DOI: 10.2478/auom-2023-0007
An. Şt. Univ. Ovidius Constanţa **Vol. 31(1),2023, 143–166**

Weighted pseudo almost automorphic functions with applications to impulsive fractional integro-differential equation

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Abstract

This paper's main motivation is to study the notion of weighted pseudo almost automorphic ($\mathcal{WPAA}$) functions and establish the existence results of piecewise continuous mild solution of fractional order integro-differential equation with instantaneous impulses. The usual $\mathcal{WPAA}$ functions may not work since the solution of impulsive differential equations may not be continuous. Thus in order to give a broader spectrum, we introduce this concept. We establish main results by using the Banach contraction mapping principle and Sadovskii's fixed point theorem. An example is shown to exhibit our analytic findings.

1 Introduction

Many real-world problems can be modeled more precisely through the formulation of fractional derivatives. Also this theory can be applied in the field of fluid flow, vis-co-elasticity, rheology, electrical circuits, fractional multipoles, optics, electroanalytical chemistry, diffusive transport, neuron modeling and so on. Fractional differential equations have seen tremendous growth in the past few decades, see the articles and monographs [6, 7, 10, 12, 13, 15, 17–19, 21, 22, 24] and the references therein.

Key Words: Integro-differential equation, Fractional order differential equation, Impulsive conditions, Weighted pseudo almost automorphic function.
2010 Mathematics Subject Classification: Primary 26A33; Secondary 34A37, 47G20.
Received: 29.01.2022
Accepted: 09.08.2022

The study of impulsive integro-differential equations enlarged rapidly over the recent years and it has several applications in mathematical models for example, chemical technology, population dynamics, electrical engineering, medicine, physics, ecology and economics, biology and so on. The authors Milman and Myshkis [20] were first introduced the concept of impulsive differential equations in the year 1960. Followed by Milman and Myshkis work, numerous monographs and articles are published by more number of authors [1, 2, 11, 13, 14, 17, 18, 25].

Bohr developed the concept of almost periodic functions in the year 1924-25 which are prompted to various fundamental generalization of this theory. One crucial extension is the notion of almost automorphic ($\mathcal{AA}$) function introduced by Bochner [4]. This notion is further generalized to various other concepts out of which one important generalization is the notion of $\mathcal{WPAA}$ functions introduced by Blot et al. [3].

Mahto and Abbas in [17] have proved the existence results of $\mathcal{AA}$ solution to fractional order differential equations with impulsive condition. The existence results are established using the fixed point theorems due to Sadovskii and Schaefer. Cao et al. [5] have investigated existence of asymptotically $\mathcal{AA}$ mild solutions to the semilinear nonautonomous evolution equation. The result is based on the nonlinearity term does not have to meet a locally Lipchitz condition. Also the main approach is based on well known Krasnoselskii's fixed point theorem. Recently, Wang et al. [26] studied the notion of pseudo $\mathcal{AA}$ functions on changeable periodic time scales with semilinear dynamic equations as an application. The main finding is derived by utilizing the Banach contraction principle. In [25], Wang and Agarwal established the exponential stability and existence and uniqueness result of piecewise $\mathcal{WPAA}$ mild solution to ∇-dynamic equations with impulses. The main finding is derived by utilizing the contraction principle and Gronwall-Bellman inequality. Moreover, for fractional integro-differential equations with impulsive effects that is (1.1), the study of asymptotic behaviour of the solutions is rare; particularly for the $\mathcal{WPAA}$ of (1.1), it is an untreated topic and this is the main focus of this article. Motivated by the works [5, 17, 25–27], our main aim is to analyze properties and the existence of piecewise $\mathcal{WPAA}$ solution for the fractional impulsive integro-differential equation given below

$$\begin{cases} D_t^{\alpha} z(t) = Az(t) + D_t^{\alpha-1} G_1(t, z(t), Kz(t)), \quad t \neq t_j, \\ Kz(t) = \displaystyle\int_{-\infty}^{t} k_1(t-s) h_1(s, z(s)) ds, \quad t \in \mathcal{R} \\ \triangle z(t_j) = I_j(z(t_j)), \quad j = 1, 2, \ldots, \alpha \in (1,2). \end{cases} \tag{1.1}$$

Here $A : D(A) \subseteq Y \to Y$ is a densely described and linear close form of sectorial operator on a complex Banach space Y. The fractional derivative D_t^{α}

is recognized by Caputo's sense. $G_1 : \mathcal{R} \times Y \times Y \to Y$ and $h_1 : \mathcal{R} \times Y \to Y$ are $\mathcal{WPAA}$ function in t satisfying suitable conditions. $I_j : Y \to Y$,$\Delta z(t)|_{t=t_j} = I_j(z(t_j)) = z(t_j^+) - z(t_j^-), 0 = t_0 < t_1 < ... < t_n < ...$, here left and right limits of $z(t)$ at $t = t_j$ are denoted by $z(t_j^-)$ and $z(t_j^+)$ respectively.

The highlights of this research work is summarized in brief as follows:

(i) We investigate a new composition theorem of $\mathcal{WPAA}$ functions due to the fact that the forced term G_1 does not always meet only the Lipschitz continuity.

(ii) We establish the existence proof of the model (1.1) when the forced term G_1 does not always meet only the Lipschitz continuity for the first time in the literature.

(iii) Depends on the Sadovskii's fixed point technique, we analyze the existence proof for the addressed model (1.1).

(iv) Depends on the Banach contraction principle, we establish the existence and uniqueness proof for the addressed model (1.1).

The remaining scheme of this work is structured as follows. The next section is devoted to define some terminologies, definitions, basic properties, assumptions and previous results of $\mathcal{WPAA}$ functions. In third Section, we first prove a new composition theorem of $\mathcal{WPAA}$ functions and other auxiliary results which are required to demonstrate the existence results. Fourth Section is dedicated to derive the existence and uniqueness of $\mathcal{WPAA}$ mild solutions to the problem (1.1). Fifth section is devoted to present an example to demonstrate our analytical findings.

2 Preliminaries

Notations, basic definitions and lemmas which are stated in this part will be employed extensively in this article.

2.1 Definitions and terminology

In this subsection, we first describe the fundamental definitions.

Consider $(Y, \|\cdot\|_Y)$ as a Banach space. The notation $C(\mathcal{R}, Y)$(respectively $C(\mathcal{R} \times Y, Y)$) denotes the collection of all continuous functions from $\mathcal{R}$ to Y(respectively from $\mathcal{R} \times Y$ to Y) and $PC(\mathcal{R}, Y)$(respectively $PC(\mathcal{R} \times Y, Y)$) denotes the collection of all piecewise continuous functions from $\mathcal{R}$ to Y (respectively from $\mathcal{R} \times Y$ to Y).

The fractional derivative in the RiemannLiouville sense of order $p > 0$ of a function f is described as

$$I^p f(s) = \frac{1}{\Gamma(p)} \int_0^s (s-\eta)^{p-1} f(\eta) d\eta.$$

The fractional derivative in the Caputo sense of a function f of order $p > 0$ is described as

$$D_s^p = \frac{1}{\Gamma(n-p)} \int_0^s (s-\eta)^{n-p-1} \frac{d^n f(\eta)}{d\eta^n} d\eta,$$

here we employed the gamma function as $\Gamma(p)$.

We define the subsequent class of space:

- $PC(\mathcal{R}, Y) = \{\varphi_1 : \mathcal{R} \to Y : \varphi_1$ is continuous for all $t \notin \{t_k\}$, $\lim_{h\to 0} \varphi_1(t_k + h) = \varphi_1(t_k^+)$, $\lim_{h\to 0} \varphi_1(t_k - h) = \varphi_1(t_k^-)$ exist and $\varphi_1(t_k) = \varphi_1(t_k^-)\}$.
- $PC(\mathcal{R} \times Y, Y) = \{\varphi_1 : \mathcal{R} \times Y \to Y : \varphi_1$ is continuous for all $t \notin \{t_k\}$, $\lim_{h\to 0} \varphi_1(t_k + h, y) = \varphi_1(t_k^+, y)$, $\lim_{h\to 0} \varphi_1(t_k - h, y) = \varphi_1(t_k^-, y)$ exist and $\varphi_1(t_k, y) = \varphi_1(t_k^-, y)\}$.

