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Residuated skew lattices with modal operator

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Abstract

In this paper, we define modal operators in residuated skew lattices and prove some fundamental properties of monotone modal operators on residuated skew lattices (RSL). We prove that the composition of two modal operators is a modal operator if and only if they commute. We investigate strong modal operators in RSL and get a characterization of them. Deductive systems under a modal operator are investigated.

1 Introduction

Modal operators on Heyting algebras were introduced and studied by Macnab in [11]. In 2006, Harlenderova and Rachunek proved some algebraic properties of modal operators on MV-algebras in [3]. Modal operators on commutative residuated lattices were studied by Kondo in [5]. Skew lattices are a generalization of lattices in which $\vee, \wedge$ are not commutative. A. Borumand Saeid and R. Koohnavard defined RSL as a non-commutative generalization of residuated lattices [2]. In a RSL two different order concepts can be defined: the natural preorder, denoted by $\leq$ and the natural partial order denoted by $\preceq$. Also, they defined some types of filters and skew filters in RSL [6, 7]. The authors defined pseudo RSL as a non-commutative generalization of RSL in which $\vee, \wedge, \odot$ are not commutative [8] and defined (skew) filters in pseudo RSL. Also, they defined state operator on RSL [10].

In this paper, we prove some fundamental results of monotone modal operators on RSL and prove that the composition of two modal operators is a modal

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operator if and only if they commute. Deductive systems under a modal operator are investigated. We show that if D is a deductive system in A, then $f(D^{**}) \subseteq D^{**}$. A normal RSL A is defined and A is normal if and only if ** is a strong modal operator. It is shown that if A is normal RSL, then A^{**} is a RSL.

2 Preliminaries

In this section, we summarize some definitions and results about residuated skew lattices and skew lattices, which will be used in the sequel.

Definition 2.1. *[1] A skew lattice is an algebra $(A, \vee, \wedge)$ of type $(2, 2)$ such that satisfies in the following identities:*
(1) $(x \vee y) \vee z = x \vee (y \vee z)$ *and* $(x \wedge y) \wedge z = x \wedge (y \wedge z)$,
(2) $x \wedge x = x$ *and* $x \vee x = x$,
(3) $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ *and* $(x \wedge y) \vee y = y = (x \vee y) \wedge y$,
The identities found in $(1-3)$ are known as the associative law, the idempotent laws and absorption laws respectively. In view of the associativity (1), we can omit parentheses when no ambiguity arises.

On a given skew lattice A the natural partial order $\leq$ and natural preorder $\preceq$ respectively are defined by $x \leq y$ iff $x \wedge y = x = y \wedge x$ or dually $x \vee y = y = y \vee x$ and $x \preceq y$ if and only if $y \vee x \vee y = y$ or equivalently $x \wedge y \wedge x = x$. Relation $\mathbb{D}$ is defined by $x \,\mathbb{D}\, y$ iff $x \vee y \vee x = x$ and $y \vee x \vee y = y$ or dually, $x \wedge y \wedge x = x$ and $y \wedge x \wedge y = y$. $\mathbb{D}$ is called the natural equivalence and it coincides with Green's relation $\mathbb{D}$ on both semigroups $(A, \wedge)$ and $(A, \vee)$ [4].

Definition 2.2. *[2] A residuated skew lattice is a nonempty set A with operations $\vee, \wedge, \odot$ and hyperoperation $\rightarrow$ and constant element 1 that satisfying the following:*
(1) $(A, \vee, \wedge, 1)$ *is a skew lattice with top 1 (for all $x \in A$, $x \leq 1$),*
(2) $(A, \odot, 1)$ *is a commutative monoid,*
(3) $\odot$ *and* $\rightarrow$ *form an adjoint pair, i.e.* $z \preceq x \rightarrow y$ *iff* $x \odot z \preceq y$, *for all* $x, y, z \in A$.

The relation between the pair of operations $\odot$ and $\rightarrow$ expressed by (3), is a special case of the law of residuation and for every $x, y \in A, x \rightarrow y = \sup\{z \in A | x \odot z \preceq y\}$. Supremum of a set in a preordered set is not a unique element, $x \rightarrow y$ may be a $\mathbb{D}$-class. Two $\mathbb{D}$-classes have $\mathbb{D}$-relationship when all of their members have $\mathbb{D}$-relationship with each other. Relation $\preceq$ between two $\mathbb{D}$-classes is defined member to member (i.e. $B \preceq C$ iff $\forall c \in C, \forall b \in B, b \preceq c$). Also each of the $\vee, \wedge, \odot, \rightarrow$, between two $\mathbb{D}$-classes is defined member to member $(B \rightarrow C = \{b \rightarrow c | b \in B, c \in C\})$ [2].

Definition 2.3. *[2] Let A be a residuated skew lattice. A nonempty subset $D \subseteq A$ is called a deductive system of A, if the following conditions are satisfied:*
(1) $1 \in D$,
(2) *If* $x \in D, x \to y \subseteq D$, *then* $y \in D$.

