscr{P}$ and $z \notin \mathscr{P}$. Let us show that $\mathscr{X}_1 \subseteq \mathscr{P}$. Let $x \in \mathscr{X}_1$. We have $0 \neq x\mathscr{R}y\mathscr{R}z \in \mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \subseteq \mathscr{P}$. This implies that $x \in \mathscr{P}$. Therefore $\mathscr{X}_1 \subseteq \mathscr{P}$.

Theorem 3.8. *If* $\mathscr{P}$ *is a ideal of* $\mathscr{R}$, *then* $\mathscr{P}$ *is a strongly* 2-*AWPI if and only if* $\mathscr{R} \setminus \mathscr{P}$ *is a weak* m_{a1}-*system.*

Proof. Let $\mathscr{P}$ be a strongly 2-AWPI. In order to show that $\mathscr{M} = \mathscr{R} \setminus \mathscr{P}$ is an weak m_{a1}-system, let $\mathscr{X}_1, \mathscr{X}_2$ and $\mathscr{X}_3$ be ideals of $\mathscr{R}$. If $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 = 0$, there is nothing to prove. Assume that $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \neq 0$. Suppose that $\mathscr{X}_1 \cap \mathscr{M} \neq \phi$, $\mathscr{X}_2 \cap \mathscr{M} \neq \phi$ and $\mathscr{X}_3 \cap \mathscr{M} \neq \phi$. We claim that $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \nsubseteq \mathscr{P}$. Suppose that $0 \neq \mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \subseteq \mathscr{P}$. Since $\mathscr{P}$ is a strongly 2-AWPI of $\mathscr{R}$ this implies that $\mathscr{X}_1 \subseteq \mathscr{P}$ or $\mathscr{X}_2 \subseteq \mathscr{P}$ or $\mathscr{X}_3 \subseteq \mathscr{P}$. Thus $\mathscr{X}_1 \cap \mathscr{M} = \phi$ or $\mathscr{X}_2 \cap \mathscr{M} = \phi$ or $\mathscr{X}_3 \cap \mathscr{M} = \phi$, which is contradiction . Hence $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \nsubseteq \mathscr{P}$ implies $(\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3) \cap \mathscr{M} \neq \phi$. Therefore $\mathscr{R} \setminus \mathscr{P}$ is a weak m_{a1}-system.

Conversely, Let $\mathscr{R} \setminus \mathscr{P}$ be a weak m_{a1}-system. Let $\mathscr{X}_1$, $\mathscr{X}_2$ and $\mathscr{X}_3$ be the ideals of $\mathscr{R}$ such that $0 \neq \mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \subseteq \mathscr{P}$. Let us shows that $\mathscr{X}_1 \subseteq \mathscr{P}$ or $\mathscr{X}_2 \subseteq \mathscr{P}$ or $\mathscr{X}_3 \subseteq \mathscr{P}$. Suppose that $\mathscr{X}_1 \nsubseteq \mathscr{P}$, $\mathscr{X}_2 \nsubseteq \mathscr{P}$ and $\mathscr{X}_3 \nsubseteq \mathscr{P}$. This implies that $\mathscr{X}_1 \cap \mathscr{M} \neq \phi$, $\mathscr{X}_2 \cap \mathscr{M} \neq \phi$ and $\mathscr{X}_3 \cap \mathscr{M} \neq \phi$. Since $\mathscr{M}$ is a weak m_{a1}-system and $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \neq 0$, then $(\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3) \cap \mathscr{M} \neq \phi$. Since $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \subseteq \mathscr{P}$, implies that $(\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3) \cap \mathscr{M} = \phi$, which is a contradiction. Thus $\mathscr{X}_1 \subseteq \mathscr{P}$ or $\mathscr{X}_2 \subseteq \mathscr{P}$ or $\mathscr{X}_3 \subseteq \mathscr{P}$. Hence $\mathscr{P}$ is a strongly 2-AWPI of $\mathscr{R}$.