Definition 2.1. *[1] Let $t_1 : \mathbb{Z}^+ \to Y$ be a bounded sequence. t_1 is called $\mathcal{AA}$, if for each real sequence $\{j'_n\}$, there exists a subsequence $\{j_n\} \subseteq \{j'_n\}$ such that*

$$\lim_{n\to\infty} t_1(j + j_n) = f(j), \quad \textit{for all} \quad n \in \mathbb{Z}$$

is well defined and

$$\lim_{n\to\infty} f(j - j_n) = t_1(j)$$

for all $j \in \mathbb{Z}^+$. Represent this class of all sequences as $\mathcal{AA}_S^o(\mathbb{Z}, Y)$.

Definition 2.2. *[1] Let $G_1 \in PC(\mathcal{R}, Y)$ be a piecewise continuous and bounded function. G_1 is called $\mathcal{AA}$ if*

- *the sequence of impulsive terms $\{t_j\}$ is an $\mathcal{AA}$ sequence*
- *for all sequence of real numbers $\{u_n\}$, there exists a subsequence $\{u_{n_k}\} \subseteq \{u_n\}$ such that*

$$F_1(t) = \lim_{n\to\infty} G_1(t + u_{n_k}), \quad \forall\ t \in \mathcal{R}$$

is well defined and

$$\lim_{n\to\infty} F_1(t - u_{n_k}) = G_1(t), \quad \forall\ t \in \mathcal{R}.$$

Represent this class of all functions as $\mathcal{AA}^o_\Omega(\mathcal{R}, Y)$.

Definition 2.3. *[1] Let* $G_1 \in PC(\mathcal{R} \times Y, Y)$ *be a piecewise continuous and bounded function.* G_1 *is called* $\mathcal{AA}$ *in* t *uniformly for* t_1 *in compact subsets of* Y *if*

- *the impulsive moments of sequence* $\{t_j\}$ *is an* $\mathcal{AA}$ *sequence*
- *for all compact set* $Q \subseteq Y$ *and each sequence of real numbers* $\{u_n\}$*, there exists a subsequence* $\{u_{n_k}\} \subseteq \{u_n\}$ *such that*

$$F_1(t, t_1) = \lim_{n\to\infty} G_1(t + u_{n_k}, t_1), \quad \forall\ t \in \mathcal{R},\ t_1 \in Q$$

is well defined and

$$\lim_{n\to\infty} F_1(t - u_{n_k}, t_1) = G_1(t, t_1), \quad \forall\ t \in \mathcal{R},\ t_1 \in Q.$$

Represent this class of all functions as $\mathcal{AA}^o_\Omega(\mathcal{R} \times Y, Y)$.

Let $\mathcal{V}_* = \{\wp^* : \mathcal{R} \to (0, \infty) :\ \wp^*$ is locally integrable and positive over $\mathcal{R}$ $\}$. For each $\wp^* \in \mathcal{V}_*$ and $q^* > 0$, we define $m_*(l_*, \wp_*) = \displaystyle\int_{-l_*}^{l_*} \wp_*(t)dt$ and

$$\mathcal{V}_{*_\infty} = \{\wp_* \in \mathcal{V}_* : \lim_{l_*\to\infty} m_*(l_*, \wp_*) = \infty\},\ \mathcal{V}_{*_B} = \{\wp_* \in \mathcal{V}_{*_\infty} : \wp_* \text{ is bounded and } \inf_{x\in\mathcal{R}} \wp_*(x) > 0\}.$$

Clearly, $\mathcal{V}_{*_B} \subset \mathcal{V}_{*_\infty} \subset \mathcal{V}_*$. For $\wp_* \in \mathcal{V}_{*_\infty}$, set

$$\begin{aligned}&\mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \wp_*)\\ &= \{G_* \in PC(\mathcal{R}, Y) : \lim_{l_*\to\infty} \frac{1}{m_*(l_*, \wp_*)} \int_{-l_*}^{l_*} \|G_*(s)\|\, \wp_*(s)ds = 0\}\end{aligned}$$

$$\mathcal{PAA}^o_\Omega(\mathcal{R} \times Y, Y, \wp_*) = \{G_* \in PC(\mathcal{R} \times Y, Y) : \lim_{l_*\to\infty} \frac{1}{m_*(l_*, \wp_*)} \int_{-l_*}^{l_*} \|G_*(s, t)\|\, \wp_*(s)ds = 0, \text{ uniformly in } t \in \mathcal{M}, \text{ here } \mathcal{M} \text{ is an arbitrary compact subset of } Y\ \}.$$

Now we ready to introduce $\mathcal{WPAA}$ functions:

$$\mathcal{WPAA}^o_\Omega(\mathcal{R}, Y, \wp_*) = \{G_1 = H_1 + H_2 \in PC(\mathcal{R}, Y) : H_1 \in \mathcal{AA}^o_\Omega(\mathcal{R}, Y) \text{ and } H_2 \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \wp_*)\}$$

$$\mathcal{WPAA}^o_\Omega(\mathcal{R}\times Y, Y, \wp_*) = \{G_1 = H_1 + H_2 \in PC(\mathcal{R}\times Y, Y) : H_1 \in \mathcal{AA}^o_\Omega(\mathcal{R}\times Y, Y) \text{ and } H_2 \in \mathcal{PAA}^o_\Omega(\mathcal{R} \times Y, Y, \wp_*)\}.$$

Let $h : \mathbb{Z} \times Y \to Y$ be a bounded sequence. Then $h \in \mathcal{PAA}_S^o(\mathbb{Z} \times Y, Y, \wp_*)$ for some $\wp_* \in \mathcal{V}_{*_\infty}$ if

$$\lim_{l_* \to \infty} \frac{1}{m(l_*, \wp_*)} \sum_{-l_*}^{l_*} \|h(k_*, x)\| \wp_*(k_*) = 0 \quad \text{for all} \quad x \in Y.$$

Definition 2.4. *A function $x : \mathbb{Z} \times Y \to Y$ is called a $\mathcal{WPAA}$ sequence if it can be deformed as $x = x_1 + x_2$ where $x_1 \in \mathcal{AA}_S^o(\mathbb{Z} \times Y \times Y, Y)$ and $x_2 \in \mathcal{PAA}_S^o(\mathbb{Z} \times Y, Y, \wp_*)$. Represent this class of functions by $\mathcal{WPAA}_S^o(\mathbb{Z} \times Y, Y, \wp_*)$.*

2.2 Previous results

We first provide some basic results in this subsection which are required in the sequel.

Lemma 2.1. *[9, Lemma 3.2] Let $\wp_* \in \mathcal{V}_{*_\infty}$ and $G_1 \in PC(\mathcal{R}, Y)$. Then $G_1 \in \mathcal{PAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$ if and only if for every $\varepsilon > 0$,*

$$\lim_{l_* \to \infty} \frac{1}{m_*(l_*, \wp_*)} \int_{M_{l_*,\varepsilon}(G_1)} \wp_*(t) dt = 0,$$

where $M_{l_*,\varepsilon}(G_1) = \{t \in [-l_*, l_*] : \|G_1(t)\| \geq \varepsilon\}$.