Lemma 2.1. *[2] If A is a residuated skew lattice and $x, y, z \in A$, then*
(1) $1 \to x = \mathbb{D}_x$ *and* $x \to x = 1$,
(2) $x \odot y \preceq x, y$ *hence* $x \odot y \preceq x \wedge y, y \wedge x,\ y \preceq x \to y$,
(3) $x \odot y \preceq x \to y$,
(4) $x \preceq y$ *iff* $x \to y = 1$ *and* $x \,\mathbb{D}\, y$ *iff* $x \to y = y \to x = 1$,
(5) $x \to 1 = 1$,
(6) $x \odot (x \to y) \preceq y$, $x \preceq (x \to y) \to y$ *and* $((x \to y) \to y) \to y) \,\mathbb{D}\, (x \to y)$,
(7) $x \to y \preceq x \odot z \to y \odot z$,
(8) $x \preceq y$ *implies* $x \odot z \preceq y \odot z$,
(9) $x \to y \preceq (z \to x) \to (z \to y)$,
(10) $x \to y \preceq (y \to z) \to (x \to z)$,
(11) $x \preceq y$ *implies* $z \to x \preceq z \to y$ *and* $y \to z \preceq x \to z$,
(12) $x \odot (y \to z) \preceq y \to (x \odot z) \preceq x \odot y \to x \odot z$,
(13) $(x \to (y \to z)) \,\mathbb{D}\, (x \odot y \to z) \,\mathbb{D}\, (y \to (x \to z))$,
(14) $x_1 \to y_1 \preceq (y_2 \to x_2) \to [(y_1 \to y_2) \to (x_1 \to x_2)]$,
(15) $x \vee y \preceq (x \to y) \to y \wedge (y \to x) \to x \wedge (x \to y) \to y$,
(16) $x \odot (x \to y) \preceq (x \wedge y)$, $x \odot (x \to y) \preceq (y \wedge x)$.

3 Modal RSL

From here until the end of this paper, let A be a RSL. In this section, we define the modal operators on RSL and we investigate their properties.

Definition 3.1. *A mapping $f : A \longrightarrow A$ is called a modal operator on A, if it satisfies conditions: for all $x, y \in A$,*
(1) $x \preceq f(x)$,
(2) $f(f(x)) \,\mathbb{D}\, f(x)$,
(3) $f(x \odot y) \,\mathbb{D}\, f(x) \odot f(y)$.
The pair (A, f) is called a modal residuated skew lattice.

Denote $MOD(A)$ the set of all modal operators on A. A modal operator f is called monotone if it satisfies:

$$If\ x \preceq y,\ \ then\ \ f(x) \preceq f(y).$$

We can show that every modal operator is monotone on A with divisibility $x \wedge y \,\mathbb{D}\, x \odot (x \to y)$. Indeed if $x \preceq y$, then $f(x) \,\mathbb{D}\, f(x \wedge y) \,\mathbb{D}\, f(y \wedge x) \,\mathbb{D}\, f(y \odot (y \to x)) \,\mathbb{D}\, f(y) \odot f(y \to x) \preceq f(y)$.

Example 3.1. *Let* $A = \{0, a, b, n, c, d, m, 1\}$ *be such that* $0 < a, b < n < c, d < m < 1$*,* $a \mathbb{D} b$ *and* $\mathbb{D}_a = \{a, b\}$*.* $\mathcal{A} = (A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ *is a residuated skew lattice with* 0*, with the following operations:*

$\rightarrow$	0	a	b	n	c	d	m	1
0	1	1	1	1	1	1	1	1
a	$\mathbb{D}_a$	1	1	1	1	1	1	1
b	$\mathbb{D}_a$	1	1	1	1	1	1	1
n	0	$\mathbb{D}_a$	$\mathbb{D}_a$	1	1	1	1	1
c	0	$\mathbb{D}_a$	$\mathbb{D}_a$	d	1	d	1	1
d	0	$\mathbb{D}_a$	$\mathbb{D}_a$	c	c	1	1	1
m	0	$\mathbb{D}_a$	$\mathbb{D}_a$	n	c	d	1	1
1	0	$\mathbb{D}_a$	$\mathbb{D}_a$	n	c	d	m	1

$\odot$	0	a	b	n	c	d	m	1
0	0	0	0	0	0	0	0	0
a	0	0	0	a	a	a	a	a
b	0	0	0	b	b	b	b	b
n	0	a	b	n	n	n	n	n
c	0	a	b	n	c	n	c	c
d	0	a	b	n	n	d	d	d
m	0	a	b	n	c	d	m	m
1	0	a	b	n	c	d	m	1

