DOI: 10.2478/auom-2023-0027
An. Şt. Univ. Ovidius Constanţa **Vol. 31(2),2023, 191–199**

Inequalities for pq^{th}-dual mixed volumes

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Abstract

In the paper, our main aim is to generalize the q^{th} dual volume to L_p space, and introduce pq^{th}-*dual mixed volume* by calculating the first order variation of q^{th} dual volumes. We establish the L_{pq}-Minkowski inequality for pq^{th}-dual mixed volumes and L_{pq}-Brunn-Minkowski inequality for the q^{th}-dual volumes, respectively. The new inequalities in special case yield some new dual inequalities for the q^{th}-dual volumes.

1 Introduction

The q^{th} dual volume was defined by for $q \neq 0$ (see e.g. [1])

$$\mu_q(K) = \left(\frac{1}{|\mu|} \int_{S^{n-1}} \rho(K,u)^q d\mu(u) \right)^{1/q}, \tag{1.1}$$

where K is a convex body (compact, convex subsets with nonempty interior) that contain the origin in their interiors, μ is a Borel measure on S^{n-1} and $\rho(K,u)$ is the radial function of K. The radial function of convex body K is defined by (see e.g. [2])

$$\rho(K,u) = \max\{c \geq 0 : cu \in K\},$$

for $u \in S^{n-1}$.

Key Words: Log-volume, q^{th} dual volume, q^{th}-dual mixed volume, pq^{th}-dual mixed volume.
2010 Mathematics Subject Classification: Primary 46E30; Secondary 52A40.
Received: 14.08.2022
Accepted: 20.12.2022

Recall that $\mu_q(K)$ is monotone nondecreasing and continuous in q. Define the log-volume of K with respect to μ by $\mu_0(K) = \lim_{q\to 0}\mu_q(K)$. Obviously, the log-volume $\mu(K)$ of K with respect to μ is the following (see also [3]):

$$\mu_0(K) = \exp\left(\frac{1}{|\mu|}\int_{S^{n-1}} \log \rho(K,u)d\mu(u)\right). \tag{1.2}$$

The log-volume $\mu(K)$ of a convex body K with respect to μ plays a very important role in solving the Gauss image problem.

In the paper, our main aim is to generalize the q^{th} dual volume to L_p space, and introduce the pq^{th}-dual mixed volume of convex bodies (contain the origin in their interiors) K and L, by calculating the first order variation of of the q^{th} dual volumes with respect to the L_p-harmonic radial addition, is denoted by $\mu_{p,q}(K,L)$, is defined by

$$\mu_{p,q}(K,L) = \frac{\mu_q(K)^{1-q}}{|\mu|}\int_{S^{n-1}}\left(\frac{\rho(K)}{\rho(L)}\right)^p \rho(K)^q d\mu(u). \tag{1.3}$$

where $p \geq 1$ and $q \neq 0$. Obviously, when $K = L$, the pq^{th}-dual mixed volume $\mu_{p,q}(K,L)$ becomes the q^{th} dual volume $\mu_q(K)$. When $q \to 0$ and $K = L$, the pq^{th}-dual mixed volume $\mu_{p,q}(K,L)$ becomes the log-volume $\mu_0(K)$. Further, we establish the following L_{pq}-Minkowski, and Bunn-Minkowski inequalities for the pq^{th}-dual mixed volumes.

The L_{pq}-Minkowski inequality for pq^{th}-dual mixed volumes *If K and L are convex bodies that contain the origin in their interiors, $q \neq 0$ and $p \geq 1$, then for $q > 0$*

$$\mu_{p,q}(K,L) \geq \mu_q(K)^{\frac{p+q}{q}}\mu_q(L)^{-\frac{p}{q}}. \tag{1.4}$$

When μ is a spherical Lebesgue measure of S^{n-1}, equality holds if and only if K and L are dilates.

The inequality is reversed for $q < 0$.