Lemma 2.2. *[21, Lemma 2.13]*
Let $\wp_* \in \mathcal{V}_{*_\infty}$. If $G_1 = H_1 + H_2 \in \mathcal{WPAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$ where $H_1 \in \mathcal{AA}_\Omega^o(\mathcal{R}, Y)$ and $H_2 \in \mathcal{PAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$ then $\{H_1(t) : t \in \mathcal{R}\} \subset \overline{\{G_1(t) : t \in \mathcal{R}\}}$.

Lemma 2.3. *[3, Theorem 2.4] Let $\wp_* \in \mathcal{V}_{*_\infty}$. If $\mathcal{WPAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$ is translation invariant, then the decomposition of a piecewise $\mathcal{WPAA}$ function is unique.*

Lemma 2.4. *[3, Theorem 2.5]*
Let $\wp_* \in \mathcal{V}_{*_\infty}$. Then $(\mathcal{WPAA}_\Omega^o(\mathcal{R}, Y, \wp_*), \|.\|_\infty)$ is a Banach space.

Theorem 2.1. *[3, Theorem 2.10] Let $\wp_* \in \mathcal{V}_{*_\infty}$ and $G_1 = G_{1_*} + G_{2_*} \in \mathcal{WPAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$. On some bounded subset $K_1 \subset Y$ uniformly in $t \in \mathcal{R}$, assume that $G_1(t, z)$ and $G_{1_*}(t, z)$ are uniformly continuous. Then $\Phi(\cdot) \in \mathcal{WPAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$ implies that $G_1(\cdot, \Phi(\cdot)) \in \mathcal{WPAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$.*

Remark 2.1. *We need to present the following important estimate from [7, Theorem 1], in order to prove our main results :*

$$\|E_\alpha(t)\|_{\mathcal{L}(Y)} \leq \frac{CM}{1 + |\omega| t^\alpha}, \quad t \geq 0. \tag{2.1}$$

Lemma 2.5. *[11, Lemma 2.4] Let the family of bounded linear operators* $E_\alpha(t)$ *be strongly continuous and satisfies* (2.1) *and let* $z \in \mathcal{AA}^o_\Omega(\mathcal{R}, Y)$. *If* $u_0 : \mathcal{R} \to Y$ *is described as* $u_0(t) = \sum_{t>t_j} E_\alpha(t-t_j)I_j(z(t_j))$ *then* $u_0(\cdot) \in \mathcal{AA}^o_\Omega(\mathcal{R}, Y)$.

Theorem 2.2. *(Sadovskii fixed point theorem [23]) If* $\mathcal{F}$ *is a bounded, convex and closed subset of a Banach space* Y *and* $\Psi : \mathcal{F} \to \mathcal{F}$ *is a condensing map then* Ψ *has a fixed point in* $\mathcal{F}$.

Lemma 2.6. *(Example 11.7, [28]) A map* $\Psi = \Psi_1 + \Psi_2 : Y \to Y$ *is* k_1-*contraction with* $0 \le k_1 < 1$ *if*

- Ψ_1 *is* k_1-*contraction, ie.,* $\|\Psi_1(x_1) - \Psi_1(x_2)\|_Y \le k_1\|x_1 - x_2\|_Y$ *and*
- Ψ_2 *is compact*

then Ψ *is a condensing map.*

2.3 Assumptions

Let the $\mathcal{WPAA}$ forcing term G_1 of model (1.1) in a perturbed form $G_1 = G_{1_*} + G_{2_*}$ which satisfies the following hypotheses:

(H1) G_{1_*} is Lipschitz and bounded, specifically, there exists a non-negative function $\mathcal{N}_1 \in BS^{p_1}(\mathcal{R})$ with $p_1 > 1$ and $L^*_{G_1} > 0$ such that

$$\|G_{1_*}(t, x_1, x_2)\| \le \mathcal{N}_1(t) \quad \text{and}$$

$$\|G_{1_*}(t, x_1, x_2) - G_{1_*}(t, x_3, x_4)\| \le L^*_{G_1}[\|x_1 - x_3\| + \|x_2 - x_4\|]$$

for all $\quad t \in \mathcal{R}, x_i \in Y, i = 1, 2, 3, 4.$

(H2) G_{2_*} is bounded and compact, specifically, there exists a non-negative function $\mathcal{N}_2 \in BS^{p_1}(\mathcal{R})$ with $p_1 > 1$ such that

$$\|G_{2_*}(t, x_1, x_2)\| \le \mathcal{N}_2(t) \quad \text{for all } t \in \mathcal{R}, x_i \in Y, i = 1, 2.$$

(H3) Let $k_1 \in C(\mathcal{R}, Y)$ and $|k_1(t)| \le C_k e^{-\mu t}$ for $t \ge 0$ and C_k, μ are some positive constants.

(H4) $h_1 : \mathcal{R} \times Y \to Y$. Let $L_{h_1} > 0$ be such that $\|h_1(t, x_1) - h_1(t, x_2)\| \le L_{h_1}\|x_1 - x_2\|$ for all $t \in \mathcal{R}$ and $x_i \in Y, i = 1, 2$.

(H5) $I_j : \mathbb{Z} \to \mathbb{Z}$. Let $L^* > 0$ be such that $\|I_j(z_1) - I_j(z_2)\| \le L^*\|z_1 - z_2\|$ for all $z_1, z_2 \in \mathbb{Z}$.

3 Useful results

First we prove a new composition theorem concerning $\mathcal{WPAA}$ function in this section and the theorem derived is then employed to demonstrate the existence of $\mathcal{WPAA}$ solution to the problem (1.1).

3.1 Composition theorem of $\mathcal{WPAA}$ function

The following two theorems are required to demonstrate the existence result for the $\mathcal{WPAA}$ function due to the fact that the forced term G_1 does not always meet only the Lipschitz continuity.

Theorem 3.1. *If assumptions (H1)-(H2) holds. Moreover, we have $\wp_* \in \mathcal{V}_{*_\infty}$ satisfying*

(H6) $\wp_* \in L^{q_1}_{loc}(\mathcal{R})$ *satisfies*

$$\lim_{l_* \to \infty} \sup \frac{l_*^{\frac{1}{p_1}} m_{*_{q_1}}(l_*, \wp_*)}{m_*(l_*, \wp_*)} < \infty,$$

where $\frac{1}{p_1} + \frac{1}{q_1} = 1$ *and* $m_{*_{q_1}}(l_*, \wp_*) = \left(\int_{-l_*}^{l_*} \wp_*^{q_1}(t) dt \right)^{\frac{1}{q_1}}$.
Then

$$\lim_{l_* \to \infty} \frac{1}{m_*(l_*, \wp_*)} \int_{-l_*}^{l_*} \|G_1(t, x, y)\| \wp_*(t) dt = 0.$$

Proof. From (H6), we find that

$$\lim_{l_* \to \infty} \sup \frac{\left(2[l_*] + 2\right)^{\frac{1}{p_1}} m_{*_{q_1}}(l_*, \wp_*)}{m_*(l_*, \wp_*)} < \infty.$$

Hence there exist constants $M_1, l_{1_*} > 0$ such that

$$\frac{\left(2[l_*] + 2\right)^{\frac{1}{p_1}} m_{*_{q_1}}(l_*, \wp_*)}{m_*(l_*, \wp_*)} < M_1, \quad l_* > l_{1_*}. \tag{3.1}$$

