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# Friedmann equations as $n$-dimensional dynamical system

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**Abstract**

In this paper we study dynamics of the standard cosmological model of the universe assuming that it is filled with $n$ types of non-interacting barotropic perfect fluids. For that purpose, a dynamical system of a class of Lotka-Volterra dynamical systems is derived, that consists of $n$ nonlinear differential equations of the first order, whose dependent variables are density parameters of the material in the universe. Analytical solution of that system represents new parametrization of density parameters. Moreover, we perceive the evolution of the universe in the frame of the linear stability theory.

## 1 Introduction

Using theory of dynamical systems, we analyzed in [4] the dynamics of the standard cosmological model of the universe, known as the ΛCDM model of the universe, filled with two barotropic perfect fluids (matter and radiation) without mutual interaction, with the equation of state parameters $\omega_m, \omega_r \in [0, 1]$, $\omega_m \neq \omega_r$, where index $m$ referes to matter, while index $r$ is related to radiation. We note that the term matter stands for both cold dark matter and baryonic matter, whereas radiation represents relativistic particles (photons, neutrinos). Here we consider $n$ barotropic perfect fluids without mutual interaction, while one of them is the cosmological constant Λ with the equation of

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state parameter $\omega_\Lambda = -1$, [2, 10, 14]. The goal of this paper is generalization of results in [4] for $n$ fluid case.

The system of the Friedmann equations with $\Lambda$ [7]

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{\kappa c^2}{a^2} + \frac{\Lambda c^2}{3}, \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda c^2}{3}, \\ \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) &= 0, \end{aligned} \tag{1.1}$$

consists of the Friedmann equation, the acceleration equation and the fluid equation, respectively. Here, $\kappa$ is the curvature of the universe, $a = a(t)$ is the scale factor that measures the expansion of the universe, $\rho = \rho(t)$ is the average density of the material in the universe and $p = p(t)$ is the pressure of the material in the universe, where $t$ is a time variable. A dot denotes the derivative in regard to $t$.

The connection between the average density of $i$-fluid, denoted with $\rho_i$, where $i \in \{1, 2, \ldots, n\}$, and the pressure of $i$-fluid, denoted with $p_i$, reflects in the assumption that every $i$-fluid is barotropic perfect fluid and therefore the equation of state for every $i$-fluid is satisfied

$$p_i = \omega_i \rho_i c^2, \tag{1.2}$$

where $\omega_i$ is the equation of state parameter for $i$-fluid, [2, 10]. For the first $(n-1)$ fluid we assume the equation of state parameter $0 \leq \omega_i \leq 1$ (see [2]), $\omega_i \neq \omega_j$ for $i \neq j$, while for the last fluid we take the cosmological constant $\Lambda$, with $\omega_n = \omega_\Lambda = -1$.

The condition that fluids do not have mutual interaction means that the fluid equation is preserved for every $i$-fluid [2, 14]

$$\dot{\rho}_i + 3\frac{\dot{a}}{a}\left(\rho_i + \frac{p_i}{c^2}\right) = 0. \tag{1.3}$$

Changing the expression for the Hubble parameter $H = \dot{a}/a$ [11] and the equation of state for $i$-fluid $p_i = \omega_i \rho_i c^2$ in (1.3), we obtain more suitable form of the fluid equation for $i$-fluid

$$\dot{\rho}_i = -3H\rho_i\left(1 + \omega_i\right). \tag{1.4}$$

The system of the Friedmann equations (1.1) for this case of $n$ fluids is in the following form

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{\kappa c^2}{a^2}, \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right), \\ \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) &= 0, \end{aligned} \tag{1.5}$$

where

$$\begin{aligned} &\rho = \sum_{j=1}^{n}\rho_j,\ \rho_n = \rho_\Lambda = \Lambda c^2/8\pi G, \quad p = \sum_{j=1}^{n} p_j,\ p_n = p_\Lambda, \\ &p_i = \omega_i \rho_i c^2,\ i \in \{1, 2, \dots, n\},\ \omega_n = \omega_\Lambda = -1. \end{aligned} \tag{1.6}$$

Density parameters are defined as usual, see [11]

$$\Omega_i(t) = \frac{8\pi G}{3H^2(t)}\rho_i(t), \quad \Omega_\kappa(t) = -\frac{\kappa c^2}{a^2(t)H^2(t)}, \tag{1.7}$$

where $\Omega_i$ is density parameter for $i$-fluid, while $\Omega_\kappa$ is density parameter of spatial curvature.

