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On the torsional energy of torus knots under infinitesimal bending

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Abstract

The article deals with the infinitesimal bending theory application to the knots theory. The impact of infinitesimal bending on the torsional energy at torus knots is considered, and the results show that it is not stationary under infinitesimal bending. The torsional energy variation is determined as well. We prove that there is no infinitesimal bending field that leaves torus curves on the torus. Besides, we define an infinitesimal bending field that does not tear the torus knots while bending. Having in mind the importance of visualization in the infinitesimal bending theory, we observed infinitesimal bending of a curve in that field using independently developed software. The graphs we obtained are presented in the paper and the torus knots are coloured according to their torsional energy. We calculated the numerical value of torsional energy under infinitesimal bending and, finally, the results are discussed using convenient specific examples.

1 Introduction

The integrals of the squared curvature and squared torsion are very important, because they are related to Willmore energy (bending energy) and torsional energy of elastic filaments [18], respectively. Willmore energy penalizes bending models the stiffness of a polymer and it has been used to model the elastic

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properties of DNA [3]. The sum of Willmore and torsional energy was studied in biophysics and, due to the fact that $\mathbf{n}_1' \cdot \mathbf{n}_1' = k^2 + \tau^2$, it is a natural function of curvature and torsion.

A knot is a closed loop in space without self-intersections. The torus knots are a special kind of knots located on the torus. They are very important in physics, biology, mechanics, etc. (for more details see [8, 10, 13, 19, 20]). A number of studies deals with relation between torus knots and DNA (see [1, 21]). The Willmore and torsional energy of torus knots were discussed in [18]. Also, elastic filaments in the shape of torus knots have been studied in relation to Willmore and torsional energy in article [4]. Here, we study the torus knots and their torsional energy under infinitesimal bending.

Given the fact that knots are curves in space, in this article, we observe the infinitesimal bending of curves, which is a part of the more general infinitesimal bending theory. Under infinitesimal bending, a curve is included in a continuous family of curves with a common property that the arc lenght is stationary. Besides the arc length, there are other stationary properties. On the other hand, the ones that change their values under infinitesimal bending, i.e. those having a variation. Infinitesimal bending of curves was widely studied in [5, 6, 14, 15, 16, 24, 25]. The infinitesimal bending theory is applied in different fields. Some of the articles on its applications are [2, 7, 11, 12, 22, 26, 27, 28]. In this article, we deal with the infinitesimal bending application in the knot theory. The Willmore energy of knots under infinitesimal bending was discussed in [9]. The total curvature of the knots under the second-order infinitesimal bending was presented in [17].

2 Preliminaries

Here we state basic definitions and theorems needed in the research of infinitesimal bending of curves, according to Efimov [5] and Velimirović [24], [25]. At some places in the article, we call upon results proven in the references we cite.

Let us consider a continuous biregular curve $C \subset \mathbb{R}^3$

$$C : \mathbf{r} = \mathbf{r}(t), \quad t \in \mathcal{J} \subseteq \mathbb{R}, \tag{1}$$

that is included in a family of the curves

$$C_\epsilon : \mathbf{r}_\epsilon = \mathbf{r}(t) + \epsilon \mathbf{z}(t), \quad \epsilon \geq 0, \ \epsilon \to 0. \tag{2}$$

Obviously, the curve C is obtained for $\epsilon = 0$, i.e. $C = C_0$.

Definition 1. *The family of curves C_ϵ is an* ***infinitesimal bending of the curve*** *C if*

$$ds_\epsilon^2 - ds^2 = o(\epsilon), \tag{3}$$

and the vector field $\mathbf{z} = \mathbf{z}(t)$*,* $\mathbf{z} \in C^1$*, is an* ***infinitesimal bending field*** *of the curve* C*.*

Considering previous propositions, one can show that the following theorem holds.

Theorem 1. *[5] The necessary and sufficient condition for vector field* $\mathbf{z}(t)$ *to be an infinitesimal bending field of a curve* C *is*

$$d\mathbf{r} \cdot d\mathbf{z} = 0 \tag{4}$$

where $\cdot$ *means the scalar product in* $\mathbb{R}^3$*.*

In the relation (4) we will also use notation $\dot{\mathbf{r}} \cdot \dot{\mathbf{z}} = 0$, where $\dot{\mathbf{r}}$ denotes a derivate of function $\mathbf{r}$ with respect to the parameter t.