By using the inequality (3.1), we find

$$\begin{aligned}
&\frac{1}{m_*(l_*,\wp_*)}\int_{-l_*}^{l_*}\|G_1(t,x,y)\|\wp_*(t)dt\\
&\leq \frac{1}{m_*(l_*,\wp_*)}\int_{-l_*}^{l_*}\mathcal{N}(t)\wp_*(t)dt, \quad \text{where } \mathcal{N}(t)=\mathcal{N}_1(t)+\mathcal{N}_2(t)\\
&\leq \frac{1}{m_*(l_*,\wp_*)}\Big(\int_{-l_*}^{l_*}\mathcal{N}^{p_1}(t)dt\Big)^{\frac{1}{p_1}}\Big(\int_{-l_*}^{l_*}\wp_*^{q_1}(t)dt\Big)^{\frac{1}{q_1}}\\
&\leq \frac{1}{m_*(l_*,\wp_*)}\Big(\int_{-[l_*]-1}^{[l_*]+1}\mathcal{N}^{p_1}(t)dt\Big)^{\frac{1}{p_1}}m_{*_{q_1}}(l_*,\wp_*)\\
&\leq \|\mathcal{N}\|_{S^{p_1}}\frac{\big(2[l_*]+2\big)^{\frac{1}{p_1}}m_{*_{q_1}}(l_*,\wp_*)}{m_*(l_*,\wp_*)}\\
&< M_1\|\mathcal{N}\|_{S^{p_1}}.
\end{aligned}$$

For all $l_* > \max\{l_{*_0}, l_{*_1}\}$, we find

$$\lim_{l_*\to\infty}\frac{1}{m_*(l_*,\wp_*)}\int_{-l_*}^{l_*}\|G_1(t,x,y)\|\wp_*(t)dt = 0.$$

□

Theorem 3.2. *(Composition Theorem) Let $\wp_* \in \mathcal{V}_{*_\infty}$. Assume that (H2) and (H6) holds and let $G_1 = G_{1_*} + G_{2_*} \in \mathcal{WPAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$ with $G_{1_*} \in \mathcal{AA}_\Omega^o(\mathcal{R}, Y)$ and $G_{2_*} \in \mathcal{PAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$. Suppose G_{1_*} satisfies the following condition*

(i) For every bounded subset $K_1 \subset Y, \{G_{1_*}(\cdot, x) : x \in K_1\}$ is bounded in $\mathcal{WPAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$.

If $Q = U + H \in \mathcal{WPAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$, where $U \in \mathcal{AA}_\Omega^o(\mathcal{R}, Y)$ and $H \in \mathcal{PAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$ then $G_1(t, Q(t)) \in \mathcal{WPAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$ with $\mathcal{B}_1(t) = G_{1_*}(t, U(t)) \in \mathcal{AA}_\Omega^o(\mathcal{R}, Y)$, $\mathcal{B}_2(t) = G_{1_*}(t, Q(t)) - G_{1_*}(t, U(t)) \in \mathcal{PAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$ and $\mathcal{B}_3(t) = G_{2_*}(t, Q(t)) \in \mathcal{PAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$.

Proof. The function G_1 can be deformed as

$$\begin{aligned}
G_1(t,Q(t)) &= G_{1_*}(t,U(t)) + G_1(t,Q(t)) - G_{1_*}(t,U(t))\\
&= G_{1_*}(t,U(t)) + [G_{1_*}(t,Q(t)) - G_{1_*}(t,U(t))] + G_{2_*}(t,Q(t)).
\end{aligned}$$

Define $\mathcal{B}_1(t) = G_{1_*}(t, U(t))$, $\mathcal{B}_2(t) = G_{1_*}(t, Q(t)) - G_{1_*}(t, U(t))$ and $\mathcal{B}_3(t) = G_{2_*}(t, Q(t))$. Then $G_1(t, Q(t)) = \mathcal{B}_1(t) + \mathcal{B}_2(t) + \mathcal{B}_3(t)$.
In view of [16, Lemma 2.2], $\mathcal{B}_1(t) = G_{1_*}(t, U(t)) \in \mathcal{AA}_\Omega^o(\mathcal{R}, Y)$.

Next we assert that $\mathcal{B}_2(t) \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$. Since $U(\cdot)$ and $Q(\cdot)$ are bounded, so we can select a bounded subset $K_1 \subseteq Y$ such that $Q(\mathcal{R}), U(\mathcal{R}) \subseteq K_1$. By condition (i), $\mathcal{B}_2(t) \in BS^{p_1}(Y)$. Also from Theorem 2.1, we see that G_{1_*} is uniformly continuous in the bounded subset $K_1 \subseteq Y$ uniformly for $t \in \mathcal{R}$. Hence for given $\epsilon > 0$, there exists $\delta > 0$ such that for all $t \in \mathcal{R}$

$$\|Q - U\| \leq \delta \Rightarrow \|G_{1_*}(t, Q(t)) - G_{1_*}(t, U(t))\| \leq \epsilon,$$

where $H(t) = Q(t) - U(t)$.

Let $M_{l_*,\delta}(H) = \{t \in [-l_*, l_*] : \|H(t)\| \geq \delta\}$.

Hence we get

$$M_{l_*,\epsilon}(\mathcal{B}_2) = M_{l_*,\epsilon}(G_{1_*}(\cdot, Q(\cdot)) - G_{1_*}(\cdot, U(\cdot))) \subseteq M_{l_*,\delta}(H).$$

Since $G_{1_*} \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$, by Lemma 2.1 we find that

$$\lim_{l_* \to \infty} \frac{1}{m_*(l_*, \wp_*)} \int_{M_{l_*,\delta}(H)} \wp_*(t) dt = 0.$$

Hence

$$\lim_{l_* \to \infty} \frac{1}{m_*(l_*, \wp_*)} \int_{M_{l_*,\epsilon}(\mathcal{B}_2)} \wp_*(t) dt = 0.$$

This proves that $\mathcal{B}_2(t) \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$.

It remains to assert that $\mathcal{B}_3(t) \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$. Clearly $\mathcal{B}_3$ is bounded. So, assume that $\|\mathcal{B}_3(t)\| \leq M_1$ for all $t \in \mathcal{R}$. By (H2), for any $\epsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{m_*(l_*, \wp_*)} \int_{-l_*}^{l_*} \|\mathcal{B}_3(t)\| \wp_*(t) dt \\
&\leq \frac{1}{m_*(l_*, \wp_*)} \int_{M_{l_*,\epsilon}(Q)} \|\mathcal{B}_3(t)\| \wp_*(t) dt \\
&\quad + \frac{1}{m_*(l_*, \wp_*)} \int_{[-l_*, l_*] \setminus M_{l_*,\epsilon}(Q)} \|G_{2_*}(t, Q(t))\| \wp_*(t) dt \\
&\leq \frac{M_1}{m_*(l_*, \wp_*)} \int_{M_{l_*,\epsilon}(Q)} \wp_*(t) dt + \frac{1}{m_*(l_*, \wp_*)} \int_{[-l_*, l_*] \setminus M_{l_*,\epsilon}(Q)} \mathcal{N}_2(t) \wp_*(t) dt \\
&\leq \frac{M_1}{m_*(l_*, \wp_*)} \int_{M_{l_*,\epsilon}(Q)} \wp_*(t) dt + \frac{1}{m_*(l_*, \wp_*)} \int_{-l_*}^{l_*} \mathcal{N}_2(t) \wp_*(t) dt \\
&\leq \frac{M_1}{m_*(l_*, \wp_*)} \int_{M_{l_*,\epsilon}(Q)} \wp_*(t) dt + \|\mathcal{N}_2\|_{S^{p_1}} \frac{\left(2[l_*] + 2\right)^{\frac{1}{p_1}} m_{*_{q_1}}(l_*, \wp_*)}{m_*(l_*, \wp_*)}. \qquad (3.2)
\end{aligned}$$

From Lemma 2.1, we get

$$\lim_{l_* \to \infty} \frac{1}{m_*(l_*, \wp_*)} \int_{M_{l_*,\epsilon}(Q)} \wp_*(t)dt = 0. \tag{3.3}$$

From (3.2), (3.3) and (H6), we get

$$\lim_{l_* \to \infty} \frac{1}{m_*(l_*, \wp_*)} \int_{-l_*}^{l_*} \|\mathcal{B}_3(t)\| \wp_*(t)dt = 0.$$

So, $\mathcal{B}_3(t) \in \mathcal{PAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$.
Hence, $G_1(t, Q(t)) \in \mathcal{WPAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$. $\square$

3.2 Auxiliary Results

We employ four major results in this subsection, which are used to establish our main findings.