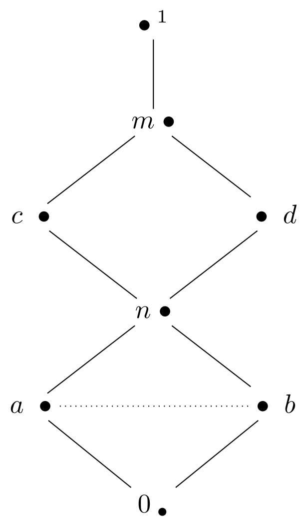
$\vee$	0	a	b	n	c	d	m	1
0	0	a	b	n	c	d	m	1
a	a	a	b	n	c	d	m	1
b	b	a	b	n	c	d	m	1
n	n	n	n	n	c	d	m	1
c	c	c	c	c	c	m	m	1
d	d	d	d	d	m	d	m	1
m	m	m	m	m	m	m	m	1
1	1	1	1	1	1	1	1	1

$\wedge$	0	a	b	n	c	d	m	1
0	0	0	0	0	0	0	0	0
a	0	a	a	a	a	a	a	a
b	0	b	b	b	b	b	b	b
n	0	a	b	n	n	n	n	n
c	0	a	b	n	c	n	c	c
d	0	a	b	n	n	d	d	d
m	0	a	b	n	c	d	m	m
1	0	a	b	n	c	d	m	1

Define the maps $f_i : A \longrightarrow A$*,* $i = 1, 2, 3, 4, 5, 6, 7, 8$ *given in the table below:*

x	0	a	b	n	c	d	m	1
f_1	1	1	1	1	1	1	1	1
f_2	0	a	b	n	c	d	m	1
f_3	0	b	a	n	c	d	m	1
f_4	m	m	m	m	m	m	m	1
f_5	n	n	n	n	c	d	m	1
f_6	0	a	a	m	m	m	m	1
f_7	0	b	b	m	m	m	m	1
f_8	0	a	b	n	c	d	1	1

Then $MOD(\mathcal{A}) = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$*. In this example,* f_i*,* $i = 1, 2, 3, 4, 5, 6, 7, 8$ *is a monotone.*



Example 3.2. *Let* $A = \{0, a, b, n, c, d, m, 1\}$ *be such that* $0 < a, b < n < c, d < m < 1$*,* $c \, \mathbb{D} \, d$ *and* $\mathbb{D}_c = \{c, d\}$*.* $A = (A, \vee, \wedge, \odot, \rightarrow, 0, 1)$ *is a residuated skew lattice with* 0*, with the following operations:*

$\rightarrow$	0	a	b	n	c	d	m	1
0	1	1	1	1	1	1	1	1
a	b	1	b	1	1	1	1	1
b	a	a	1	1	1	1	1	1
n	0	a	b	1	1	1	1	1
c	0	a	b	$\mathbb{D}_c$	1	1	1	1
d	0	a	b	$\mathbb{D}_c$	1	1	1	1
m	0	a	b	n	$\mathbb{D}_c$	$\mathbb{D}_c$	1	1
1	0	a	b	n	$\mathbb{D}_c$	$\mathbb{D}_c$	m	1

$\odot$	0	a	b	n	c	d	m	1
0	0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a	a
b	0	0	b	b	b	b	b	b
n	0	a	b	n	n	n	n	n
c	0	a	b	n	n	n	c	c
d	0	a	b	n	n	n	d	d
m	0	a	b	n	c	d	m	m
1	0	a	b	n	c	d	m	1

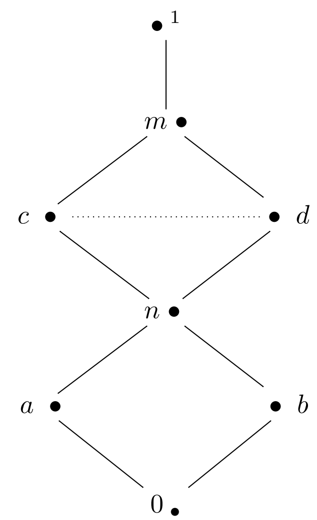
$\vee$	0	a	b	n	c	d	m	1
0	0	a	b	n	c	d	m	1
a	a	a	n	n	c	d	m	1
b	b	n	b	n	c	d	m	1
n	n	n	n	n	c	d	m	1
c	c	c	c	c	c	d	m	1
d	d	d	d	d	c	d	m	1
m	m	m	m	m	m	m	m	1
1	1	1	1	1	1	1	1	1

$\wedge$	0	a	b	n	c	d	m	1
0	0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a	a
b	0	0	b	b	b	b	b	b
n	0	a	b	n	n	n	n	n
c	0	a	b	n	c	c	c	c
d	0	a	b	n	d	d	d	d
m	0	a	b	n	c	d	m	m
1	0	a	b	n	c	d	m	1

Define the maps $f_i : A \longrightarrow A$*,* $i = 1, 2, 3, 4, 5, 6$ *given in the table below:*

x	*0*	a	b	n	c	d	m	*1*
f_1	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*
f_2	*0*	a	b	n	c	d	m	*1*
f_3	*0*	a	b	n	d	c	m	*1*
f_4	m	m	m	m	m	m	m	*1*
f_5	n	n	n	n	c	d	m	*1*
f_6	*0*	a	b	n	c	d	*1*	*1*

Then $MOD(A) = \{f_1, f_2, f_3, f_4, f_5, f_6\}$.