Theorem 2. *[24] Infinitesimal bending field for a curve* C *is*

$$\mathbf{z}(t) = \int (p(t)\mathbf{n}_1(t) + q(t)\mathbf{n}_2(t))\, dt, \tag{5}$$

where $p(t)$ *and* $q(t)$ *are arbitrary integrable functions and vectors* $\mathbf{n}_1(t)$ *and* $\mathbf{n}_2(t)$ *are unit principal normal and binormal vector fields of the curve* C*, respectively.*

Under the infinitesimal bending some of the curve properties have variations (for example, curvature, torsion, etc.), while others (such as arc length and geodesic curvature) remain stationary.

Definition 2. *[23] Let* $\mathcal{A} = \mathcal{A}(t)$ *be the magnitude that characterizes a geometric property on a curve* C *and* $\mathcal{A}_\epsilon = \mathcal{A}_\epsilon(t)$ *the corresponding magnitude on the curve* C_ϵ *being infinitesimal bending of the curve* C*. Then*

$$\delta\mathcal{A} = \left.\frac{d(\mathcal{A}_\epsilon(t))}{d\epsilon}\right|_{\epsilon=0}. \tag{6}$$

is called the ***variation*** *of the geometric magnitude* $\mathcal{A}$ *under infinitesimal bending* C_ϵ *of the curve* C*.*

Further information about variation of geometric properties under infinitesimal bending can be found in articles [9], [14] - [17], [24]-[27].

3 Variation of torsional energy

We will consider a biregular curve

$$C : \mathbf{r} = \mathbf{r}(s) = \mathbf{r}(t(s)), \quad s \in \mathcal{I} = [0, S] \subseteq \mathbb{R} \tag{7}$$

parametrized by the arc length s. The torsional energy of the curve C is given with the equation

$$E_\tau = \frac{1}{2}\int\limits_{\mathfrak{I}} \tau^2(s)ds. \tag{8}$$

Theorem 3. *Under infinitesimal bending of the curve C, variation of its torsional energy is*

$$\delta E_\tau = \frac{1}{2}\int\limits_{\mathfrak{I}} \left(\tau^2 z\right)' ds + \delta E_{\tau_1} + \delta E_{\tau_2}, \tag{9}$$

whereby it is

$$\delta E_{\tau_1} = \int\limits_{\mathfrak{I}} \left(2\tau\left(\frac{\tau'}{k}\right)' + \frac{\tau'^2}{k} + \frac{3}{2}k\tau^2\right) z_1 ds + \int\limits_{\mathfrak{I}} \left(2\frac{\tau^2}{k}z_1' - \frac{\tau\tau'}{k}z_1\right)' ds, \tag{10}$$

and

$$\begin{aligned}\delta E_{\tau_2} = &\int\limits_{\mathfrak{I}} \left(-\left(\frac{\tau'}{k}\right)'' + \frac{\tau^2\tau'}{k} - (k\tau)'\right) z_2 ds\\ &+ \int\limits_{\mathfrak{I}} \left(\frac{\tau}{k}z_2'' - \frac{\tau'}{k}z_2' + (k^2-\tau^2)\frac{\tau}{k}z_2 + \left(\frac{\tau'}{k}\right)' z_2\right)' ds.\end{aligned} \tag{11}$$

where k and τ are the curvature and the torsion of the curve C (7), respectively, and $\mathbf{z} = (z, z_1, z_2)$ is infinitesimal bending field of the curve C. Prime denotes derivative with respect to the arc length s.