Lemma 3.1. *Let $\wp_* \in \mathcal{V}_{*_B}$. If uniformly continuous function $G_1 \in \mathcal{PAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$ then a bounded sequence $\{a_m\} \in \mathcal{PAA}_\Omega^o(\mathbb{Z}, Y, \wp_*)$ such that $G_1(t_m) = a_m, t_m \in \mathcal{R}, m \in \mathbb{Z}$.*

Proof. Let $0 < \zeta < 1$ and $t \in (t_m - \delta, t_m)$, where $\delta > 0, m \in \mathbb{Z}$, we get $\|G_1(t)\| \wp_*(t) \geq (1-\zeta) \|G_1(t_m)\| \wp_*(t_m)$. With no loss of generality, assume $t_m \geq 0$ and $t_{-m} < 0$, there exist $l_{*_m}, l_{*_{-m}} \in \mathcal{R}^+$ such that $l_{*_m} = t_m, -l_{*_{-m}} = t_{-m}$. Let $l'_{*_m} = \max(l_{*_m}, l_{*_{-m}})$, we have

$$\begin{aligned} \int_{-l'_{*_m}}^{l'_{*_m}} \|G_1(t)\| \wp_*(t)dt &\geq \int_{-l_{*_{-m}}}^{l_{*_m}} \|G_1(t)\| \wp_*(t)dt = \int_{t_{-m}}^{t_m} \|G_1(t)\| \wp_*(t)dt \\ &\geq \sum_{j=-m+1}^{m} \int_{t_{j-\delta}}^{t_j} \|G_1(t)\| \wp_*(t)dt \geq \delta(1-\zeta) \sum_{j=-m+1}^{m} \|G_1(t_j)\| \wp_*(t_j). \end{aligned}$$

Thus we obtain

$$\begin{aligned} &\frac{1}{m_*(l'_{*_m}, \wp_*)} \int_{-l_{*_m}}^{l_{*_m}} \|G_1(t)\| \wp_*(t)dt \\ &\geq \delta(1-\zeta) \frac{1}{m_*(l'_{*_m}, \wp_*)} \sum_{j=-m+1}^{m} \|G_1(t_j)\| \wp_*(t_j), \end{aligned} \tag{3.4}$$

as $m \to \infty$. Since $G_1 \in \mathcal{PAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$, it follows from the inequality (3.4) that $G_1(t_m) = a_m \in \mathcal{PAA}_\Omega^o(\mathbb{Z}, Y, \wp_*)$. $\square$

Theorem 3.3. *Assume that $I_j : Y \to Y$ is a $\mathcal{WPAA}$ sequence and satisfies (H5). If $\phi \in \mathcal{WPAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$ then $I_j(\phi(t_j))$ is a $\mathcal{WPAA}$ sequence.*

Proof. Since $\phi = h_1+h_2$, where $h_1 \in \mathcal{AA}^o_\Omega(\mathcal{R}, Y), h_2 \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$. It follows that $I_j(\phi(t_j)) = I_j(h_1(t_j))+I_j(h_2(t_j))$. By [17, Lemma 3.2], the sequence $I_j(h_1(t_j))$ is $\mathcal{AA}$. Next it remains to assert that $I_j(h_2(t_j)) \in \mathcal{PAA}^o_\Omega(\mathbb{Z}, Y, \wp_*)$. Let $t_j \geq 0, t_{-j} < 0, j \in \mathbb{Z}$ there exist $l_{*_j}, l_{*_{-j}} \in \mathcal{R}^+$ such that $l_{*_j} = t_j, -l_{*_{-j}} = t_{-j}$ and $l'_{*_j} = \max(l_{*_j}, l_{*_{-j}})$, we have

$$\begin{aligned}\int_{-l'_{*_j}}^{l'_{*_j}} \|h_2(t)\|\wp_*(t)dt &\geq \int_{-l_{*_{-j}}}^{l_{*_j}} \|h_2(t)\|\wp_*(t)dt = \int_{t_{-j}}^{t_j} \|h_2(t)\|\wp_*(t)dt \\ &\geq \sum_{i=-j+1}^{j} \int_{t_i-\delta}^{t_i} \|h_2(t)\|\wp_*(t)dt \geq \delta(1-\zeta) \sum_{i=-j+1}^{j} \|h_2(t_i)\|\wp_*(t_i).\end{aligned}$$

Thus we obtain

$$\frac{1}{m_*(l'_{*_j}, \wp_*)} \int_{-l'_{*_j}}^{l'_{*_j}} \|h_2(t)\|\wp_*(t)dt \geq \delta(1-\zeta)\frac{1}{m_*(l'_{*_j}, \wp_*)} \sum_{i=-j+1}^{j} \|h_2(t_i)\|\wp_*(t_i), \tag{3.5}$$

as $j \to \infty$, since $h_2 \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$, the result follows from the inequality (3.5), that $h_2(t_j) \in \mathcal{PAA}^o_\Omega(\mathbb{Z}, Y, \wp_*)$.
Now from (H5),

$$\|I_j(h_2(t_j))\| \leq \|I_j(h_2(t_j)) - I_j(0)\| + \|I_j(0)\| \leq L^*\|h_2(t_j)\| + \|I_j(0)\|$$

we see that the sequence $I_j(\phi(t_j))$ is a $\mathcal{WPAA}$ sequence. □

Theorem 3.4. *If $G_1 = H_1+H_2$ with $H_1 \in \mathcal{AA}^o_\Omega(\mathcal{R}, Y), H_2 \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$ is a $\mathcal{WPAA}$ function, then $Q_1(t) = \int_{-\infty}^{t} E_\alpha(t-s)G_1(s)ds + \sum_{t>t_j} E_\alpha(t-t_j)I_j(z(t_j))$ is a $\mathcal{WPAA}$ function.*

Proof. Let $\int_{-\infty}^{t} E_\alpha(t-s)G_1(s)ds = R_1(t) + S_1(t)$, where

$$R_1(t) = \int_{-\infty}^{t} E_\alpha(t-s)H_1(s)ds, \quad S_1(t) = \int_{-\infty}^{t} E_\alpha(t-s)H_2(s)ds.$$

By [12, Lemma 3.3], $R_1(t) \in \mathcal{AA}^o_\Omega(\mathcal{R}, Y)$ and $S_1(t) \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$.
Next it remains to show that $\sum_{t>t_j} E_\alpha(t-t_j)I_j(z(t_j)) \in \mathcal{WPAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$.

By Theorem 3.3, $I_j(z(t_j)) \in \mathcal{WPAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$.
Thus we have:
$I_j(z(t_j)) = \beta_j + \gamma_j$, where $\beta_j \in \mathcal{AA}^o_\Omega(\mathcal{R}, Y)$ and $\gamma_j \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$, then

$$\sum_{t>t_j} E_\alpha(t-t_j)I_j(z(t_j)) = \sum_{t>t_j} E_\alpha(t-t_j)\beta_j + \sum_{t>t_j} E_\alpha(t-t_j)\gamma_j = R_2(t) + S_2(t).$$

By Lemma 2.5, $R_2(t) \in \mathcal{AA}^o_\Omega(\mathcal{R}, Y)$.
Presently, we establish that $S_2(t) \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$.
Since $\gamma_j \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$ by Lemma 3.1, there exists $g_1(t) = \gamma_k, t \in [k, k+1)$ such that $g_1 \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$ and $g_1(k) = \gamma_k, k \in Z$.