$1_A, Id_A : A \longrightarrow A$, defined by $1_A(x) = 1$ and $Id_A(x) = x$, for all $x \in A$. It is clear that $1_A, Id_A \in MOD(A)$.

A binary operator $\oplus$ is defined by $x \oplus y = (x^* \odot y^*)^*$ where $x^* = x \to 0$.

Proposition 3.1. *Let f be any monotone modal operator on A. Then*
(1) $f(x \to y) \preceq f(x) \to f(y) \mathbb{D} f(f(x) \to f(y)) \mathbb{D} x \to f(y) \mathbb{D} f(x \to f(y))$,
(2) $f(x) \preceq (x \to f(0)) \to f(0)$,
(3) $f(x) \odot x^* \preceq f(0)$,
(4) $f(x) \preceq f(x^{**}) \preceq x \oplus f(0)$,
(5) $f(x \vee y) \mathbb{D} f(x \vee f(y)) = f(f(x) \vee f(y))$,
(6) $f(x) \wedge f(y) \mathbb{D} f(f(x) \wedge f(y))$.

Proof. We only show some of cases $(1), (2)$ and (6) for the sake of simplicity. (1) It follows from $x \odot (x \to y) \preceq y$ that $f(x \odot (x \to y)) \preceq f(y)$ and thus $f(x) \odot f(x \to y) \preceq f(y)$. This means that $f(x \to y) \preceq f(x) \to f(y)$. Moreover,

we have $f(x) \to f(y) \preceq x \to f(y)$ by $x \preceq f(x)$. This yields that $f(x) \to f(y) \preceq x \to f(y) \preceq f(x \to f(y) \preceq f(x) \to f(f(y)) \mathbb{D} f(x) \to f(y)$. Thus, $f(x) \to f(y) \preceq f(f(x) \to f(y)) \preceq f(f(x)) \to f(f(y)) \mathbb{D} f(x) \to f(y)$. We get that $f(x \to y) \preceq f(x) \to f(y) \mathbb{D} f(f(x) \to f(y)) \mathbb{D} x \to f(y) \mathbb{D} f(x \to f(y))$.
(2) We have $f(x) \to f(0) \mathbb{D} x \to f(0)$ by (1) and $f(x) \odot (f(x) \to f(0)) \preceq f(0)$, thus $f(x) \preceq (f(x) \to f(0)) \to f(0) \mathbb{D} (x \to f(0))$.
(6) Since $f(x) \wedge f(y) \preceq f(x), f(y)$, it is obvious that $f(f(x) \wedge f(y)) \preceq f(f(x))$, $f(f(y))$ and thus $f(f(x) \wedge f(y)) \preceq f(x) \wedge f(y)$.
Conversely, since f is the monotone modal operator, it is clear that $f(x) \wedge f(y) \preceq f(f(x) \wedge f(y))$. Thus we have $f(x) \wedge f(y) \mathbb{D} f(f(x) \wedge f(y))$. □

Proposition 3.2. *Let f be a monotone modal operator on A. If $x \oplus f(0) \mathbb{D} f(x \oplus 0)$, then $f(x) \oplus f(0) \mathbb{D} x \oplus f(0)$, for all $x \in A$.*

Proof. It follows from the proposition above that $f(x) \preceq x \oplus f(0)$ and thus $f(x) \oplus f(0) \preceq x \oplus f(0) \oplus f(0)$. Since $f(0) \oplus f(0) \mathbb{D} f(f(0) \oplus 0) \mathbb{D} f(0 \oplus f(0)) \mathbb{D} f(f(0 \oplus 0)) \mathbb{D} f(0 \oplus 0) \mathbb{D} f(0)$ by assumption, we have $f(x) \oplus f(0) \preceq x \oplus f(0)$.
Conversely, it is obvious that $x \oplus f(0) \preceq f(x) \oplus f(0)$. Thus we get $f(x) \oplus f(0) \mathbb{D} x \oplus f(0)$. □

A monotone modal operator f on A is called strong if $f(x \oplus y) \mathbb{D} f(x \oplus f(y))$, for all $x, y \in A$. Denote $MOD^s(A)$ the set of all strong modal operators on A. In Example 3.1 , $MOD^s(A) = \{f_1, f_2, f_4, f_5\}$ is a set of strong modal operators. We can get a characterization of strong modal operators.