Proof. Given the fact that $\delta(ds) = 0$, the torsional energy of bent curves C_ϵ will be

$$\begin{aligned}E_{\tau_\epsilon} &= \frac{1}{2}\int\limits_{\mathfrak{I}} \tau_\epsilon^2 ds_\epsilon = \frac{1}{2}\int\limits_{\mathfrak{I}} (\tau + \epsilon\delta\tau)^2 (ds + \epsilon\delta(ds))\\ &= \frac{1}{2}\int\limits_{\mathfrak{I}} \left(\tau^2 + 2\epsilon\tau\delta\tau + \epsilon^2(\delta\tau)^2\right) ds.\end{aligned} \tag{12}$$

After neglecting the terms $\epsilon^\alpha, \alpha \geq 2$, we get

$$E_{\tau_\epsilon} = E_\tau + \epsilon\int\limits_{\mathfrak{I}} \tau\delta\tau ds = E_\tau + \epsilon\delta E_\tau, \tag{13}$$

and from here

$$\delta E_\tau = \int_{\mathcal{J}} \tau \delta\tau ds. \tag{14}$$

Considering (32) in [14], for variation of the torsion we have

$$\delta\tau = \tau' z + k(z_2' + 2\tau z_1) + \left(\frac{1}{k}\left(2\tau z_1' + \tau' z_1 + z_2'' - \tau^2 z_2\right)\right)'. \tag{15}$$

After necessary rearranging, it follows

$$\begin{aligned}\tau\delta\tau =& \tau\tau' z + k\tau z_2' + 2k\tau^2 z_1 + \frac{1}{k^2}\left(3k\tau\tau' z_1' + 2k\tau^2 z_1'' + k\tau\tau'' z_1 + k\tau z_2'''\right.\\ &\left.-2k\tau^2\tau' z_2 - k\tau^3 z_2' - 2k'\tau^2 z_1' - k'\tau\tau' z_1 - k'\tau z_2'' + k'\tau^3 z_2\right).\end{aligned} \tag{16}$$

From equation

$$(\tau^2 z)' = 2\tau\tau' z + \tau^2 z', \tag{17}$$

and based on equation (14) in [14], we obtain

$$\tau\tau' z = \frac{1}{2}(\tau^2 z)' - \frac{1}{2}k\tau^2 z_1. \tag{18}$$

Further, we have

$$\frac{1}{k^2}\left(3k\tau\tau' z_1' + 2k\tau^2 z_1'' - 2k'\tau^2 z_1'\right) = 2\left(\frac{\tau^2}{k} z_1'\right)' - \frac{1}{k^2}k\tau\tau' z_1'. \tag{19}$$

Also, the next equation holds

$$\begin{aligned}\frac{1}{k^2}\left(k\tau\tau'' z_1 - k'\tau\tau' z_1\right) &= \frac{2}{k^2}\left(k\tau\tau'' z_1 - k'\tau\tau' z_1\right) - \frac{1}{k^2}\left(k\tau\tau'' z_1 - k'\tau\tau' z_1\right)\\ &= 2\tau z_1\left(\frac{\tau'}{k}\right)' - \left(\frac{\tau\tau'}{k} z_1\right)' + \frac{1}{k^2}\left(k\tau'^2 z_1 + k\tau\tau' z_1'\right),\end{aligned} \tag{20}$$

because it is

$$2\tau z_1\left(\frac{\tau'}{k}\right)' = \frac{2}{k^2}\left(k\tau\tau'' z_1 - k'\tau\tau' z_1\right), \tag{21}$$

and

$$-\left(\frac{\tau\tau'}{k} z_1\right)' = -\frac{1}{k^2}\left(k\tau\tau'' z_1 - k'\tau\tau' z_1\right) - \frac{1}{k^2}\left(k\tau'^2 z_1 + k\tau\tau' z_1'\right). \tag{22}$$

As it is $(k\tau z_2)' = k'\tau z_2 + k\tau' z_2 + k\tau z_2'$, we have

$$k\tau z_2' = (k\tau z_2)' - k'\tau z_2 - k\tau' z_2 = (k\tau z_2)' - (k\tau)' z_2. \tag{23}$$