$$\begin{aligned}
&\frac{1}{m_*(l_*, \wp_*)} \int_{-l_*}^{l_*} \|S_2(t)\| \, \wp_*(t)dt \le \frac{1}{m_*(l_*, \wp_*)} \int_{-l_*}^{l_*} \sum_{t>t_j} \|E_\alpha(t-t_j)\| \|\gamma_j\| \wp_*(t)dt \\
&\le \frac{CM}{m_*(l_*, \wp_*)} \int_{-l_*}^{l_*} \left[\sum_{0<t-t_j\le 1} \frac{1}{1+|\omega|(t-t_j)^\alpha} + \sum_{i=1}^{\infty} \sum_{i<t-t_j\le i+1} \frac{1}{1+|\omega|(t-t_j)^\alpha} \right] \\
&\quad \|g_1(t)\| \wp_*(t)dt \\
&\le \frac{CM}{m_*(l_*, \wp_*)} \int_{-l_*}^{l_*} \left[\frac{1}{1+|\omega| m_1^\alpha} + \sum_{n=2}^{\infty} \frac{1}{1+|\omega| n^\alpha} \right] \|g_1(t)\| \wp_*(t)dt \\
&= \left[\frac{CM}{1+|\omega| m_1^\alpha} + CMN \right] \frac{1}{m_*(l_*, \wp_*)} \int_{-l_*}^{l_*} \|g_1(t)\| \wp_*(t)dt \to 0 \quad \text{as} \quad l_* \to \infty,
\end{aligned}$$

where $m_1 = \{\min(t-t_j) : 0 < t-t_j \le 1\}$ and $N = \displaystyle\sum_{n=2}^{\infty} \frac{1}{1+|\omega| n^\alpha}$.
Thus, $S_2(t) \in \mathcal{PAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$. □

Lemma 3.2. *[27, Lemma 3.1] If $z \in \mathcal{WPAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$ and assume that (H3)-(H4) holds, then $Kz \in \mathcal{WPAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$*

4 Existence Results

First, we present the following definition.

Definition 4.1. *Let $z : \mathcal{R} \to Y$ be a function. Then z is called a mild solution of the problem (1.1) if for any $t \in \mathcal{R}$*

$$z(t) = \int_{-\infty}^{t} E_\alpha(t-s)G_1(s, z(s), Kz(s))ds + \sum_{t>t_j} E_\alpha(t-t_j)I_j(z(t_j)). \quad (4.1)$$

Theorem 4.1. *If assumptions (H1)-(H5) holds then the problem (1.1) has at least one piecewise $\mathcal{WPAA}$ mild solution provided*

$$\sum_{t>t_j} \frac{L^*}{1+|\omega|(t-t_j)^\alpha} < 1. \tag{4.2}$$

Proof. Set $B_r = \{z \in \mathcal{WPAA}^o_\Omega(\mathcal{R}, Y, \wp_*) : \|z\| \leq r\}$ for each $r > 0$, here B_r is the bounded, closed and convex subset of $\mathcal{WPAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$.

Now introduce an operator $J_1 : B_r \to \mathcal{WPAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$ as follows:

$$(J_1 z)(t) = \int_{-\infty}^t E_\alpha(t-s)G_1(s, z(s), Kz(s))ds + \sum_{t>t_j} E_\alpha(t-t_j)I_j(z(t_j)). \tag{4.3}$$

Let us decompose $J_1 = J_1^* + J_2^*$ as

$$(J_1^* z)(t) = \int_{-\infty}^t E_\alpha(t-s)G_{1_*}(s, z(s), Kz(s))ds + \sum_{t>t_j} E_\alpha(t-t_j)I_j(z(t_j)),$$

$$(J_2^* z)(t) = \int_{-\infty}^t E_\alpha(t-s)G_{2_*}(s, z(s), Kz(s))ds.$$

Now, to assert the existence of a $\mathcal{WPAA}$ solution, we need to prove the results given below.

(i) J_1 is well defined and self mapping

(ii) J_1^* is contraction and continuous

(iii) J_2^* is compact

(iv) J_1 is condensing.

$*$ *To prove J_1 is well defined*

By Lemma 3.2, $Kx \in \mathcal{WPAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$, thus $G_1(\cdot, z(\cdot), Kz(\cdot)) \in \mathcal{WPAA}^o_\Omega(\mathcal{R}, Y, \wp_*)$ from the composition Theorem 3.2. By Theorem 3.4, the operator J_1 is well defined.

$*$ *To show that J_1 is self-mapping*

We assert that there exists a positive number r such that $J_1(B_r) \subset B_r$. If this result is not true then for all $r > 0$, there exist $z \in B_r$ and $t \in \mathcal{R}$ such that $\|(J_1z)(t)\| > r$. On the other hand, from (H1), (H2) and (2.1), we get

$$\begin{aligned} r < \|(J_1z)(t)\| &\leq \Big\| \int_{-\infty}^{t} E_\alpha(t-s)G_1(s,z(s),Kz(s))ds \\ &\quad + \sum_{t>t_j} E_\alpha(t-t_j)I_j(z(t_j))\Big\| \\ (1+|\omega|t^\alpha)\|(J_1z)(t)\| &\leq CM\Big\| \int_{-\infty}^{t} \frac{1+|\omega|t^\alpha}{1+|\omega|(t-s)^\alpha}\mathcal{N}(s)ds\Big\| \\ &\quad + (1+|\omega|t^\alpha)\Big\| \sum_{t>t_j} E_\alpha(t-t_j)I_j(z(t_j))\Big\|, \end{aligned}$$

where $\mathcal{N}(t) = \mathcal{N}_1(t) + \mathcal{N}_2(t)$.

$$\begin{aligned} &(1+|\omega|t^\alpha)\|(J_1z)(t)\| \\ &\leq CM2^\alpha\Big\| \int_{-\infty}^{t} (1+|\omega|s^\alpha)\mathcal{N}(s)ds\Big\| + (1+|\omega|t^\alpha)\Big\| \sum_{t>t_j} E_\alpha(t-t_j)I_j(z(t_j))\Big\|, \end{aligned}$$

since $\frac{1+|\omega|t^\alpha}{1+|\omega|(t-s)^\alpha} \leq 2^\alpha(1+|\omega|s^\alpha)$.
Now,

$$\begin{aligned} &(1+|\omega|t^\alpha)\|(J_1z)(t)\| \\ &\leq CM2^\alpha(1+|\omega|t^\alpha)\|\mathcal{N}\|_1 \\ &\quad + (1+|\omega|t^\alpha)\sum_{t>t_j} \frac{1}{1+|\omega|(t-t_j)^\alpha}\Big[\|I_j(z(t_j)) - I_j(0)\| + \|I_j(0)\|\Big] \\ &\leq CM2^\alpha(1+|\omega|t^\alpha)\|\mathcal{N}\|_1 \\ &\quad + (1+|\omega|t^\alpha)\Big[\sum_{t>t_j} \frac{L^*\|z(t_j)\|}{1+|\omega|(t-t_j)^\alpha} + \sum_{t>t_j} \frac{\|I_j(0)\|}{1+|\omega|(t-t_j)^\alpha}\Big]. \end{aligned}$$

Thus,

$$r < \|(J_1z)(t)\| \leq CM2^\alpha\|\mathcal{N}\|_1 + \sum_{t>t_j} \frac{\|I_j(0)\|}{1+|\omega|(t-t_j)^\alpha} + L^*r\sum_{t>t_j} \frac{1}{1+|\omega|(t-t_j)^\alpha}$$

On dividing both sides by r and taking the limit as $r \to \infty$, we get

$$1 \leq L^*\sum_{t>t_j} \frac{1}{1+|\omega|(t-t_j)^\alpha},$$

this contradicts (4.2). Hence for some positive number r, $J_1(B_r) \subset B_r$. Therefore, J_1 is self-mapping.
$*$ *To establish J_1^* is contraction on B_r*