The L_{pq}-Brunn-Minkowski inequality for q^{th} dual volumes *If K and L are convex bodies that contain the origin in their interiors, $q \neq 0$, $\varepsilon > 0$ and $p \geq 1$, then for $q > 0$*

$$\mu_q(K \widehat{+}_p \varepsilon \cdot L)^{-\frac{p}{q}} \geq \mu_q(K)^{-\frac{p}{q}} + \varepsilon \cdot \mu_q(L)^{-\frac{p}{q}}. \tag{1.5}$$

When μ is a spherical Lebesgue measure of S^{n-1}, equality holds if and only if K and L are dilates, and where $\widehat{+}_p$ is the L_p-harmonic radial addition (see Section 2).

The inequality is reversed for $q < 0$.

2 Notations and Preliminaries

A body in the n-dimensional Euclidean space $\mathbb{R}^n$ is a compact set equal to the closure of its interior. For a compact set $K \subset \mathbb{R}^n$, we denote by $V(K)$ the (n-dimensional) Lebesgue measure of K, called the volume of K. The unit ball in $\mathbb{R}^n$ and its surface are denoted by B and S^{n-1}, respectively. Let $\mathcal{K}^n$ denote the class of nonempty compact convex subsets containing the origin in their interiors in $\mathbb{R}^n$. The radial function sssociated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin and contains the origin, is $\rho(K,\cdot) : S^{n-1} \to [0,\infty)$. If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Let $\mathcal{S}^n$ denote the set of star bodies about the origin in $\mathbb{R}^n$. Two star bodies K and L are dilates if $\rho(K,u)/\rho(L,u)$ is independent of $u \in S^{n-1}$. For $K, L \in \mathcal{S}^n$, the radial Hausdorff metric is given by (see e.g. [4])

$$\tilde{\delta}(K,L) = |\rho(K,u) - \rho(L,u)|_\infty.$$

2.1 L_p-harmonic radial addition

The L_p-harmonic radial addition was defined by Lutwak [5]: If K, L are star bodies, the L_p-harmonic radial addition, defined by

$$\rho(K \widehat{+}_p L, x)^{-p} = \rho(K,x)^{-p} + \rho(L,x)^{-p}, \tag{2.1}$$

for $p \geq 1$ and $x \in \mathbb{R}^n$. The L_p-harmonic radial addition of convex bodies was first studied by Firey [6]. The operation of the L_p-harmonic radial addition and L_p-dual Minkowski, Brunn-Minkwski inequalities are the basic concept and inequalities in the L_p-dual Brunn-Minkowski theory.

2.2 L_p-dual mixed volume

The dual mixed volume $\widetilde{V}_{-1}(K,L)$ of star bodies K and L is defined by ([5])

$$\widetilde{V}_{-1}(K,L) = \lim_{\varepsilon \to 0^+} \frac{V(K) - V(K \widehat{+} \varepsilon \cdot L)}{\varepsilon}, \tag{2.2}$$

where $\widehat{+}$ is the harmonic addition. The following is a integral representation for the dual mixed volume $\widetilde{V}_{-1}(K,L)$:

$$\widetilde{V}_{-1}(K,L) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n+1} \rho(L,u)^{-1} dS(u). \tag{2.3}$$

The dual Minkowski inequality for the dual mixed volume states that

$$\widetilde{V}_{-1}(K,L)^n \geq V(K)^{n+1} V(L)^{-1}, \tag{2.4}$$

with equality if and only if K and L are dilates. (see ([7]))

The dual Brunn-Minkowski inequality for the harmonic addition (due to Firey [6]) states that

$$V(K\widehat{+}L)^{-1/n} \geq V(K)^{-1/n} + V(L)^{-1/n}, \tag{2.5}$$

with equality if and only if K and L are dilates.

The L_p-dual mixed volume $\widetilde{V}_{-p}(K,L)$ of K and L is defined by ([5])

$$\widetilde{V}_{-p}(K,L) = -\frac{p}{n}\lim_{\varepsilon\to 0^+}\frac{V(K\widehat{+}_p\varepsilon\cdot L)-V(K)}{\varepsilon}, \tag{2.6}$$

where $K, L \in \mathcal{S}^n$ and $p \geq 1$.