All discussions in this paper are considered for flat universe ($\kappa = 0$), or open universe ($\kappa < 0$). With respect to physical restrictions, our assumption is that all functions in this paper are continuously differentiable, as many times as required.

## 2 The dynamics of the universe filled with $n$ non-interacting barotropic perfect fluids

Here we formulate and prove Theorem 2.1, which is a version of the appropriate theorem in [4], but for $n$ non-interacting barotropic perfect fluids. That result enables us to study the system of the Friedmann equations (1.5) in the frame of the dynamical systems theory.

In order to obtain the equivalent system, we follow the procedure from [4], that is based on [2, 14, 8, 16, 9], by introducing $\xi(t) = \ln(a(t))$. The derivatives regarding to $\xi$ we denote with $'$.

**Theorem 2.1.** *Under the assumptions (1.6), the system of the Friedmann equations (1.5) with the equation (1.4) for every $i \in \{1, 2, \dots, n\}$ fluid, is* *equivalent to the following system*

$$\begin{aligned}
&{\Omega_i}' = \Omega_i \left( -\left(1+3\omega_i\right) + \sum_{j=1}^{n} \Omega_j \left(1+3\omega_j\right) \right), \\
&\sum_{j=1}^{n} \Omega_j + \Omega_\kappa = 1, \\
&{\Omega_i}' = \frac{\dot{\Omega}_i}{H},
\end{aligned} \tag{2.1}$$

*where* $\Omega_i$*,* $i \in \{1, 2, \ldots, n\}$ *are from (1.7).*

**Proof**. First, we prove that the system (2.1) follows from the system (1.5).

Changing $\rho = \sum\limits_{j=1}^{n} \rho_j$ in the Friedmann equation in (1.5) and dividing it with $H^2 \neq 0$, follows

$$\sum_{j=1}^{n} \Omega_j + \Omega_\kappa = 1. \tag{2.2}$$

Calculating the derivatives of the density parameters $\Omega_i$ regarding to $\xi$, we obtain

$${\Omega_i}' = \frac{8\pi G}{3H} \left( \frac{\dot{\rho}_i}{H^2} - 2\dot{H} \frac{\rho_i}{H^3} \right). \tag{2.3}$$

Substituting (1.4) (which is justified, since the fluids are non-interacting) and the definition of $\Omega_i$ in (2.3), we infer

$${\Omega_i}' = -3\Omega_i(1+\omega_i) + \Omega_i \cdot \frac{-2\dot{H}}{H^2}. \tag{2.4}$$

If we substitute $\rho = \sum\limits_{j=1}^{n} \rho_j$ in the Friedmann equation in (1.5) and then differentiate it with respect to time t, we infer

$$2H\dot{H} = \frac{8\pi G}{3} \sum_{j=1}^{n} \dot{\rho}_j + 2\kappa c^2 \cdot \frac{\dot{a}}{a^3}. \tag{2.5}$$

Replacing $\Omega_\kappa = 1 - \sum\limits_{j=1}^{n} \Omega_j$ and (1.4) into (2.5), leads to

$$\frac{-2\dot{H}}{H^2} = 2 + \sum_{j=1}^{n} \Omega_j \left(1+3\omega_j\right). \tag{2.6}$$

Substituting (2.6) into (2.4), we derive

$$\Omega_i{}' = \Omega_i \left( -\left(1 + 3\omega_i\right) + \sum_{j=1}^{n} \Omega_j \left(1 + 3\omega_j\right) \right), \tag{2.7}$$

which is exactly the system (2.1).

Now we prove the other direction.

From $\sum\limits_{j=1}^{n} \Omega_j + \Omega_\kappa = 1$ immediately follows the Friedmann equation in (1.5). The system (1.5) is not determined and since the fluid equation and the Friedmann equation imply the acceleration equation, it is enough to prove the fluid equation in order to prove the other direction of Theorem 2.1.