Furthermore, we have the following equation

$$\begin{aligned}\frac{1}{k^2}\left(k\tau z_2''' - k'\tau z_2''\right) &= \left(\frac{\tau}{k} z_2''\right)' - \frac{1}{k^2} k\tau' z_2'' \\ &= \left(\frac{\tau}{k} z_2''\right)' - \left(\frac{\tau'}{k} z_2'\right)' + \frac{1}{k^2}\left(k\tau'' z_2' - k'\tau' z_2'\right) \\ &= \left(\frac{\tau}{k} z_2''\right)' - \left(\frac{\tau'}{k} z_2'\right)' + \left(\left(\frac{\tau'}{k}\right)' z_2\right)' - \left(\frac{\tau'}{k}\right)'' z_2,\end{aligned} \tag{24}$$

since the expression $\frac{1}{k^2}\left(k\tau'' z_2' - k'\tau' z_2'\right)$ can be written as

$$\frac{1}{k^2}\left(k\tau'' z_2' - k'\tau' z_2'\right) = \left(\left(\frac{\tau'}{k}\right)' z_2\right)' - \left(\frac{\tau'}{k}\right)'' z_2. \tag{25}$$

Taking into consideration the fact that

$$-\left(\frac{\tau^3}{k} z_2\right)' = -\frac{1}{k^2}\left(3k\tau^2\tau' z_2 + k\tau^3 z_2' - k'\tau^3 z_2\right), \tag{26}$$

we have

$$\frac{1}{k^2}\left(-2k\tau^2\tau' z_2 - k\tau^3 z_2' + k'\tau^3 z_2\right) = \frac{1}{k^2} k\tau^2\tau' z_2 - \left(\frac{\tau^3}{k} z_2\right)'. \tag{27}$$

Finally, substituting (18), (19), (20), (23), (24) and (27) into (14) we get (9). □

Corollary 1. *The torsional energy of a bent curve is*

$$E_{\tau\epsilon} = E_\tau + \epsilon\delta E_\tau \tag{28}$$

where E_τ and δE_τ are given with (8) and (9), respectively.

Given that the torus knots are closed curves, we will express the previous result for them. A curve is closed if, under bending, the condition $\mathbf{z}(0) = \mathbf{z}(S)$ is met on the interval $\mathcal{I} = [0, S]$.

Corollary 2. *Under infinitesimal bending of a closed curve, variation of its torsional energy is*

$$\delta E_\tau = \int_{\mathcal{J}} \left(\left(2\tau \left(\frac{\tau'}{k}\right)' + \frac{\tau'^2}{k} + \frac{3}{2}k\tau^2 \right) z_1 + \left(-\left(\frac{\tau'}{k}\right)'' + \frac{\tau^2\tau'}{k} - (k\tau)' \right) z_2 \right) ds. \tag{29}$$

where k and τ are the curvature and the torsion of the closed curve, respectively.

This corollary applies to the torus knots as well.

4 Infinitesimal bending of a curve on the torus

A torus knot is a knot that can be drawn on the surface of an inner tube, without self-intersections. The parametric equation of the torus knot $T(p,q)$ is

$$\mathbf{r}(t) = \begin{cases} (c + a\cos(pt))\cos(qt), \\ (c + a\cos(pt))\sin(qt), \\ a\sin(pt), \end{cases} \tag{30}$$

$t \in [0, 2\pi)$, a, c, p, $q \in \mathbb{R}$. In this note, the torus knots are wrapped on a torus, where a is the helix radius and c the torus radius (Fig. 1).

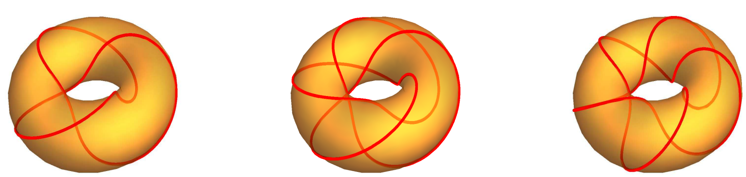

Figure 1: Torus knots $T(3,2)$, $T(4,3)$ and $T(5,2)$, respectively.

In this part we consider infinitesimal bending of an arbitrary curve on the torus. The following theorem states that an arbitrary curve on the torus does not stay on it after the infinitesimal bending.