Theorem 3.9. *If $\mathscr{P}$ is a ideal of $\mathscr{R}$, then $\mathscr{P}$ is a 2-AWPI if and only if $\mathscr{R} \setminus \mathscr{P}$ is a weak m_{a2}-system.*

Proof. Let $\mathscr{X}_1, \mathscr{X}_2$ and $\mathscr{X}_3$ are ideals of $\mathscr{R}$. If $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 = 0$, there is nothing to prove. Suppose that $\mathscr{X}_1\mathscr{X}_2 \cap \mathscr{M} \neq \phi$, $\mathscr{X}_2\mathscr{X}_3 \cap \mathscr{M} \neq \phi$ and $\mathscr{X}_1\mathscr{X}_3 \cap \mathscr{M} \neq \phi$. We claim that $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \nsubseteq \mathscr{P}$. Suppose that $0 \neq \mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \subseteq \mathscr{P}$. Since $\mathscr{P}$ is a 2-AWPI of $\mathscr{R}$ this implies that $\mathscr{X}_1\mathscr{X}_2 \subseteq \mathscr{P}$ or $\mathscr{X}_2\mathscr{X}_3 \subseteq \mathscr{P}$ or $\mathscr{X}_1\mathscr{X}_3 \subseteq \mathscr{P}$. Thus $\mathscr{X}_1\mathscr{X}_2 \cap \mathscr{M} = \phi$ or $\mathscr{X}_2\mathscr{X}_3 \cap \mathscr{M} = \phi$ or $\mathscr{X}_1\mathscr{X}_3 \cap \mathscr{M} = \phi$, which is a contradiction . Hence $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \nsubseteq \mathscr{P}$ implies that $(\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3) \cap \mathscr{M} \neq \phi$. Therefore $\mathscr{R} \setminus \mathscr{P}$ is a weak m_{a2}-system.

Conversely, let $\mathscr{R} \setminus \mathscr{P}$ be a weak m_{a2}-system. Let $\mathscr{X}_1, \mathscr{X}_2$ and $\mathscr{X}_3$ be ideals of $\mathscr{R}$ such that $0 \neq \mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \subseteq \mathscr{P}$. Let us shows that $\mathscr{X}_1\mathscr{X}_2 \subseteq \mathscr{P}$ or $\mathscr{X}_2\mathscr{X}_3 \subseteq \mathscr{P}$ or $\mathscr{X}_1\mathscr{X}_3 \subseteq \mathscr{P}$. Suppose that $\mathscr{X}_1\mathscr{X}_2 \nsubseteq \mathscr{P}$, $\mathscr{X}_2\mathscr{X}_3 \nsubseteq \mathscr{P}$ and $\mathscr{X}_1\mathscr{X}_3 \nsubseteq \mathscr{P}$. This implies that $\mathscr{X}_1\mathscr{X}_2 \cap \mathscr{M} \neq \phi$, $\mathscr{X}_2\mathscr{X}_3 \cap \mathscr{M} \neq \phi$ and $\mathscr{X}_1\mathscr{X}_3 \cap \mathscr{M} \neq \phi$. Since $\mathscr{M}$ is a weak m_{a2}-system and $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \neq 0$, then $(\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3) \cap \mathscr{M} \neq \phi$. Since $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \subseteq \mathscr{P}$, implies that $(\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3) \cap \mathscr{M} = \phi$, which is a contradiction. Thus $\mathscr{X}_1\mathscr{X}_2 \subseteq \mathscr{P}$ or $\mathscr{X}_2\mathscr{X}_3 \subseteq \mathscr{P}$ or $\mathscr{X}_1\mathscr{X}_3 \subseteq \mathscr{P}$. Hence $\mathscr{P}$ is a 2-AWPI of $\mathscr{R}$.

Lemma 3.10. *Every strongly m_{a1}-system is a weak m_{a1}-system.*

Proof. Suppose that $\mathscr{M}$ is a strongly m_{a1}-system. Let $\mathscr{X}_1 \cap \mathscr{M} \neq \phi$, $\mathscr{X}_2 \cap \mathscr{M} \neq \phi$ and $\mathscr{X}_3 \cap \mathscr{M} \neq \phi$, for the ideals $\mathscr{X}_1$, $\mathscr{X}_2$, $\mathscr{X}_3$ of $\mathscr{R}$. Then there exist $x \in \mathscr{X}_1$, $y \in \mathscr{X}_2$ and $z \in \mathscr{X}_3$ such that $x, y, z \in \mathscr{M}$. Since $\mathscr{M}$ is a m_{a1}-system, then there exist $x_1 \in < x >$, $y_1 \in < y >$ and $z_1 \in < z >$ such that $x_1y_1z_1 \in \mathscr{M}$. This implies that $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \cap \mathscr{M} \neq \phi$, since $x_1y_1z_1 \in < x >< y >< z > \subseteq \mathscr{X}_1\mathscr{X}_2\mathscr{X}_3$. Therefore $\mathscr{M}$ is a weak m_{a1}-system.

The converse of the Lemma 3.10 is not true as we can see by the next Example.