The following is an integral representation for the L_p-dual mixed volume: For $K, L \in \mathcal{S}^n$ and $p \geq 1$,

$$\widetilde{V}_{-p}(K,L) = \frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n+p}\rho(L,u)^{-p}dS(u). \tag{2.7}$$

L_p-dual Minkowski and Brunn-Minkowski inequalities were established by Lutwak [5]: If $K, L \in \mathcal{S}^n$ and $p \geq 1$, then

$$\widetilde{V}_{-p}(K,L)^n \geq V(K)^{n+p}V(L)^{-p}, \tag{2.8}$$

with equality if and only if K and L are dilates, and

$$V(K\widehat{+}_pL)^{-p/n} \geq V(K)^{-p/n} + V(L)^{-p/n}, \tag{2.9}$$

with equality if and only if K and L are dilates.

2.3 L_p-mixed harmonic quermassintegral

From (2.1), it is easy to see that if $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, then

$$-\frac{p}{n-i}\lim_{\varepsilon\to 0^+}\frac{\widetilde{W}_i(K\widehat{+}_p\varepsilon\cdot L)-\widetilde{W}_i(L)}{\varepsilon} = \frac{1}{n}\int_{S^{n-1}}\rho(K.u)^{n-i+p}\rho(L.u)^{-p}dS(u). \tag{2.10}$$

Let $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, the mixed p-harmonic quermassintegral of star K and L, denoted by $\widetilde{W}_{-p,i}(K,L)$, defined by (see [8])

$$\widetilde{W}_{-p,i}(K,L) = \frac{1}{n}\int_{S^{n-1}}\rho(K,u)^{n-i+p}\rho(L,u)^{-p}dS(u). \tag{2.11}$$

Obviously, when $K = L$, the p-harmonic quermassintegral $\widetilde{W}_{-p,i}(K,L)$ becomes the dual quermassintegral $\widetilde{W}_i(K)$. The Minkowski and Brunn-Minkowski inequalities for the mixed p-harmonic quermassintegral are following (see [9]): If $K, L \in \mathcal{S}^n$, $0 \le i < n$ and $p \ge 1$, then

$$\widetilde{W}_{-p,i}(K,L)^{n-i} \ge \widetilde{W}_i(K)^{n-i+p}\widetilde{W}_i(L)^{-p}, \tag{2.12}$$

with equality if and only if K and L are dilates. If $K, L \in \mathcal{S}^n$, $0 \le i < n$ and $p \ge 1$, then

$$\widetilde{W}_i(K \widehat{+}_p L)^{-p/(n-i)} \ge \widetilde{W}_i(K)^{-p/(n-i)} + \widetilde{W}_i(L)^{-p/(n-i)}, \tag{2.13}$$

with equality if and only if K and L are dilates.

3 Inequalities for pq^{th}-dual mixed volumes

In this section, in order to derive the L_{pq}-Minkowski inequality for the pq^{th}-dual mixed volumes and L_{pq}-Brunn-Minkowski inequality for the q^{th}-dual volumes, we need to the following definition and lemmas.

Definition 3.1 (The pq^{th}-dual mixed volumes) For $K, L \in \mathcal{K}^n$, $q \ne 0$ and $p \ge 1$, the pq^{th}-dual mixed volume of K and L, is denoted by $\mu_{p,q}(K,L)$, is defined by

$$\mu_{p,q}(K,L) = \frac{\mu_q(K)^{1-q}}{|\mu|}\int_{S^{n-1}} \left(\frac{\rho(K)}{\rho(L)}\right)^p \rho(K)^q d\mu(u). \tag{3.1}$$

When $p = 1$, the pq^{th}-dual mixed volume $\mu_{p,q}(K,L)$ becomes the q^{th} dual mixed volume $\mu_q(K,L)$, and for $q \ne 0$

$$\mu_q(K,L) = \frac{\mu_q(K)^{1-q}}{|\mu|}\int_{S^{n-1}} \rho(K)^{q+1}\rho(L)^{-1} d\mu(u).$$

When $q \to 0$, the pq^{th}-dual mixed volume $\mu_{p,q}(K,L)$ becomes the L_p log-volume $\mu_{p,0}(K)$, and for $p \ge 1$

$$\mu_{p,0}(K,L) = \frac{\mu_0(K)}{|\mu|}\int_{S^{n-1}} \left(\frac{\rho(K)}{\rho(L)}\right)^p d\mu(u).$$

When $p = 1$, the L_p log-volume $\mu_{p,0}(K)$ becomes the log-volume $\mu_0(K)$.