Since $\rho_n = \rho_\Lambda = \Lambda c^2/8\pi G$, we have $\Omega_n = \Omega_\Lambda = \Lambda c^2/3H^2$, wherefrom follows

$$\frac{\Omega_n{}'}{\Omega_n} = \frac{-2\dot{H}}{H^2}. \tag{2.8}$$

From (2.8) and the first equation in (2.1) for $i = n$, we conclude

$$\frac{-2\dot{H}}{H^2} = -\left(1 + 3\omega_n\right) + \sum_{j=1}^{n} \Omega_j \left(1 + 3\omega_j\right). \tag{2.9}$$

From $\Omega_i{}' = \dot{\Omega}_i/H$ and $\Omega_i = 8\pi G \rho_i/3H^2$, we have

$$\Omega_i{}' = \frac{8\pi G}{3H^3}\dot{\rho}_i + \frac{8\pi G \rho_i}{3H^2} \cdot \frac{-2\dot{H}}{H^2}, \tag{2.10}$$

wherefrom, substituting (2.9), follows

$$\frac{\Omega_i{}'}{\Omega_i} = \frac{\dot{\rho}_i}{\rho_i} \cdot \frac{1}{H} - \left(1 + 3\omega_n\right) + \sum_{j=1}^{n} \Omega_j \left(1 + 3\omega_j\right). \tag{2.11}$$

Comparing (2.11) with the equation from the system (1.5), as well as using $\omega_n = -1$, leads to

$$\frac{\dot{\rho}_i}{\rho_i} \cdot \frac{1}{H} = 3\left(\omega_n - \omega_i\right) = -3\left(1 + \omega_i\right), \tag{2.12}$$

wherefrom follows

$$\begin{aligned} &\dot{\rho}_i = -3H\left(1 + \omega_i\right),\ i \neq n, \\ &\dot{\rho}_n = 0, \end{aligned} \tag{2.13}$$

which is exactly the fluid equation (1.4) for every $i$-fluid, including the cosmological constant. Using (2.13) and (1.6), the fluid equation is easily obtained and therefore the system of Friedmann equations (1.5) is infered. □

Theorem 2.1 states that for $n$ barotropic perfect fluids (one of them is cosmological constant), the system of the Friedmann equations (1.5) for non-interacting fluids, is equivalent to the dynamical system (2.1).

We note that the system (2.1) is of a class of Lotka-Volterra dynamical systems [12, 17]. Lotka-Volterra dynamical systems model numerous problems, for example in medicine, physics, cosmology, space explorations [6, 1, 14, 5]. Those systems are mostly analyzed using numerical analysis [15], since obtaining their analytical solutions is not easy. The next step in our paper is exactly obtaining analytical solution of the system (2.1).

We assume that all density parameters are strictly positive. Otherwise, obtaining analytical solution of the dynamical system (2.1) is significantly easier, since the number of its equations is reduced. Since the sum of all density parameters has to be equal to 1, notice that all of the density parameters are less than 1. Therefore, we are only interested in solutions $0 < \Omega_i < 1$.

According to the three dimensional case in [4], we claim that the analytical solution of the system (2.1), for

$$0 \leq \omega_i \leq 1,\ i \in \{1, 2, \dots, n-1\},\ \omega_n = \omega_\Lambda = -1,\ \omega_i \neq \omega_j \text{ for } i \neq j, \tag{2.14}$$

is in the form

$$\Omega_i(\xi) = \frac{c_{i-1}}{\sum\limits_{j=1}^{n} c_{j-1} \mathrm{e}^{3(\omega_i - \omega_j)\xi} + c_n \mathrm{e}^{(1+3\omega_i)\xi}}, \quad c_0 := 1, \tag{2.15}$$

where $c_1, c_2, \dots, c_n$ are strictly positive constants.

Firstly, $\Omega_i$ from (2.15) are continuously differentiable for every $\xi \in \mathbb{R}$. Secondly, the condition $0 < \Omega_i < 1$ is satisfied, since $c_1, c_2, \dots, c_n$ are strictly positive and it is easy to notice that $1/\Omega_i > 1$. Finally, it can be easily verified that $\Omega_i$ from (2.15) satisfy the system (2.1). Hence, (2.15) really represents analytical solution for the system (2.1).