Example 3.11. *Consider the ring* $\mathscr{R} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Z}_2 \right\}$.
Let $\mathscr{M} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ *is a weak* m_{a1}*- system, but not a strongly* m_{a1}*- system. For* $x = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathscr{M}$*, but there is no* $r_1, r_2 \in \mathscr{R}$ *such that* $xr_1yr_2z \in \mathscr{M}$.

Lemma 3.12. *Every weak* m_{a1}*-system is a weak* m_{a2}*-system.*

Proof. Suppose that $\mathscr{M}$ is a weak m_{a1}-system and $\mathscr{X}_1\mathscr{X}_2 \cap \mathscr{M} \neq \phi$, $\mathscr{X}_2\mathscr{X}_3 \cap \mathscr{M} \neq \phi$ and $\mathscr{X}_1\mathscr{X}_3 \cap \mathscr{M} \neq \phi$ for the ideals $\mathscr{X}_1$, $\mathscr{X}_2$ and $\mathscr{X}_3$ of $\mathscr{R}$. Let us show that $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \cap \mathscr{M} \neq \phi$ or $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 = 0$. If $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 = 0$, then nothing to prove. Suppose that $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \neq 0$. Since $\mathscr{X}_1 \subseteq \mathscr{X}_1\mathscr{X}_2$ implies $\mathscr{X}_1 \cap \mathscr{M} \subseteq \mathscr{X}_1\mathscr{X}_2 \cap \mathscr{M} \neq \phi$. Hence $\mathscr{X}_1 \cap \mathscr{M} \neq \phi$, similarly $\mathscr{X}_2 \cap \mathscr{M} \neq \phi$ and $\mathscr{X}_3 \cap \mathscr{M} \neq \phi$. Since $\mathscr{M}$ is a weak m_{a1}-system, thus $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \cap \mathscr{M} \neq \phi$.

The converse of the Lemma 3.12 is not true as we can see by the following Example.

Example 3.13. *Consider the ring* $\mathscr{R} = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \middle| a, b, c \in \mathbb{Z}_2 \right\}$.
Let $\mathscr{M} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ *is a weak* m_{a2}*-system, but not a weak* m_{a1}*-system.*

Theorem 3.14. *Let* I *be an ideal of* $\mathscr{R}$ *and* $\mathscr{M}$ *be a weak* m_{a1}*-system with* $I \cap \mathscr{M} = \phi$*. Then there exists a strongly* 2*- AWPI* $\mathscr{P}$ *of* $\mathscr{R}$ *containing* I *with* $\mathscr{P} \cap \mathscr{M} = \phi$*.*

Proof. Let $\mathscr{A}_1 = \{J | J$ is an ideal with $I \subseteq J$ and $J \cap \mathscr{M} = \phi\}$. Clearly $\mathscr{A}_1$ is non-empty. By Zorn's lemma, there exists an maximal element $\mathscr{P}$ in $\mathscr{R}$ with $I \subseteq \mathscr{P}$. We claim that $\mathscr{P}$ strongly 2- API of $\mathscr{R}$. Suppose that $0 \neq < x >< y >< z > \subseteq \mathscr{P}$. Let us show that $x \in \mathscr{P}$ or $y \in \mathscr{P}$ or $z \in \mathscr{P}$. Let $x, y, z \notin \mathscr{P}$. We have $\mathscr{P} \subset \mathscr{P} + < x >$, $\mathscr{P} \subset \mathscr{P} + < y >$ and $\mathscr{P} \subset \mathscr{P} + < z >$. By the maximal property, $(\mathscr{P} + < x >) \cap \mathscr{M} \neq \phi$, $(\mathscr{P} + < y >) \cap \mathscr{M} \neq \phi$ and $(\mathscr{P} + < z >) \cap \mathscr{M} \neq \phi$. Let $\mathscr{X}_1 = \mathscr{P} + < x >$, $\mathscr{X}_2 = \mathscr{P} + < y >$ and $\mathscr{X}_3 = \mathscr{P} + < z >$. We have $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 = (\mathscr{P} + < x >)(\mathscr{P} + < y >)(\mathscr{P} + < z >) \neq 0$, since $< x >< y >< z > \neq 0$. Since

$\mathscr{M}$ is a m_{a1}-system, and $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \neq 0$, then $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \cap \mathscr{M} \neq \phi$. But $(\mathscr{P}+ < x >)(\mathscr{P}+ < y >)(\mathscr{P}+ < z >) = \mathscr{P}+ < x >< y >< z >\subseteq \mathscr{P}$. Thus $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 = (\mathscr{P}+ < x >)(\mathscr{P}+ < y >)(\mathscr{P}+ < z >) \subseteq \mathscr{P}$. This implies that $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \cap \mathscr{P}^c = \phi$ and hence $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \cap \mathscr{M} = \phi$, which is a contradiction. Hence $< x >< y >< z >\nsubseteq \mathscr{P}$. Therefore $\mathscr{P}$ is a strongly 2- AWPI of $\mathscr{R}$.