Proposition 3.3. *Let f be a monotone modal operator on A. Then it is strong if and only if it satisfies the condition $x \oplus f(0) \mathbb{D} f(x \oplus 0)$, for all $x \in A$.*

Proof. We assume that f is a strong modal operator. It is obvious from Proposition 3.1(4) that $f(x \oplus 0) \mathbb{D} f(x^{**}) \preceq x \oplus f(0)$ and $x \oplus f(0) \preceq f(x \oplus f(0)) \mathbb{D} f(x \oplus 0)$. Thus $x \oplus f(0) \mathbb{D} f(x \oplus 0)$.
Conversely, suppose that $x \oplus f(0) \mathbb{D} f(x \oplus 0)$. In general, we have $x \oplus y \mathbb{D} x \oplus y \oplus 0$, for all $x, y \in A$. This yields that $f(x \oplus fy) \mathbb{D} f((x \oplus f(y)) \oplus 0) \mathbb{D} x \oplus f(y) \oplus f(0) \mathbb{D} x \oplus y \oplus f(0) \mathbb{D} f(x \oplus y \oplus 0) \mathbb{D} f(x \oplus y)$. □

Proposition 3.4. *Let $f : A \longrightarrow A$ a mapping. Then f is a monotone modal operator on A if and only if it satisfies the following two conditions:*
(1) $f(x) \to f(y) \mathbb{D} x \to f(y)$,
(2) $f(x \odot y) \preceq f(x) \odot f(y)$.

Proof. It is easy to prove that if f is a monotone modal operator, then it satisfies the conditions. We only show the converse, that is, if a mapping f satisfies the conditions, then it is the monotone modal operator. Since $1 = f(x) \to f(x) \mathbb{D} x \to f(x)$ by (1), we have $x \preceq f(x)$. Next if $x \preceq y$, since $y \preceq f(y)$ which is just proved above, then $x \preceq f(y)$ and thus $1 = x \to f(y) \mathbb{D} f(x) \to f(y)$. This means that $f(x) \preceq f(y)$ and that f is an order preserving mapping. The fact that $1 = f(x) \to f(x) \mathbb{D} f(f(x)) \to f(x)$ by (1) implies $f(f(x)) \mathbb{D} f(x)$. Finally, since $x \odot y \preceq f(x \odot y)$ and thus $y \preceq x \to f(x \odot y) \mathbb{D} f(x) \to f(x \odot y)$, we have $f(x) \preceq y \to f(x \odot y) \mathbb{D} f(y) \to f(x \odot y)$ and $f(x) \odot f(y) \preceq f(x \odot y)$. It follows from (2) that $f(x \odot y) \mathbb{D} f(x) \odot f(y)$. □

Proposition 3.5. *Let f a monotone modal operator on A. Then we have $f(0) = 0$ if and only if $f(x^*) \mathbb{D} x^*$, for all $x \in A$.*

Proof. Suppose that $f(0) = 0$. It follows from Proposition 3.1(2) that $f(x) \preceq (x \to f(0)) \to f(0) \mathbb{D} (x \to 0) \to 0 \mathbb{D} x^{**}$ and hence that $f(x) \preceq x^{**}$, for all $x \in A$. This means that $f(x^*) \preceq (x^*)^{**} \mathbb{D} x^*$. On the other hand, it is clear that $x^* \preceq f(x^*)$. Thus we have $f(x^*) \mathbb{D} x^*$, for all $x \in A$.
Conversely, if we assume that $f(x^*) \mathbb{D} x^*$, then $f(0) \mathbb{D} f(1^*) = 1^* = 0$. From the above, in case of RSL with meeting the condition $x^{**} \mathbb{D} x$. □

Proposition 3.6. *Let $x^{**} \mathbb{D} x$, for all $x \in A$ and f a mapping on A with $f(0) = 0$. Then f is a monotone modal operator if and only if it is an identity map.*

We get another characterization of $f(0) = 0$.

Proposition 3.7. *Let $x^{**} \mathbb{D} x$, for all $x \in A$ and f a monotone modal operator on A. Then $f(0) = 0$ if and only if $x \to y^* \mathbb{D} f(x) \to (f(y))^*$.*

Proof. By above corollary, we have $x \to y^* \mathbb{D} \to (f(y))^*$. On the other hand, since $x \preceq f(x)$ and $y \preceq f(y)$, $f(x) \to (f(y))^* \preceq x \to y^*$. This implies that $x \to y^* \mathbb{D} f(x \to y^*) \preceq f(x) \to (f(y))^* \preceq x \to y^*$ and thus $x \to y^* \mathbb{D} f(x \to y^*) \mathbb{D} f(x) \to (f(y))^*$.
Conversely, suppose that $x \to y^* \mathbb{D} f(x) \to (f(y))^*$ and take $y \mathbb{D} x^*$. Then we have $1 = x \to x^{**} \mathbb{D} f(x) \to (f(x^*))^*$. This means that $f(x) \preceq (f(x^*))^*$. Thus if we put $x = 0$, then $f(0) \preceq (f(0^*))^* \mathbb{D} (f(1))^* = 1^* = 0$. □

Proposition 3.8. *Let f, g be monotone modal operators on A and $f \preceq g$. Then $gf \mathbb{D} g$.*

Proof. Assume that f, g are monotone modal operators on A such that $f \preceq g$ and $x \in A$. We have $g(x) \preceq g(f(x)) \mathbb{D} gf(x)$ and so $g \preceq gf$. Moreover $g(f(x)) \preceq g(g(x)) \mathbb{D} g(x)$. Thus $gf \mathbb{D} g$. □

The following example show that the condition $f \preceq g$ from Proposition 3.8 is necessary.