Moreover, from $\sum\limits_{j=1}^{n} \Omega_j + \Omega_\kappa = 1$ and (2.15), we infer the following

$$\begin{aligned} \Omega_i(\xi) &= c_{i-1} \mathrm{e}^{3(\omega_1 - \omega_i)\xi} \Omega_1(\xi), \\ \Omega_j(\xi) &= \frac{c_{j-1}}{c_{i-1}} \mathrm{e}^{3(\omega_i - \omega_j)\xi} \Omega_i(\xi), \\ \Omega_\kappa(\xi) &= 1 - \Omega_1(\xi) \sum_{j=1}^{n} c_{j-1} \mathrm{e}^{3(\omega_1 - \omega_j)\xi}, \quad c_0 := 1. \end{aligned} \tag{2.16}$$

Substituting $\xi = \ln a$ in (2.15) and (2.16), the analytical solution of the $n$ fluid case can be written as

$$\Omega_i\left(a\right) = \frac{c_{i-1}}{\sum\limits_{j=1}^{n} c_{j-1} a^{3(\omega_i - \omega_j)} + c_n a^{1+3\omega_i}}, \quad c_0 := 1, \tag{2.17}$$

while the appropriate relations are

$$\begin{aligned} &\Omega_i\left(a\right) = c_{i-1} a^{3(\omega_1 - \omega_i)} \Omega_1\left(a\right), \\ &\Omega_j\left(a\right) = \frac{c_{j-1}}{c_{i-1}} a^{3(\omega_i - \omega_j)} \Omega_i\left(a\right), \\ &\Omega_\kappa\left(a\right) = 1 - \Omega_1(a) \sum_{j=1}^{n} c_{j-1} a^{3(\omega_1 - \omega_j)}, \quad c_0 := 1. \end{aligned} \tag{2.18}$$

The solution (2.17) describes the relation between the density parameters $0 < \Omega_i < 1$ and the scale factor, for every $\Lambda$CDM model of the universe filled with arbitrary number of non-interacting barotropic perfect fluids with $0 \leq \omega_i \leq 1$, $\omega_i \neq \omega_j$ for $i \neq j$, with cosmological constant being also non-interacting barotropic fluid with $\omega_n = \omega_\Lambda = -1$. One should just know the number of such fluids, as well as the concrete values of $\omega_i$ and substitute it into (2.17). Therefore, we found solutions for various cases described in literature and, moreover, we proved that all such solutions are contained in one formula, which is (2.17).

In Sec. 3, we describe evolution for every such $\Lambda$CDM model of the universe.

First, we briefly discuss the three fluid case.

### 2.1 Case n=3

Substituting

$$n = 3,\ \Omega_1 \to \Omega_m,\ \Omega_2 \to \Omega_r,\ \Omega_3 \to \Omega_\Lambda,\ \omega_1 \to \omega_m,\ \omega_2 \to \omega_r,\ \omega_3 \to \omega_\Lambda, \tag{2.19}$$

in the system (2.1), we obtain the following system

$$\begin{aligned} &{\Omega_m}' = \Omega_m\left(-\left(1 + 3\omega_m\right) + \Omega_m\left(1 + 3\omega_m\right) + \Omega_r\left(1 + 3\omega_r\right) + \Omega_\Lambda\left(1 + 3\omega_\Lambda\right)\right), \\ &{\Omega_r}' = \Omega_r\left(-\left(1 + 3\omega_r\right) + \Omega_m\left(1 + 3\omega_m\right) + \Omega_r\left(1 + 3\omega_r\right) + \Omega_\Lambda\left(1 + 3\omega_\Lambda\right)\right), \\ &{\Omega_\Lambda}' = \Omega_\Lambda\left(-\left(1 + 3\omega_\Lambda\right) + \Omega_m\left(1 + 3\omega_m\right) + \Omega_r\left(1 + 3\omega_r\right) + \Omega_\Lambda\left(1 + 3\omega_\Lambda\right)\right), \\ &\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_\kappa = 1, \\ &{\Omega_i}' = \frac{\dot{\Omega}_i}{H}. \end{aligned} \tag{2.20}$$