Theorem 4. *Let $C:\ \mathbf{r}:(t_1,t_2)\to\mathbb{R}^3$ be a regular continuous curve on the torus*

$$\mathcal{S}:\mathbf{r}(u,v)=\left((c+a\cos v)\cos u,(c+a\cos v)\sin u,a\sin v\right), \tag{31}$$

$u,v\in[0,2\pi)$, $a,c\in\mathbb{R}$. There is no non-trivial vector field $\mathbf{z}(t)$ that includes the given curve under infinitesimal bending into the family of curves $C_\epsilon:\ \mathbf{r}_\epsilon=\mathbf{r}(t)+\epsilon\mathbf{z}(t)$, $\epsilon\geq 0$, $\epsilon\to 0$, on the torus $\mathcal{S}$ (31).

Proof. The equation of the torus in cartesian coordinates is

$$\mathcal{S}:\left(c-\sqrt{x^2+y^2}\right)^2+z^2=a^2. \tag{32}$$

The equation of the curve C on the torus (31) will be

$$C:\ \mathbf{r}(t)=\begin{cases}(c+a\cos v(t))\cos u(t),\\(c+a\cos v(t))\sin u(t),\\a\sin v(t),\end{cases} \tag{33}$$

$t\in[0,2\pi)$ and the family of the curves C_ϵ on the torus will be

$$C_\epsilon:\ \mathbf{r}_\epsilon(t)=\mathbf{r}(t)+\epsilon\mathbf{z}(t)=\begin{cases}(c+a\cos v(t))\cos u(t)+\epsilon z(t),\\(c+a\cos v(t))\sin u(t)+\epsilon z_1(t),\\a\sin v(t)+\epsilon z_2(t),\end{cases} \tag{34}$$

where $\mathbf{z}(t)=(z(t),z_1(t),z_2(t))$ and $z(t),z_1(t),z_2(t)$ are real continuous functions. Since all the curves of the C_ϵ family are on the torus (31), based on (32), it must be valid

$$\left(c-\sqrt{((c+a\cos v(t))\cos u(t)+\epsilon z(t))^2+((c+a\cos v(t))\sin u(t)+\epsilon z_1(t))^2}\right)^2$$
$$+\left(a\sin v(t)+\epsilon z_2(t)\right)^2=a^2. \tag{35}$$

From here, after necessary calculations, we get

$$z^2(t)+z_1^2(t)+z_2^2(t)=0, \tag{36}$$

i.e. $z(t)=z_1(t)=z_2(t)=0$. □

One can show that this theorem holds for the torus knots alike, so the bent knots do not stay on the torus. Attention should be paid that, after bending knots, a closed curve is still obtained. Therefore, finding an infinitesimal bending field that does not tear closed curves poses an interesting problem in the theory of the infinitesimal bending. In the following theorem we define an infinitesimal bending field that transforms torus knots into knots.

Theorem 5. *The infinitesimal bending field, under which torus knots after bending remain closed, is defined with the equation*

$$\mathbf{z}(t) = \begin{cases} \frac{\sin((q+p)t)}{2(q+p)} + \frac{\sin((q-p)t)}{2(q-p)}, \\ -\frac{\cos((q+p)t)}{2(q+p)} - \frac{\cos((q-p)t)}{2(q-p)}, \\ -\frac{\cos(pt)}{p}. \end{cases} \tag{37}$$

Proof. Since

$$\dot{\mathbf{r}} = \begin{cases} -ap\sin(pt)\cos(qt) - q(c + a\cos(pt))\sin(qt), \\ -ap\sin(pt)\sin(qt) + q(c + a\cos(pt))\cos(qt), \\ ap\cos(pt), \end{cases}$$

and

$$\dot{\mathbf{z}} = \begin{cases} \cos(pt)\cos(qt), \\ \cos(pt)\sin(qt), \\ \sin(pt), \end{cases}$$

it follows that $\dot{\mathbf{r}} \cdot \dot{\mathbf{z}} = 0$, where $\dot{\mathbf{r}}$ and $\dot{\mathbf{z}}$ are derivatives of the torus knot (30) and of the bending field (37), respectively, with respect to the parameter t. It can easily be proven that $\mathbf{z}(0) = \mathbf{z}(2\pi)$, i.e. under infinitesimal bending of the torus knots in field (37) the bent curves are closed. □

5 Torsional energy of torus knots and bent knots

We will consider here influence of infinitesimal bending on torus knots with an aim to visualise changes in shape and torsional energy. In the following figures colours are used to indicate the values of torsional energy at different points of the knots, together with a colour-value scale. In addition the torsional energy is also calculated.