Example 3.3. *Consider the modal operators f_3 and f_4 given in Example 3.2. Indeed, $f_4 \npreceq f_3$, since $f_4(0) \mathbb{D}\, m \npreceq f_3(0) = 0$, we have*

$$(f_3(f_4))(0) \,\mathbb{D}\, f_3(f_4(0)) \,\mathbb{D}\, f_3(m) \,\mathbb{D}\, m \npreceq f_3(0) = 0.$$

Proposition 3.9. *Let f, g be monotone modal operators on A and $f \preceq g$. The following are equivalent:*
(1) $fg \,\mathbb{D}\, gf$,
(2) $fg, gf \in MOD(A)$,
(3) $fgfg \,\mathbb{D}\, fg$ and $gfgf \,\mathbb{D}\, gf$.

Proof. Assume that f, g are monotone modal operators on A and $x, y \in A$.
(1) $\Rightarrow$ (2) Suppose that $fg \,\mathbb{D}\, gf$. We have by modal operator:
($\bullet$) $x \preceq g(x) \preceq f(g(x))$,
($\bullet\bullet$) $f(g(f(g(x)))) \,\mathbb{D}\, f(f(g(g(x)))) \,\mathbb{D}\, f(f(g(x))) \,\mathbb{D}\, f(g(x))$,
($\bullet\bullet\bullet$) $fg(x \to y) \,\mathbb{D}\, f(g(x \to y)) \preceq f(g(x) \to g(y)) \preceq f(g(x)) \to f(g(y))$.
Thus, $fg \in MOD(A)$. By a similar argument we get $gf \in MOD(A)$.
(2) $\Rightarrow$ (3) Assume that $fg, gf \in MOD(A)$. Since $fg \preceq fgfg$, we have $fgfg \,\mathbb{D}\, fgfgfg \,\mathbb{D}\, fg$, by ($\bullet\bullet$) and Proposition 3.8. By a similar argument we obtain $gfgf \,\mathbb{D}\, gf$.
(3) $\Rightarrow$ (1) Assume that $fgfg \,\mathbb{D}\, fg$ and $gfgf \,\mathbb{D}\, gf$. Then we have: for all $x \in A$, $f(g(x)) \preceq f(g(f(x))) \preceq g(f(g(f(x)))) \,\mathbb{D}\, g(f(x))$. Similarly, $g(f(x)) \preceq f(g(x))$. Thus $fg \,\mathbb{D}\, gf$. □

4 Deductive systems under a modal operator

In this section, we define the modal upper set of two elements of a modal RSL and we investigate some properties of this set.
Let (A, f) be a modal RSL and $x, y \in A$. We define the notion of a modal upper set $mA(x, y)$ as follows: $mA(x, y) = \{z \in A | x \to (y \to f(z)) = 1\}$. Obviously, it is a non-empty set, since $1 \in mA(x, y)$.

Remark 4.1. *In general, the upper set $A(x, y) = \{z \in A | x \to (y \to z) = 1\}$ is not equal to modal upper set $mA(x, y)$. Indeed, in Example 3.2, if we take $f := f_4$, then $mA(1, a) = \{0, a, b, n, c, d, m, 1\} \neq \{a, n, c, d, m, 1\} = A(1, a)$.*

Proposition 4.1. *Let (A, f) be a monotone modal RSL. Then for all $x, y \in A$:*
(1) if $f(y) \,\mathbb{D}\, y$, then $A(1, y) \subseteq mA(1, y)$,
(2) $mA(x, 1) \subseteq mA(x, y)$,
(3) $mA(f(x), 1) \subseteq mA(f(x), y)$,

(4) *if* $D \in DS(A)$*, then* $f(mA(x,y)) \subseteq D$*, for all* $x, y \in D$*,*
(5) *if* $mA(f(x),1) \in DS(A)$ *and* $y \in mA(f(x),1)$*, then* $mA(f(x),y) \subseteq mA(f(x),1)$*, and so* $mA(f(x),y) = mA(f(x),1)$*,*
(6) $A(x,y) \subseteq mA(x,y)$*, for all* $x, y \in A$*.*