Substituting (2.19) in (2.15), we obtain the analytical solution of the system (2.20)

$$\begin{aligned}\Omega_m(\xi) &= \frac{1}{1+c_1 e^{3(\omega_m-\omega_r)\xi}+c_2 e^{3(\omega_m-\omega_\Lambda)\xi}+c_3 e^{(1+3\omega_m)\xi}},\\ \Omega_r(\xi) &= \frac{c_1}{e^{3(\omega_r-\omega_m)\xi}+c_1+c_2 e^{3(\omega_r-\omega_\Lambda)\xi}+c_3 e^{(1+3\omega_r)\xi}},\\ \Omega_\Lambda(\xi) &= \frac{c_2}{e^{3(\omega_\Lambda-\omega_m)\xi}+c_1 e^{3(\omega_\Lambda-\omega_r)\xi}+c_2+c_3 e^{(1+3\omega_\Lambda)\xi}}.\end{aligned} \tag{2.21}$$

Changing (2.19) in (2.17) and (2.18), we obtain the following dependences between density parameters and the scale factor $a = a(t)$

$$\begin{aligned}\Omega_m(a) &= \frac{1}{1+c_1 a^{3(\omega_m-\omega_r)}+c_2 a^{3(\omega_m-\omega_\Lambda)}+c_3 a^{1+3\omega_m}},\\ \Omega_r(a) &= \frac{c_1}{a^{3(\omega_r-\omega_m)}+c_1+c_2 a^{3(\omega_r-\omega_\Lambda)}+c_3 a^{1+3\omega_r}},\\ \Omega_\Lambda(a) &= \frac{c_2}{a^{3(\omega_\Lambda-\omega_m)}+c_1 a^{3(\omega_\Lambda-\omega_r)}+c_2+c_3 a^{1+3\omega_\Lambda}},\end{aligned} \tag{2.22}$$

as well as

$$\begin{aligned}\Omega_r(a) &= c_1 a^{3(\omega_m-\omega_r)}\Omega_m(a),\\ \Omega_\Lambda(a) &= c_2 a^{3(\omega_m-\omega_\Lambda)}\Omega_m(a),\\ \Omega_\kappa(a) &= 1-\Omega_m(a)\left(1+c_1 a^{3(\omega_m-\omega_r)}+c_2 a^{3(\omega_m-\omega_\Lambda)}\right).\end{aligned} \tag{2.23}$$

Note that, if we substitute $\omega_\Lambda = -1$ in the system (2.20) and its analytical solution, we get the dynamical system that is analyzed in [4], as well as its analytical solution. However, having in mind that $\omega_\Lambda = -1$, we keep $\omega_\Lambda$ throughout the Sec. 2.1, since that provides us with symmetry in the system (2.20) with respect to the indices $m$, $r$ and $\Lambda$. Consequently, the same symmetry holds when it comes to the analytical solution of the system (2.20). Moreover, that symmetry inspired us to generalize this three fluid case into $n$ fluid case, which we presented in Sec. 2.

We want to stress that the relations (2.23) for $\omega_m = 0$, $\omega_r = 1/3$ and $\omega_\Lambda = -1$ are obtained in [13] using algebraic approach.

The linear stability theory applied to the system (2.20) gives the same results as in [4]. We briefly list them here for the sake of generalization for $n$ fluid case.

The origin $E_0 = (0,0,0)$ is one of the equilibriums of the system (2.20). According to $\omega_m, \omega_r \in [0,1]$, $\omega_m \neq \omega_r$, $\omega_\Lambda = -1$, the rest of the equilibriums of the system (2.20) are $E_1 = (1,0,0)$, $E_2 = (0,1,0)$ and $E_3 = (0,0,1)$, that are positioned on the axes $\Omega_m$, $\Omega_r$ and $\Omega_\Lambda$, respectively.

$E_0$ represents a saddle point, for the eigenvalues of the Jacobian at this equilibrium are

$$\lambda_m = -\left(1+3\omega_m\right) < 0,\ \lambda_r = -\left(1+3\omega_r\right) < 0,\ \lambda_\Lambda = -\left(1+3\omega_\Lambda\right) = 2. \quad (2.24)$$

This model of the universe near $E_0$ is characterised as the Milne universe (empty and open model of the universe without the cosmological constant).