Lemma 3.1 *If $K, L \in \mathcal{S}^n$ and $p \ge 1$, then (see e.g.* [10])

$$K \widehat{+}_p \varepsilon \cdot L \to K \tag{3.2}$$

as $\varepsilon \to 0^+$.

Lemma 3.2 *If $K, L \in \mathcal{K}^n$, $q \neq 0$ and $p \geq 1$, then*

$$d\mu_q(K\widehat{+}_p\varepsilon \cdot L)\Big|_{\varepsilon=0} = -\frac{\mu_p(K)^{1-q}}{p|\mu|}\int_{S^{n-1}}\left(\frac{\rho(K)}{\rho(L)}\right)^p \rho(K)^q d\mu(u). \tag{3.3}$$

Proof From the hypotheses and by using Lemma 3.1, it is easy to observe that

$$\lim_{\varepsilon\to 0}\frac{\mu_q(K\widehat{+}_p\varepsilon\cdot L)-\mu_q(K)}{\varepsilon}$$

$$= \frac{1}{|\mu|^{1/q}}\lim_{\varepsilon\to 0}\frac{\left(\int_{S^{n-1}}\rho(K\widehat{+}_p\varepsilon\cdot L,u)^q d\mu(u)\right)^{1/q}-\left(\int_{S^{n-1}}\rho(K,u)^q d\mu(u)\right)^{1/q}}{\varepsilon}$$

$$= \frac{1}{q|\mu|^{1/q}}\left(\int_{S^{n-1}}\rho(K,u)^q d\mu(u)\right)^{1/q-1}\lim_{\varepsilon\to 0}\int_{S^{n-1}}\frac{\rho(K\widehat{+}_p\varepsilon\cdot L,u)^q-\rho(K,u)^q}{\varepsilon}d\mu(u)$$

$$= \frac{1}{q|\mu|}\mu_q(K)^{1-q}\int_{S^{n-1}}\lim_{\varepsilon\to 0}\frac{(\rho(K,u)^{-p}+\varepsilon\rho(L,u)^{-p})^{-q/p}-\rho(K,u)^q}{\varepsilon}d\mu(u)$$

$$= -\frac{1}{p|\mu|}\mu_q(K)^{1-q}\int_{S^{n-1}}\left(\frac{\rho(K)}{\rho(L)}\right)^p \rho(K)^q d\mu(u).$$

□

Lemma 3.3 *If $K, L \in \mathcal{K}^n$, $q \neq 0$ and $p \geq 1$, then*

$$\mu_{p,q}(K,L) = p\lim_{\varepsilon\to 0}\frac{\mu_q(K)-\mu_q(K\widehat{+}_p\varepsilon\cdot AL)}{\varepsilon}. \tag{3.4}$$

Proof This yields immediately from the Definition 3.1 and Lemma 3.1. □

Lemma 3.4 *If $K, L \in \mathcal{S}^n$ and $p \geq 1$, then for $A \in O(n)$ (see e.g.* [10]*)*

$$A(K\widehat{+}_p\varepsilon\cdot L) = AK\widehat{+}_p\varepsilon\cdot AL. \tag{3.5}$$