Proof. (1) Let $y \in A$ and $z \in A(1,y)$. Then $1 \to (y \to z) \mathbb{D} y \to z = 1$, and so $y \preceq z$. Therefore $f(y) \preceq f(z)$. Hence $1 = f(y) \to f(z) \mathbb{D} 1 \to (f(y) \to f(z)) \mathbb{D} 1 \to (y \to f(z))$. Thus, $z \in mA(1,y)$.
(2) Let $z \in mA(x,1)$. Then $1 = x \to (1 \to f(z)) \mathbb{D} x \to f(z)$. Now, we get $x \to (y \to f(z)) \mathbb{D} y(x \to f(z)) = 1$. Therefore, $z \in mA(x,y)$.
(3) Let $z \in mA(f(x),1)$. Then $f(x) \to (1 \to f(z)) = 1$, i.e. $f(x) \to f(z) = 1$. Hence $f(x) \to (y \to f(z)) \mathbb{D} y \to (f(x) \to f(z)) \mathbb{D} y \to 1 = 1$, i.e. $z \in mA(f(x),y)$.
(4) Assume that $D \in DS(A)$ and $z \in f(mA(x,y))$. Then there exists $a \in mA(x,y)$ such that $z \mathbb{D} f(a)$. Hence $x \to (y \to f(a)) = 1 \in D$. Since D is a deductive system and $x, y \in D$, we get $y \to f(a) \in D$, and so $z \mathbb{D} f(a) \in D$. Therefore, $f(mA(x,y)) \subseteq D$.
(5) Since $f(x) \to (1 \to f(x)) \mathbb{D} f(x) \to (1 \to f(f(x))) = 1$, we get $f(x) \in mA(f(x),1)$. Now, let $y \in mA(f(x),1)$. Since $1 = f(x) \to f(y) \mathbb{D} f(x) \to (1 \to f(y)) \in mA(f(x),1)$, we have $f(y) \in mA(f(x),1)$. Let $z \in mA(f(x),y)$. Then by Lemma 2.1, $1 = f(x) \to (y \to f(z)) \mathbb{D} y \to (f(x) \to f(z))$. Now, by Definition 3.1 and f is a monotone, we get $1 = f(1) \mathbb{D} f(y \to (f(x) \to f(z)))(y) \to f(f(x) \to f(z))(y) \to (f(f(x)) \to f(f(z))) \mathbb{D} f(y) \to (f(x) \to f(z)) \in mA(f(x),1)$. Now, since $mA(f(x),1)$ is a deductive system and $f(x), f(y) \in mA(f(x),1)$, we get $f(z) \in mA(f(x),1)$, and so $1 = f(x) \to (1 \to f(f(z))) \mathbb{D} f(x) \to (1 \to f(z))$. Hence $z \in mA(f(x),1)$. Thus, $mA(f(x),y) \subseteq mA(f(x),1)$. By (3), we get $mA(f(x),y) = mA(f(x),1)$.
(6) Let $z \in A(x,y)$. Then $x \to (y \to z) = 1$. We have $z \preceq f(z)$. Applying Lemma 2.1, we get $x \to (y \to z) \preceq x \to (y \to f(z))$, and so $x \to (y \to f(z)) = 1$. Thus, $z \in mA(x,y)$. □

Proposition 4.2. *Let* (A, f) *be a monotone modal RSL and* $D \in DS(A)$*. Then* $f(D) = \bigcup_{x,y \in D} f(mA(f(x),y))$*.*

Proof. Assume that $D \in DS(A)$ and consider $f(z)$, for $z \in D$. Since $f(z) \to (1 \to f(z)) \mathbb{D} f(z) \to (1 \to f(f(z))) = 1$, by Definition 3.1 and Lemma 2.1, we have $f(z) \in mA(f(z),1)$. Now, by Proposition 4.1, we have $f(z) \in mA(f(z),1) \subseteq mA(f(z),y)$. Thus, $f(z) \mathbb{D} f(f(z)) \in f(mA(f(z),1)) \subseteq f(mA(f(z),y))$. Therefore, $f(D) \subseteq f(mA(f(z),y)) \subseteq \bigcup_{y \in D} f(mA(f(z),y))$. On the other hand, by Proposition 4.1, $f(mA(x,y)) \subseteq D$, for all $x, y \in D$. Thus, $f(mA(f(x),y)) \subseteq f(D)$, for all $x, y \in D$. Therefore, $\bigcup_{x,y \in D} f(mA(f(x), y)) \subseteq f(D)$. □

The following examples shows that the condition $D \in DS(A)$ is necessary, in Proposition 4.2.

Example 4.1. *In Example 3.2, taking $D = 0$, and f_3 is a monotone modal operator on A, and so $f_3(0) = \{0\} \neq \bigcup_{x,y \in \{0\}} f(mA(f(x), y)) = mA(f(0), 0) = \{0, a, b, n, c, d, m, 1\}$.*

Proposition 4.3. *Let (A, f) be a modal RSL and $D \subseteq A$ containing 1. $f(D) \in DS(A)$ if and only if $x \preceq y \to z$ implies $z \in f(D)$, for all $x, y \in f(D)$.*

Proof. Let $f(D) \in DS(A)$ and $x \preceq y \to z$, for all $x, y \in f(D)$. Since $x, y \in f(D)$ and $f(D)$ is a deductive system, we have $y \to z \in f(D)$, and so $z \in f(D)$. Conversely, $1 \in f(D)$, since $1 \in D$. Let $x, x \to y \in f(D)$. Since $x \to y \preceq x \to y$ we can see that by hypothesis $y \in f(D)$. $\square$