The eigenvalues of the Jacobian at $E_1$ are

$$\begin{aligned} &\lambda_m = 1+3\omega_m > 0,\ \lambda_r = 3\left(\omega_m-\omega_r\right) \neq 0, \\ &\lambda_\Lambda = 3\left(\omega_m-\omega_\Lambda\right) = 3\left(\omega_m+1\right) > 0, \end{aligned} \quad (2.25)$$

which indicates that this equilibrium is a saddle if $\omega_r > \omega_m$, otherwise is a repeller. The universe near $E_1$ behaves as the Einstein-de Sitter universe (matter dominated and flat model of the universe, without the cosmological constant).

The nature of the equilibrium $E_2$ can be inferred interchanging places of indices $m$ and $r$ in the discussion for the equilibrium $E_1$. The universe near $E_2$ acts as radiation dominated and flat universe without the cosmological constant.

Near $E_3$, the universe behaves as the de-Sitter universe (flat model of the universe dominated by the cosmological constant). This equilibrium is attractor, according to the appropriate eigenvalues

$$\begin{aligned} &\lambda_m = 3\left(\omega_\Lambda-\omega_m\right) = -3\left(1+\omega_m\right) < 0, \\ &\lambda_r = 3\left(\omega_\Lambda-\omega_r\right) = -3\left(1+\omega_r\right) < 0,\ \lambda_\Lambda = 1+3\omega_\Lambda = -2. \end{aligned} \quad (2.26)$$

Further detailed analysis for the three fluid case can be found in [4], while for more detailed physical interpretation one may also consult [3].

## 3 The evolution of the universe filled with $n$ non-interacting barotropic perfect fluids

We analyze the system (2.1) with $0 \le \Omega_i \le 1$. One of the equilibriums is the origin $E_0$. Given that $0 \le \omega_i \le 1$ for $i \neq n$, $\omega_n = -1$, as well as $\omega_i \neq \omega_j$ for $i \neq j$, it is easy to realize that we have $n$ more equilibriums of the system (2.1). We denote them with $E_i$, where $i$ suggests us that the $i^{\text{th}}$ coordinate of the equilibrium $E_i$ is equal to 1, while the rest $(n-1)$ coordinates are equal to 0. Consequently, we have one equilibrium on each of $n$ coordinate axes $\Omega_i$.

For the sake of generalization, we introduce the substitution $m \to 1$, $r \to 2$ and $\Lambda \to 3$ into the eigenvalues of the Jacobian calculated at critical points obtained for the three fluid case. The results for $i,j \in \{1,2,3\}$ are the following:

$$\begin{aligned} &\text{the eigenvalues at } E_0 \text{ are } \lambda_j = -\left(1+3\omega_j\right), \\ &\text{the eigenvalues at } E_i \text{ are } \lambda_j = \begin{cases} 3\left(\omega_i - \omega_j\right), & i \neq j, \\ 1+3\omega_i, & j = i. \end{cases} \end{aligned} \tag{3.1}$$

We stress that the eigenvalues at $E_i$, $i \in \{1,2,3\}$, arise in the analytical solution (2.21) of the three fluid case dynamical system (2.20) as the multipliers of the exponential function arguments. We expect the similar situation for $i \in \{1,2,\ldots,n\}$.

Let denote with $F_i = F_i\left(\Omega_1, \Omega_2, \ldots, \Omega_n\right)$ the right hand side of the system (2.1), i.e.

$$F_i = \Omega_i \left( -\left(1+3\omega_i\right) + \sum_{j=1}^{n} \Omega_j \left(1+3\omega_j\right) \right). \tag{3.2}$$

Then, the elements of the Jacobian matrix $\left[\dfrac{\partial F_i}{\partial \Omega_l}\right]_{n\times n}$ are

$$\frac{\partial F_i}{\partial \Omega_l} = \begin{cases} \Omega_i\left(1+3\omega_l\right), & i \neq l, \\ -\left(1+3\omega_i\right) + \Omega_i\left(1+3\omega_i\right) + \sum\limits_{j=1}^{n} \Omega_j\left(1+3\omega_j\right), & i = l. \end{cases} \tag{3.3}$$

We split the discussion into two parts.