Lemma 3.5 *If $K, L \in \mathcal{K}^n$, $q \neq 0$ and $p \geq 1$, then* for $A \in O(n)$,

$$\mu_{p,q}(AK, AL) = \mu_{p,q}(K,L). \tag{3.6}$$

Proof From (3.4) and (3.5), we have

$$\begin{aligned}
\mu_{p,q}(AK,AL) &= p\lim_{\varepsilon\to 0}\frac{\mu_q(AK)-\mu_q(AK\widehat{+}_p\varepsilon\cdot AL)}{\varepsilon}\\
&= p\lim_{\varepsilon\to 0}\frac{\mu_q(AK)-\mu_q(A(K\widehat{+}_p\varepsilon\cdot L))}{\varepsilon}\\
&= p\lim_{\varepsilon\to 0}\frac{\mu_q(K)-\mu_q(K\widehat{+}_p\varepsilon\cdot L)}{\varepsilon}\\
&= \mu_{p,q}(K,L),
\end{aligned}$$

where μ is a spherical Lebesgue measure of S^{n-1}. □

Lemma 3.6 *Let $K, L \in \mathcal{K}^n$, $\varepsilon > 0$ and $p \geq 1$.*

(1) If K and L are dilates, then K and $K \widehat{+}_p \varepsilon \cdot L$ are dilates.

(2) If K and $K \widehat{+}_p \varepsilon \cdot L$ are dilates, then K and L are dilates.

Proof Suppose exist a constant $\delta > 0$ such that $L = \delta K$, for $\varepsilon > 0$ and $p \geq 1$, we have

$$\rho(K \widehat{+}_p \varepsilon \cdot L) = [1 + \varepsilon \delta^{-p}]^{-1/p} \cdot \rho(K, u).$$

On the other hand, the exist unique constant $\eta > 0$ such that

$$\rho(\eta K, u) = [1 + \varepsilon \delta^{-p}]^{-1/p} \cdot \rho(K, u),$$

where η satisfies that

$$\eta = [1 + \varepsilon \delta^{-p}]^{-1/p} \cdot \rho(K, u).$$

This shows that $(1 - \lambda)K +_p \varepsilon \cdot L = \eta K$.

For $p \geq 1$, suppose exist a constant $\delta > 0$ such that $K \widehat{+}_p \varepsilon \cdot L = \delta K$. Then

$$\left(\frac{\rho(K, u)}{\rho(L, u)}\right)^{-p} = \frac{\varepsilon}{\delta^{-p} - 1}.$$

This shows that K and L are homothetic. □

Theorem 3.1 (The L_{pq}-Minkowski inequality for $pq^{\rm th}$-dual volumes) *If $K, L \in \mathcal{K}^n$, $q \neq 0$ and $p \geq 1$, then for $q > 0$*

$$\mu_{p,q}(K, L) \geq \mu_q(K)^{\frac{p+q}{q}} \mu_q(L)^{-\frac{p}{q}}. \tag{3.7}$$

When μ is a spherical Lebesgue measure of S^{n-1}, equality holds if and only if K and L are dilates. The inequality is reversed for $q < 0$.

Proof From (1.1), (3.1) and by using Hölder inequality for $p > 0$

$$\begin{aligned}
\mu_{p,q}(K, L) &= \frac{\mu_q(K)^{1-q}}{|\mu|} \int_{S^{n-1}} \left(\frac{\rho(K)}{\rho(L)}\right)^p \rho(K)^q d\mu(u) \\
&= \frac{\mu_q(K)^{1-q}}{|\mu|} \int_{S^{n-1}} (\rho(K)^q)^{\frac{p+q}{q}} (\rho(L)^q)^{\frac{-p}{q}} d\mu(u) \\
&\geq \frac{\mu_q(K)^{1-q}}{|\mu|} \left(\int_{S^{n-1}} \rho(K)^q d\mu(u)\right)^{\frac{p+q}{q}} \left(\int_{S^{n-1}} \rho(L)^q d\mu(u)\right)^{-\frac{p}{q}} \\
&= \mu_q(K)^{\frac{p+q}{q}} \mu_q(L)^{-\frac{p}{q}}.
\end{aligned}$$

When μ is a spherical Lebesgue measure of S^{n-1}, from the equality of Hölder's inequality, it yields the equality holds if and only if K and L are dilates. Obviously, the inequality is reversed for $q < 0$. □

Theorem 3.2 (The L_{pq}-Brunn-Minkowski inequality for q^{th} dual volumes) *If $K, L \in \mathcal{K}^n$, $q \neq 0$, $\varepsilon > 0$ and $p \geq 1$, then for $q > 0$*

$$\mu_q(K \widehat{+}_p \varepsilon \cdot L)^{-\frac{p}{q}} \geq \mu_q(K)^{-\frac{p}{q}} + \varepsilon \cdot \mu_q(L)^{-\frac{p}{q}}. \tag{3.8}$$

When μ is a spherical Lebesgue measure of S^{n-1}, equality holds if and only if K and L are dilates.