We start from parametric knot representations given in equation (30), then we will apply the bending field given by theorem 5.

The curve is visualized as polygonal line which connect points on the curve. At every such point, as well as, every subdivision point for the purpose of numerically integral calculation we should calculate functions: the curve, the first, the second and the third derivative, torsion, also local values of torsional energy. This is necessary to obtain transformed shape of curve and for aimed curve colouring. We have developed our own software tool using *Microsoft* Visual C++. We have aimed tool for dealing with arbitrary mathematical functions, so tool is developed for manipulating explicitly defined functions. It starts from usual symbolic definitions as a string, then parsing it to obtain an internal, tree like, function form. On that way we can efficiently calculated function many times. Another additional important benefits of the tree like form: make derivatives, combine more functions to obtain a compound function like sub integral function for calculating torsional energy. We do not calculate integral symbolically, instead we are using ability for fast calculation of sub integral function foro numerically calculate integral.

Knot visualization and obtaining 3D model is done by using OpenGL. In the following examples the knot is represented as a tube around a curve.

5.1 Torus knot $T(3,2)$

The first example is torus knot $T(3,2)$. Equation of the corresponding bent knots $T_\epsilon(3,2)$ in the field (37) is

$$\mathbf{r}_\epsilon(t)=\begin{cases}(c+a\cos(3t))\cos(2t)+\frac{\epsilon}{10}(\sin(5t)+5\sin t),\\(c+a\cos(3t))\sin(2t)-\frac{\epsilon}{10}(\cos(5t)-5\cos t),\\a\sin(3t)-\frac{\epsilon}{3}\cos(3t),\end{cases}\quad t\in[0,2\pi). \tag{38}$$

The numerically calculated torsional energy of the torus knot $T(3,2)$ is $E_\tau=13.34818$ and of the bent knot $T_\epsilon(3,2)$ is $E_{\tau_\epsilon}=9.94781$, for $\epsilon=2.0$ $(a=1,\ c=2)$.

Graphical presentation of the torus knot $T(3,2)$ and infinitesimally bent knot $T_\epsilon(3,2)$, for $\epsilon=2.0$, is given in Fig. 2 and Fig. 3.

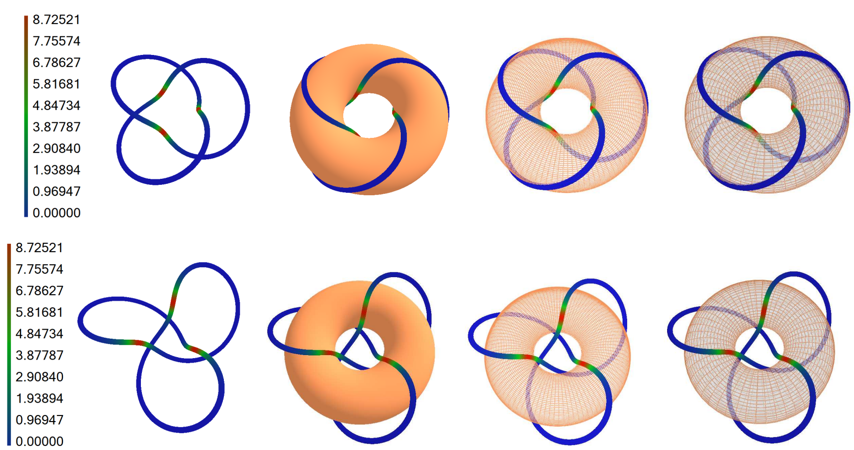

Figure 2: Torus knot $T(3,2)$ (the first row) and infinitesimally bent knot $T_\epsilon(3,2)$ for $\epsilon=2.0$ (the second row)

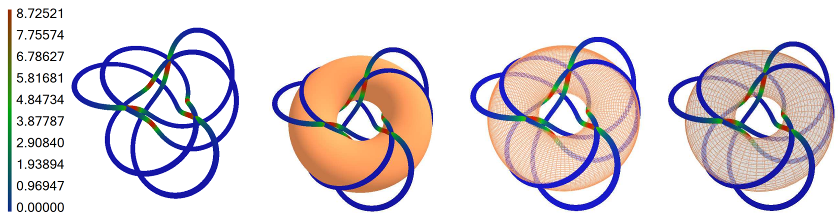

Figure 3: Torus knot $T(3,2)$ with infinitesimally bent knot $T_\epsilon(3,2)$ for $\epsilon = 2.0$.