**The equilibrium $\boldsymbol{E_0}$.** Here, all $\Omega_i = 0$ and hence for the curvature density parameter stands $\Omega_\kappa = 1$. Consequently, the model of the universe near this equilibrium point acts as the Milne universe and now we determine its stability.

The elements of the Jacobian matrix at $E_0$ are

$$\left.\frac{\partial F_i}{\partial \Omega_l}\right|_{E_0} = \begin{cases} 0, & i \neq l, \\ -\left(1+3\omega_i\right), & i = l. \end{cases} \tag{3.4}$$

Accordingly, the Jacobian matrix at $E_0$ is the diagonal matrix and therefore its eigenvalues are $\lambda_i = -\left(1+3\omega_i\right)$. The equilibrium $E_0$ is unstable, since

$$\lambda_i = \begin{cases} -\left(1+3\omega_i\right) < 0, & i \neq n, \\ -\left(1+3\omega_n\right) = 2, & i = n. \end{cases} \tag{3.5}$$

Thus, the universe that acts as the Milne universe is unstable.

**The equilibrium $\boldsymbol{E_p}$, $\boldsymbol{p \in \{1,2,\ldots,n\}}$.** In this case, only $\Omega_p = 1$, whereas all others $\Omega_i$ (as well as $\Omega_\kappa$) are equal to zero. The model of the universe near the equilibrium point $E_p$ is flat and dominated by the $p$-fluid.

The elements of the Jacobian matrix at the critical point $E_p$ have the following values

$$\left.\frac{\partial F_i}{\partial \Omega_l}\right|_{E_p} = \begin{cases} 1+3\omega_l, & l \neq i = p, \\ 3\left(\omega_p - \omega_i\right), & l = i \neq p, \\ 1+3\omega_p, & l = i = p, \\ 0, & \text{otherwise.} \end{cases} \tag{3.6}$$

Thus, the Jacobian matrix at $E_p$ has the form

$$\left.\left[\frac{\partial F_i}{\partial \Omega_l}\right]\right|_{E_p} = \begin{bmatrix} 3\left(\omega_p-\omega_1\right) & 0 & \dots & 0 & \dots & 0 \\ 0 & 3\left(\omega_p-\omega_2\right) & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1+3\omega_1 & 1+3\omega_2 & \dots & 1+3\omega_p & \dots & 1+3\omega_n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 3\left(\omega_p-\omega_n\right) \end{bmatrix}. \tag{3.7}$$

Characteristic polynomial of the Jacobian matrix at $E_p$ can be obtained as

$$P\left(\lambda\right) = \det\left(\left.\left[\frac{\partial F_i}{\partial \Omega_l}\right]\right|_{E_p} - \lambda I_n\right). \tag{3.8}$$

If we apply the Laplace expansion in (3.8) along the first row for $(p-1)$ times consecutively, we obtain

$$P\left(\lambda\right) = D_{n-p+1} \cdot \prod_{i=1}^{p-1} \left(3\left(\omega_p - \omega_i\right) - \lambda\right), \tag{3.9}$$

where

$$D_{n-p+1} = \begin{vmatrix} 1+3\omega_p-\lambda & 1+3\omega_{p+1} & \dots & 1+3\omega_n \\ 0 & 3\left(\omega_p-\omega_{p+1}\right)-\lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 3\left(\omega_p-\omega_n\right)-\lambda \end{vmatrix}. \tag{3.10}$$

We note that in (3.10) we have the determinant of upper triangular matrix of order $(n-p+1)$. Therefore, we infer

$$P\left(\lambda\right) = \left(1+3\omega_p-\lambda\right) \prod_{\substack{i=1 \\ i\neq p}}^{n} \left(3\left(\omega_p-\omega_i\right)-\lambda\right). \tag{3.11}$$

Consequently, the eigenvalues of the Jacobian matrix at $E_p$ are

$$\lambda_i = \begin{cases} 3\left(\omega_p - \omega_i\right), & i \neq p, \\ 1 + 3\omega_p, & i = p. \end{cases} \tag{3.12}$$