Proposition 4.4. *Let (A, f) be a modal RSL and $D \subseteq A$ containing 1. Then $f(D) \in DS(A)$ if and only if $x \in f(D), y \in X \setminus f(D)$ imply $x \to y \in X \setminus f(D)$.*

Proof. Assume that $f(D) \in DS(A)$ and let $x, y \in A$ be such that $x \in f(D)$ and $y \in X \setminus f(D)$. If $x \to y \notin X \setminus f(D)$. Then $x \to y \in f(D)$, i.e. $y \in f(D)$, which is a contradiction. Thus $x \to y \in X \setminus f(D)$.
Conversely, $1 \in D$ by hypothesis. Let $x, x \to y \in f(D)$. If $y \notin f(D)$, then $x \to y \in X \setminus f(D)$ by assumption. This is a contradiction and so $y \in f(D)$. Hence there exists $z \in D$ such that $y \mathbb{D} f(z)$. Thus, $f(y) \mathbb{D} f(f(z)) \mathbb{D} f(z) \mathbb{D} y \in f(D)$. By a similar argument $xy \in X \setminus f(D)$. $\square$

For any non-empty subset B of A, we define a subset B^{**} as follows: $B^{**} := \{x \in A | x \to a \in B,\ for\ some\ \ a \in B\}$.

Proposition 4.5. *Let (A, f) be a monotone modal RSL and $D \in DS(A)$. Then $f(D^{**}) \subseteq D^{**}$.*

Proof. Let $D \in DS(A)$ and $x \in D^{**}$. There exists $a \in D$ such that $x \to a \in D$. It follows that $x \to a \preceq f(x \to a) \preceq f(x) \to f(a)$, hence $f(x) \to f(a) \in D$. Since $a \preceq f(a)$, then $f(a) \in D$, so $f(x) \in D^{**}$. Thus $f(D^{**}) \subseteq D^{**}$. $\square$

5 Normal RSL

A is called normal if $(x \odot y)^{**} \mathbb{D} x^{**} \odot y^{**}$. We note that, for A, it is normal if and only if $(x \oplus y)^{*} \mathbb{D} x^{*} \odot y^{*}$ and $(x \odot y)^{*} \mathbb{D} x^{*} \oplus y^{*}$. In Example 3.2, A is normal.

In case of normal RSL, we show that the double negation operator $f_N(x) = x^{**}$ is a strong modal operator.

Proposition 5.1. *A is normal if and only if $f_N(x) = x^{**}$ is a strong modal operator.*

Proof. Suppose that A is a normal RSL. It is easy to verify that
(1) $x \preceq x^{**}$,
(2) If $x \preceq y$ then $x^{**} \preceq y^{**}$,
(3) $x^{****} \mathbb{D}\, x^{**}$,
(4) $(x \odot y)^{**} \mathbb{D}\, x^{**} \odot y^{**}$.
This means that the double negation operator $f_N(x) = x^{**}$ is the monotone modal operator. Moreover, since $x \oplus 0^{**} \mathbb{D}\, x \oplus 0 \,\mathbb{D}\, x^{**} \mathbb{D}\, (x^{**})^{**}$, it follows that $f_N(x) = x^{**}$ is the strong modal operator.
Conversely, if the double negation $f_N(x) = x^{**}$ is a strong modal operator, then it is clear that $(x \odot y)^{**} \mathbb{D}\, x^{**} \odot y^{**}$. Therefore A is normal. □

Corollary 5.1. *If A is normal, then A^{**} is a RSL.*

Let $I(A)$ be the set of all idempotent elements of A with respect to $\odot$, that is $I(A) = \{a | a \odot a \,\mathbb{D}\, a\}$. It is familiar that if $a \in I(A)$, then $a \wedge x \,\mathbb{D}\, a \odot x$ for all $x \in A$, in case of RSL with divisibility. In case of normal RSL, we can characterize the set $I(A)$ of idempotent.

Proposition 5.2. *Let A be a normal RSL. Then $a^* \in I(A)$ if and only if $a \oplus a \,\mathbb{D}\, a^{**}$.*

Proof. If $a^* \in I(A)$, then we have $a \oplus a \,\mathbb{D}\, (a^* \odot a^*)^* \,\mathbb{D}\, (a^*)^* \,\mathbb{D}\, a^{**}$.
Conversely, if $a^{**} \,\mathbb{D}\, a \oplus a$, then $a^* \,\mathbb{D}\, a^{***} \,\mathbb{D}\, (a \oplus a)^* \,\mathbb{D}\, a^* \odot a^*$. Thus $a^* \in I(A)$. □

6 CONCLUSION

Modal operators in RSL were defined. It was investigated if f, g be monotone modal operators and $f \preceq g$, then $gf \,\mathbb{D}\, g$ and with a example was shown that condition $f \preceq g$ is necessary. We showed that the double negation operator $f_N(x) = x^{**}$ is a strong modal operator. It was shown that if A is a normal RSL, then A^{**} is a RSL.

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