The inequality is reversed for $q < 0$.

Proof From (2.1), (3.1) and (3.7) , for $p > 0$, $\varepsilon > 0$ and any $M \in \mathcal{K}^n$

$$\mu_{p,q}(M, K \widehat{+}_p \varepsilon \cdot L) = \tfrac{\mu_q(K)^{1-q}}{|\mu|} \int_{S^{n-1}} \rho(M)^{p+q} \left(\rho(K)^{-p} + \varepsilon \rho(L)^{-p}\right) d\mu(u)$$

$$= \tfrac{\mu_q(M)^{1-q}}{|\mu|} \left(\int_{S^{n-1}} \left(\tfrac{\rho(M)}{\rho(K)}\right)^p \rho(K)^q d\mu(u) + \varepsilon \cdot \int_{S^{n-1}} \left(\tfrac{\rho(M)}{\rho(L)}\right)^p \rho(M)^q d\mu(u) \right)$$

$$= \mu_{p,q}(M, K) + \varepsilon \cdot \mu_{p,q}(M, L)$$

$$\geq \quad \mu_q(M)^{\frac{p+q}{q}} \mu_q(K)^{-\frac{p}{q}} + \varepsilon \cdot \mu_q(M)^{\frac{p+q}{q}} \mu_q(L)^{-\frac{p}{q}}. \tag{3.9}$$

Putting $M = K \widehat{+}_p \varepsilon \cdot L$ in (3.9), and as $\mu_{p,q}(M, K \widehat{+}_p \varepsilon \cdot \varepsilon \cdot L) = \mu_q(K \widehat{+}_p \varepsilon \cdot L)$, (3.8) easily follows.

From the equality of (3.7), it follows that the equality in (3.8) holds if and only if $K \widehat{+}_p \varepsilon \cdot L$ and K, and $K \widehat{+}_p \varepsilon \cdot L$ and L are dilates, respectively. On the other hand, from the equality of Theorem 3.1, this yields that when μ is a spherical Lebesgue measure of S^{n-1}, the equality in (3.8) holds if and only if K and L are dilates. Obviously. this inequality is reversed for $q < 0$. □

The following inequalities are special cases of (3.7) and (3.8), respectively.

Corollay 3.1 (The L_q-Minkowski inequality for q^{th}-dual volumes) *If $K, L \in \mathcal{K}^n$ and $q \neq 0$, then $q > 0$*

$$\mu_q(K, L) \geq \mu_q(K)^{\frac{q+1}{q}} \mu_q(L)^{-\frac{1}{q}}. \tag{3.10}$$

When μ is a spherical Lebesgue measure of S^{n-1}, equality holds if and only if K and L are dilates. The inequality is reversed for $q < 0$.

Corollary 3.2 (The L_q-Brunn-Minkowski inequality for q^{th} dual volumes) *If $K, L \in \mathcal{K}^n$, $q \neq 0$ and $\varepsilon > 0$, then for $q > 0$*

$$\mu_q(K \widehat{+} \varepsilon \cdot L)^{-\frac{1}{q}} \geq \mu_q(K)^{-\frac{1}{q}} + \varepsilon \cdot \mu_q(L)^{-\frac{1}{q}}. \tag{3.11}$$

When μ is a spherical Lebesgue measure of S^{n-1}, equality holds if and only if K and L are dilates. The inequality is reversed for $q < 0$.

Acknowledgements. The research is supported by the National Natural Science Foundation of China (11371334, 10971205).

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