If we perform the substitution $i \to j$ and $p \to i$ in (3.12) and in the eigenvalues for the Jacobian at $E_0$, for $i, j \in \{1, 2, \dots, n\}$ we obtain

$$\begin{aligned} &\text{the eigenvalues at } E_0 \text{ are } \lambda_j = -\left(1 + 3\omega_j\right), \\ &\text{the eigenvalues at } E_i \text{ are } \lambda_j = \begin{cases} 3\left(\omega_i - \omega_j\right), & j \neq i, \\ 1 + 3\omega_i, & j = i, \end{cases} \end{aligned} \tag{3.13}$$

which is exactly the same result as in (3.1). Moreover, we notice the same situation as for the three fluid case, i.e. the eigenvalues at $E_i$ from (3.13) are the multipliers of the exponential function arguments in (2.15), which is the analytical solution for the $n$ fluid case.

When it comes to the sign of the eigenvalues at $E_i$, when $i \neq n$, we infer

$$\lambda_j = \begin{cases} 3\left(\omega_i - \omega_j\right), & j \neq i \text{ and } j \neq n, \\ 3\left(\omega_i - \omega_n\right) = 3\left(\omega_i + 1\right) > 0, & j \neq i \text{ and } j = n, \\ 1 + 3\omega_i > 0, & j = i \neq n, \end{cases} \tag{3.14}$$

wherefrom we conclude that $E_i$ is unstable equilibrium for $i \neq n$, regardless of the sign of the term $3\left(\omega_i - \omega_j\right)$ for $j \neq i$ and $j \neq n$. The appropriate physical interpretation is that flat universe without cosmological constant and dominated by some $i$-fluid, $0 \leq \omega_i \leq 1$, is unstable.

From the other hand, if $i = n$, i.e. if we inspect the nature of the equilibrium $E_n$, we have

$$\lambda_j = \begin{cases} 3\left(\omega_n - \omega_j\right) = -3\left(1 + \omega_j\right) < 0, & j \neq n, \\ 1 + 3\omega_n = -2, & j = n. \end{cases} \tag{3.15}$$

Thus, the equilibrium $E_n$ is an attractor. Hence, the de-Sitter universe is asymptotically stable.

We already saw in [4] that varying the number of fluids in the universe, we obtain different models of the universe and the evolution of such models is various. Despite that, we can guarantee this - if a chosen model of the universe contains non-zero cosmological constant, then its final future state is the de-Sitter universe, which is consistent with [14].

## 4 Conclusion

We proved that the system of the Friedmann equations for the ΛCDM model of the universe that contains $n$ non-interacting barotropic perfect fluids (one of them is assumed to be the cosmological constant) is equivalent to the system of $n$ nonlinear differential equations of the first order with the density parameters as dependent variables. The equivalence of the system of the Friedmann equations and the system of nonlinear differential equations enables us to study the dynamics of the universe using the tools of the dynamical systems. Moreover, we obtained analytical solution of that dynamical system in the case when all density parameters are strictly positive.

The observed dynamical system is of a class of $n$-dimensional Lotka-Volterra dynamical systems. Lotka-Volterra equations are mostly analyzed using tools of numerical analysis, because their analytical solutions are not easy to find. We obtained here analytical solution of a class of $n$-dimensional Lotka-Volterra dynamical systems. That analytical solution gave new algebraic connection between density parameters of the material in the universe and the scale factor, that may be useful for future cosmological research.

Furthermore, we proved that for fixed $n \in \mathbb{N}$ and for fixed values of $\omega_i \in [0, 1]$, one can determine not only analytical solution of the appropriate dynamical system, but also the nature of all equilibriums of that dynamical system and therefore the evolution of the chosen cosmological model. Despite an infinite number of possible cosmological models as well as an infinite number of evolutionary scenarious, one thing is for sure - if in the chosen model of the universe exists a non-zero cosmological constant, than its future is uniquely determined and it is represented by the de-Sitter universe.

Since we encompassed this theme in the frame of the ΛCDM model of the universe with arbitrary number of non-interacting barotropic perfect fluids, our future work is directed into considering universe's models with barotropic perfect fluids that interact mutualy.

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