5.2 Torus knot $T(4,3)$

The second example is torus knot $T(4,3)$. Equation of the corresponding bent knots $T_\epsilon(4,3)$ in the field (37) is

$$\mathbf{r}_\epsilon(t) = \begin{cases} (c + a\cos(4t))\cos(3t) + \frac{\epsilon}{14}(\sin(7t) + 7\sin t), \\ (c + a\cos(4t))\sin(3t) - \frac{\epsilon}{14}(\cos(7t) - 7\cos t), \\ a\sin(4t) - \frac{\epsilon}{4}\cos(4t), \end{cases} \quad t \in [0, 2\pi). \quad (39)$$

The numerically calculated torsional energy of the torus knot $T(4,3)$ is $E_\tau = 21.82216$ and of the bent knot $T_\epsilon(4,3)$ is $E_{\tau_\epsilon} = 18.087391$, for $\epsilon = 2.0$ ($a = 1$, $c = 2$).

Graphical presentation of the torus knot $T(4,3)$ and infinitesimally bent knot $T_\epsilon(4,3)$, for $\epsilon = 2.0$, is given in Fig. 4 and Fig. 5.

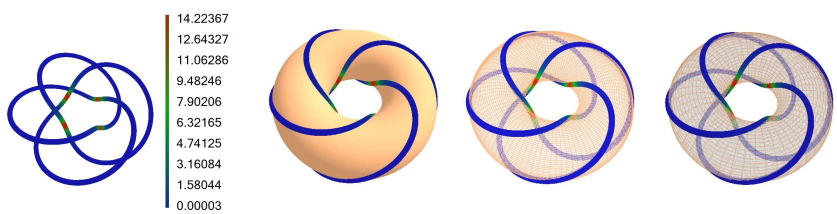

Figure 4: Torus knot $T(4,3)$

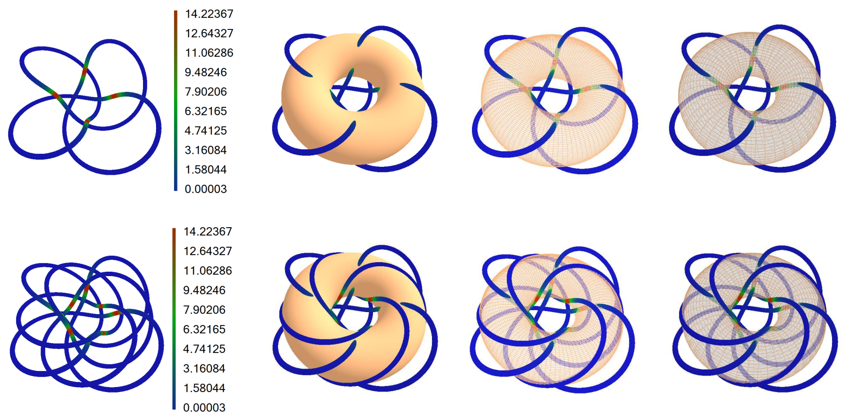

Figure 5: Infinitesimally bent knot $T_\epsilon(4,3)$ for $\epsilon = 2.0$ (the first row) and torus knot $T(4,3)$ with infinitesimally bent knot $T_\epsilon(4,3)$ for $\epsilon = 2.0$ (the second row)

5.3 Torus knot $T(5,2)$

The third example is torus knot $T(5,2)$. Equation of the corresponding bent knots $T_\epsilon(5,2)$ in the field (37) is

$$\mathbf{r}_\epsilon(t) = \begin{cases} (c + a\cos(5t))\cos(2t) + \frac{\epsilon}{14}(\sin(7t) + \frac{7}{3}\sin(3t)), \\ (c + a\cos(5t))\sin(2t) - \frac{\epsilon}{14}(\cos(7t) - \frac{7}{3}\cos(3t)), \\ a\sin(5t) - \frac{\epsilon}{5}\cos(5t), \end{cases} \quad t \in [0, 2\pi). \tag{40}$$

The numerically calculated torsional energy of the torus knot $T(5,2)$ is $E_\tau = 10.52096$ and of the bent knot $T_\epsilon(5,2)$ is $E_{\tau_\epsilon} = 6.78789$, for $\epsilon = 4.0$ ($a = 1$, $c = 2$).