For each $t \in \mathcal{R}$, let $z_1, z_2 \in B_r$ then by (H1), (H3)-(H5) and equation (2.1), we have

$$
\begin{aligned}
&\|(J_1^* z_1)(t) - (J_1^* z_2)(t)\| \\
&\leq \int_{-\infty}^{t} \|E_\alpha(t-s)\| \|G_{1_*}(s, z_1(s), Kz_1(s)) - G_{1_*}(s, z_2(s), Kz_2(s))\| ds \\
&\quad + \sum_{t>t_j} \|E_\alpha(t-t_j)\| \|I_j(z_1(t_j)) - I_j(z_2(t_j))\| \\
&\leq CM\Bigg[\int_{-\infty}^{t} \frac{1}{1+|\omega|(t-s)^\alpha} L^*_{G_1}\Big[\|z_1(s)-z_2(s)\| + \|Kz_1(s) - Kz_2(s)\|\Big] ds \\
&\quad + \sum_{t>t_j} \frac{1}{1+|\omega|(t-t_j)^\alpha} L^* \|z_1(t_j) - z_2(t_j)\|\Bigg] \\
&\leq CM\Bigg[\int_{-\infty}^{t} \frac{L^*_{G_1}}{1+|\omega|(t-s)^\alpha}\Big[\|z_1(s)-z_2(s)\| \\
&\quad + \int_{-\infty}^{s} \|k_1(s-\eta)\| \|h_1(\eta, z_1(\eta)) - h_1(\eta, z_2(\eta))\| d\eta\Big] ds \\
&\quad + \sum_{t>t_j} \frac{1}{1+|\omega|(t-t_j)^\alpha} L^* \|z_1(t_j) - z_2(t_j)\|\Bigg] \\
&\leq CM\Bigg[\int_{-\infty}^{t} \frac{L^*_{G_1}}{1+|\omega|(t-s)^\alpha}\Big[\|z_1(s)-z_2(s)\| \\
&\quad + \int_{-\infty}^{s} C_k e^{-\mu(s-\eta)} L_{h_1} \|z_1(\eta) - z_2(\eta)\| d\eta\Big] ds \\
&\quad + \sum_{t>t_j} \frac{1}{1+|\omega|(t-t_j)^\alpha} L^* \|z_1(t_j) - z_2(t_j)\|\Bigg] \\
&\leq CM\Bigg[\int_{-\infty}^{t} \frac{L^*_{G_1}}{1+|\omega|(t-s)^\alpha}\Big(1 + \frac{C_k L_{h_1}}{\mu}\Big) ds + \sum_{t>t_j} \frac{L^*}{1+|\omega|(t-t_j)^\alpha}\Bigg] \|z_1 - z_2\| \\
&\leq CM\Bigg[\frac{\pi L^*_{G_1} |\omega|^{\frac{-1}{\alpha}}}{\alpha \sin(\frac{\pi}{\alpha})}\Big(1 + \frac{C_k L_{h_1}}{\mu}\Big) + \sum_{t>t_j} \frac{L^*}{1+|\omega|(t-t_j)^\alpha}\Bigg] \|z_1 - z_2\|.
\end{aligned}
$$

Since $CM\Bigg[\frac{\pi L^*_{G_1} |\omega|^{\frac{-1}{\alpha}}}{\alpha \sin(\frac{\pi}{\alpha})}\Big(1 + \frac{C_k L_{h_1}}{\mu}\Big) + \sum_{t>t_j} \frac{L^*}{1+|\omega|(t-t_j)^\alpha}\Bigg] < 1$, we see that, J_1^* is a contraction on B_r.

$*$ *To assert that J_1^* is continuous on B_r*

Let $\{z^n(t)\}_{n=0}^{\infty} \subseteq B_r$ with $z^n \to z$ in B_r then by (H1), (H3)-(H5) and (2.1)

$$\begin{aligned}
&\|(J_1^* z^n)(t) - (J_1^* z)(t)\| \\
&\leq \int_{-\infty}^{t} \|E_\alpha(t-s)\| \|G_{1_*}(s, z^n(s), Kz^n(s)) - G_{1_*}(s, z(s), Kz(s))\| ds \\
&\quad + \sum_{t>t_j} \|E_\alpha(t-t_j)\| \|I_j(z^n(t_j)) - I_j(z(t_j))\| \\
&\leq CM\Bigg[\int_{-\infty}^{t} \frac{L_{G_1}^*}{1+|\omega|(t-s)^\alpha}\Big[\|z^n(s) - z(s)\| \\
&\quad + \int_{-\infty}^{s} \|k_1(s-\eta)\| \|h_1(\eta, z^n(\eta)) - h_1(\eta, z(\eta))\| d\eta\Big] ds \\
&\quad + \sum_{t>t_j} \frac{1}{1+|\omega|(t-t_j)^\alpha} L^* \|z^n(t_j) - z(t_j)\|\Bigg] \\
&\leq CM\Bigg[\frac{\pi L_{G_1}^* |\omega|^{\frac{-1}{\alpha}}}{\alpha \sin(\frac{\pi}{\alpha})}\Big(1 + \frac{C_k L_{h_1}}{\mu}\Big) + \sum_{t>t_j} \frac{L^*}{1+|\omega|(t-t_j)^\alpha}\Bigg] \|z^n - z\| \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Therefore, J_1^* is continuous on B_r.

$*$ *To prove J_2^* is compact*

For $t_k < \eta_1 < \eta_2 \leq t_{k+1}$, we have

$$\begin{aligned}
&\|(J_2^* z)(\eta_2) - (J_2^* z)(\eta_1)\| \\
&= \Big\| \int_{-\infty}^{\eta_2} E_\alpha(\eta_2 - s) G_{2_*}(s, z(s), Kz(s)) ds \\
&\quad - \int_{-\infty}^{\eta_1} E_\alpha(\eta_1 - s) G_{2_*}(s, z(s), Kz(s)) ds \Big\| \\
&\leq \Big\| \int_{-\infty}^{\eta_1} \big[E_\alpha(\eta_2 - s) - E_\alpha(\eta_1 - s)\big] G_{2_*}(s, z(s), Kz(s)) ds \Big\| \\
&\quad + \Big\| \int_{\eta_1}^{\eta_2} E_\alpha(\eta_2 - s) G_{2_*}(s, z(s), Kz(s)) ds \Big\| \\
&\leq \Big\| \int_{-\infty}^{\eta_1} E_\alpha(\eta_2 - s)\big[G_{2_*}(\eta_2 - s, z(\eta_2 - s), Kz(\eta_2 - s)) \\
&\quad - G_{2_*}(\eta_1 - s, z(\eta_1 - s), Kz(\eta_1 - s))\big] ds \Big\| \\
&\quad + \Big\| \int_{\eta_1}^{\eta_2} E_\alpha(\eta_2 - s) G_{2_*}(s, z(s), Kz(s)) ds \Big\|
\end{aligned}$$

$$\begin{aligned}
&\leq \|G_{2_*}\big(\eta_2 - \cdot, z(\eta_2 - \cdot), Kz(\eta_2 - \cdot)\big) \\
&\quad - G_{2_*}\big(\eta_1 - \cdot, z(\eta_1 - \cdot), Kz(\eta_1 - \cdot)\big)\|\pi CM \frac{|\omega|^{\frac{-1}{\alpha}}}{\alpha \sin(\frac{\pi}{\alpha})} \\
&\quad + \Big\| \int_{\eta_1}^{\eta_2} E_\alpha(\eta_2 - s) G_{2_*}(s, z(s), Kz(s)) ds \Big\|. \qquad (4.4)
\end{aligned}$$

We find that the expression (4.4) does not depend on z and also tends to 0 as $\eta_2 \to \eta_1$. Hence using the well known Arzela-Ascoli theorem for equicontinuous function (Diethelm, Theorem D.10 [8]), the operator J_2^* is precompact, so J_2^* is completely continuous. Hence J_2^* is compact.