Theorem 3.15. *Let I be an ideal of $\mathscr{R}$ and $\mathscr{M}$ be a weak m_{a2}-system such that $I \cap \mathscr{M} = \phi$. Then there exists a 2- AWPI $\mathscr{P}$ of $\mathscr{R}$ containing I with $\mathscr{P} \cap \mathscr{M} = \phi$.*

Proof. Let $\mathscr{A}_1 = \{J|J \text{ is an ideal with } I \subseteq J \text{ and } J \cap \mathscr{M} = \phi\}$. Clearly $\mathscr{A}_1$ is non-empty. By Zorn's lemma, there exist an maximal element $\mathscr{P}$ in $\mathscr{R}$ with $I \subseteq \mathscr{P}$. We claim that $\mathscr{P}$ is a 2- AWPI of $\mathscr{R}$. Suppose that $0 \neq< x >< y >< z >\subseteq \mathscr{P}$. Let us show that $< x >< y >\subseteq \mathscr{P}$ or $< y >< z >\subseteq \mathscr{P}$ or $< x >< z >\subseteq \mathscr{P}$. Suppose that $< x >< y >\nsubseteq \mathscr{P}$, $< y >< z >\nsubseteq \mathscr{P}$ and $< x >< z >\nsubseteq \mathscr{P}$. This imply that $< x >\nsubseteq \mathscr{P}$, $< y >\nsubseteq \mathscr{P}$ and $< z >\nsubseteq \mathscr{P}$. Hence $\mathscr{P} \subset \mathscr{P}+ < x >$, $\mathscr{P} \subset \mathscr{P}+ < y >$ and $\mathscr{P} \subset \mathscr{P}+ < z >$. By the maximal property, we have $(\mathscr{P}+ < x >) \cap \mathscr{M} \neq \phi$, $(\mathscr{P}+ < y >) \cap \mathscr{M} \neq \phi$ and $(\mathscr{P}+ < z >) \cap \mathscr{M} \neq \phi$. Let $\mathscr{X}_1 = \mathscr{P}+ < x >$, $\mathscr{X}_2 = \mathscr{P}+ < y >$ and $\mathscr{X}_3 = \mathscr{P}+ < z >$ be ideals of $\mathscr{R}$. Hence $\mathscr{X}_1\mathscr{X}_2 \cap \mathscr{M} = (\mathscr{P}+ < x >< y >) \cap \mathscr{M} \neq \phi$, $\mathscr{X}_2\mathscr{X}_3 \cap \mathscr{M} = (\mathscr{P}+ < y >< z >) \cap \mathscr{M} \neq \phi$ and $\mathscr{X}_1\mathscr{X}_3 \cap \mathscr{M} = (\mathscr{P}+ < x >< z >) \cap \mathscr{M} \neq \phi$. Since $\mathscr{M}$ is a weak m_{a2}-system and $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \neq 0$, it follows that $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 \cap \mathscr{M} \neq \phi$. But $\mathscr{X}_1\mathscr{X}_2\mathscr{X}_3 = (\mathscr{P}+ < x >)(\mathscr{P}+ < y >)(\mathscr{P}+ < z >) = \mathscr{P}+ < x >< y >< z >\subseteq \mathscr{P}$. Thus $\mathscr{P} \cap \mathscr{M} \neq \phi$, which is a contradiction. Thus, $< x >< y >< z >\nsubseteq \mathscr{P}$. Hence $\mathscr{P}$ is a 2- AWPI of $\mathscr{R}$.

4 Conclusion

In this article, the strongly 2-API, 2-API, strongly 2-AWPI and 2-AWPI are studied. The PI in a non-commutative ring is also characterized. We discuss some of their key properties and provide descriptions of some of them in terms of their m-systems. We also analyze connections between strongly 2-API, 2-API and strongly 2-AWPIs. As a future study, we intend to study several classes of semirings and ternary semirings using strongly 2-absorbing prime bi-ideals, 2-absorbing prime bi-ideals, strongly 2-absorbing weak prime bi-ideals and 2-absorbing weak prime bi-ideals and 2-absorbing maximal prime bi-ideals.

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