Graphical presentation of the torus knot $T(5,2)$ and infinitesimally bent knot $T_\epsilon(5,2)$, for $\epsilon = 4.0$, is given in Fig. 6.

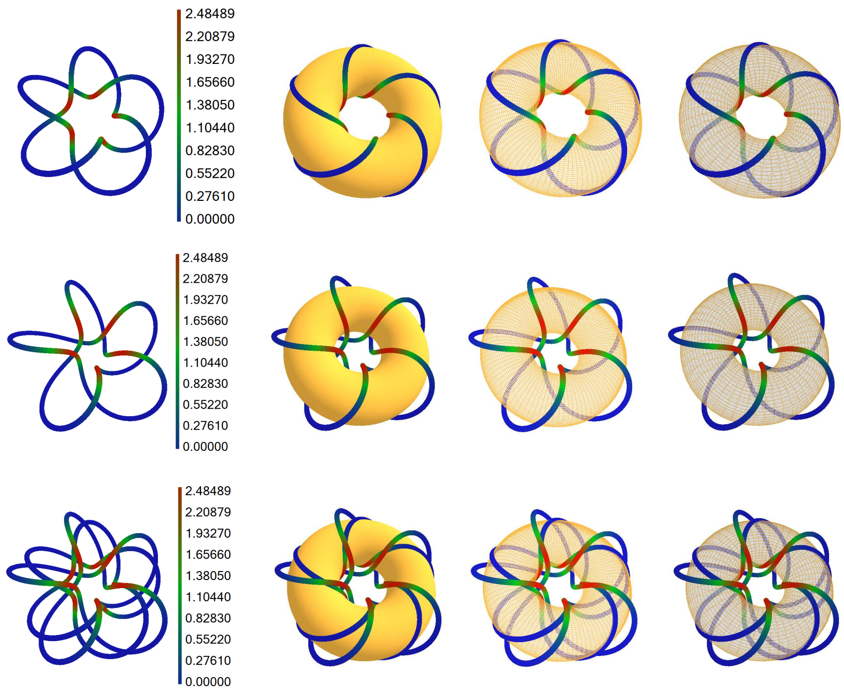

Figure 6: Torus knot $T(5,2)$, infinitesimally bent knot $T_\epsilon(5,2)$ for $\epsilon = 4.0$ and all knots together

6 Conclusion

Having in mind the importance of the torsional and Willmore energy in the elasticity theory, in this artcile we considered the impact that the infinitesimal bending of torus knots makes on their torsional energy.

Considering a biregular curve parametrized by the arc length, we proved that the torsional energy is not stationary under the infinitesimal bending and in the theorem 3 we determined its variation. A corollary to this theorem for closed curves is also given. The variation of torsional energy under infinitesimal bending of torus knots is given with equation (29). We also considered the infinitesimal bending of arbitrary curves on the torus and in the theorem 4 we proved that there is no infinitesimal bending field under which the bent curves stay on the torus. For that reason we determined the bending field (37) that does not tear the torus knots, i.e. closed curves are obtained after bending in that field. Visualization plays an important role in the infinitesimal bending theory, therefore in the section 5 we visualized the results obtained. For the purposes of this research we used independently developed software to observe and determine the impact that the infinitesimal bending makes on torsional energy at every point of the torus knots. The graphs we obtained are presented in the paper. The torus knots in the graphs are coloured according to their torsional energy values at different points. One can clearly see in the graphs that the highest value of the torsional energy (coloured in red) is at the inner facet of the torus. The torsional energy variation was calculated numerically as well, and, considering the examples given in the article, we draw a conclusion that the torsional energy declines when the torus knots are infinitesimally bent in the field (37).

7 Acknowledgement

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