$*$ *To assert* J_1 *is condensing*

Since $J_1 = J_1^* + J_2^*$, J_1^* is contraction, continuous and J_2^* is compact, hence by Lemma 2.6, J_1 is a condensing map on B_r.

Also by using Theorem 2.2, we find that the model (1.1) has a piecewise $\mathcal{WPAA}$ solution on B_r.

□

Theorem 4.2. *Suppose (H3)-(H5) hold. Also if* G_1 *is Lipschitz and bounded especially* $\|G_1(t, x_1, x_2) - G_1(t, x_3, x_4)\| \leq L_{G_1}[\|x_1 - x_3\| + \|x_2 - x_4\|]$ *for every* $t \in \mathcal{R}$, $x_i \in Y, i = 1, 2, 3, 4$ *then the model (1.1) has a unique* $\mathcal{WPAA}$ *mild solution provided*

$$\Lambda = CM \left\{ \frac{\pi L_{G_1} |\omega|^{-\frac{1}{\alpha}}}{\alpha \sin(\frac{\pi}{\alpha})} \left[1 + \frac{C_k L_{h_1}}{\mu}\right] + \sum_{t > t_j} \frac{L^*}{1 + |\omega|(t - t_j)^\alpha} \right\} < 1. \qquad (4.5)$$

Proof. Define an operator $J_1 : \mathcal{WPAA}_\Omega^o(\mathcal{R}, Y, \wp_*) \to \mathcal{WPAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$ as given in (4.3).

$*$ *To prove* J_1 *is well defined:* This proof is same as the proof given in the first step of Theorem 4.1.

Now, for every $t \in \mathcal{R}, z_1, z_2 \in \mathcal{WPAA}_\Omega^o(\mathcal{R}, Y, \wp_*)$, we have

$$\begin{aligned}
&\|J_1 z_1(t) - J_1 z_2(t)\| \\
&\quad \leq \int_{-\infty}^{t} \|E_\alpha(t - s)\| \|G_1(s, z_1(s), Kz_1(s)) - G_1(s, z_2(s), Kz_2(s))\| ds \\
&\qquad + \sum_{t > t_j} \|E_\alpha(t - t_j)\| \|I_j(z_1(t_j)) - I_j(z_2(t_j))\| \\
&\quad \leq \int_{-\infty}^{t} \frac{CM \|G_1(s, z_1(s), Kz_1(s)) - G_1(s, z_2(s), Kz_2(s))\| ds}{1 + |\omega|(t - s)^\alpha}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{t>t_j}\frac{CM}{1+|\omega|(t-t_j)^{\alpha}}\|I_j(z_1(t_j))-I_j(z_2(t_j))\|\\
&\leq CM\left\{\frac{\pi L_{G_1}|\omega|^{-\frac{1}{\alpha}}\left[1+\frac{C_kL_{h_1}}{\mu}\right]}{\alpha\sin(\frac{\pi}{\alpha})}+\sum_{t>t_j}\frac{L^*}{1+|\omega|(t-t_j)^{\alpha}}\right\}\|z_1-z_2\|\\
&\leq \Lambda\|z_1-z_2\|.
\end{aligned}
$$

By (4.5), J_1 is contraction. Therefore, by utilizing the Banach contraction principle, there exist a unique fixed point $z\in\mathcal{WPAA}^o_{\Omega}(\mathcal{R},Y,\wp_*)$. Thus the model (1.1) has a unique piecewise $\mathcal{WPAA}$ mild solution. $\square$

5 Example

We consider the subsequent model:

$$
\begin{cases}
\partial_t^{\alpha}z(t,\chi)=\partial_{\chi}^2\left(z(t,\chi)-uz(t,\chi)\right)+\partial_t^{\alpha-1}\nu_1\Bigg[\sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)z(t,\chi)\\
+e^{-|t|}\sin z(t,\chi)+\sin\left(\displaystyle\int_0^t e^{(t-s)}h_1(s,z(s,\chi))ds\right)\Bigg], t\in\mathcal{R}, z\in[0,\pi],\\
\triangle z(t_j,\chi)=I_j(z(t_j,\chi))=\nu_2\Big[\sin\left(\frac{1}{2+\sin j}\right)z(t_j,\chi)+e^{-j}\cos z(t_j,\chi)\Big]\\
z(t,0)=z(t,\pi)=0,\quad t\in\mathcal{R},
\end{cases}
\tag{5.1}
$$

where ν_1,ν_2 are constants. Consider $Y=L^2[0,\pi]$ with norm $\|.\|$ and inner product $(\cdot,\cdot)_2$ and describe the operator $A:D(A)\subset Y\to Y$ given by $Az=\dfrac{\partial^2 z(\chi)}{\partial\chi^2}-2z(\chi)$ with domain $D(A)=\{z(.)\in Y: z^{'}\in Y, z^{''}\in Y$ is absolutely continuous on $[0,\pi], z(0)=z(\pi)=0\}$.
It is prominent that $G_1(t,y_1(\chi),y_2(\chi))=P(t,y_1(\chi))+y_2(\chi)\in\mathcal{PAA}^o_{\Omega}(\mathcal{R},Y,\wp_*)$, where

$$
P(t,y_1(\chi))=\nu_1\Big[\sin\left(\frac{1}{2+\sin t+\sin\pi t}\right)y_1(\chi)+e^{-|t|}\sin y_1(\chi)\Big]
$$

and

$$
y_2(\chi)=\nu_1\sin\left(\int_0^t e^{t-s}h_1(s,y_2(\chi))ds\right).
$$

Now

$$
\begin{aligned}
&\|P(t,x_1(\chi)) - P(t,x_2(\chi))\| \\
&\le \nu_1 \Bigg[\Big\| \sin\left(\frac{1}{2+\sin t+\sin \pi t}\right) x_1(\chi) - \sin\left(\frac{1}{2+\sin t+\sin \pi t}\right) x_2(\chi)\Big\| \\
&\quad + \| \sin x_1(\chi) - \sin x_2(\chi)\| \Bigg] \\
&\le \nu_1\|x_1 - x_2\| + \nu_1\|x_1 - x_2\| \le 2\nu_1\|x_1 - x_2\|.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\|G_1(t,x_1,x_2) - G_1(t,x_3,x_4)\| \\
&\le 2\nu_1\|x_1 - x_3\| + \nu_1\|\sin x_2 - \sin x_4\| \\
&\le 2\nu_1\|x_1 - x_3\| + \nu_1\|x_2 - x_4\| \le 3\nu_1\Big[\|x_1 - x_3\| + \|x_2 - x_4\|\Big], \\
&\qquad \text{for all } t \in \mathcal{R}, x_i \in Y, i = 1,2,3,4.
\end{aligned}
$$

Thus, $G_1(t,y_1,y_2) \in \mathcal{WPAA}^o_\Omega(\mathcal{R},Y,\wp_*)$.

Moreover,

$$
\begin{aligned}
&\|I_j(x_1) - I_j(x_2)\| \\
&\le \nu_2\Big[\Big\|\cos\left(\frac{1}{\sin j+\sin\sqrt{2j}}\right)\Big\|\|x_1(\chi) - x_2(\chi)\| \\
&\quad + |e^{-j}|\Big\|\cos x_1(\chi) - \cos x_2(\chi)\Big\|\Big] \\
&\le \nu_2\|x_1 - x_2\| + \nu_2\|x_1 - x_2\| \le 2\nu_2\|x_1 - x_2\|.
\end{aligned}
$$

Hence, the problem (5.1) can be restructured as a descriptive problem (1.1) and assumptions (H1), (H3)-(H5) hold with $C = M = C_k = \mu = 1, \omega = -1, \nu_1 = \nu_2 = \frac{1}{15}, L_{G_1} = 3\nu_1 = \frac{1}{5}, L_{h_1} = \frac{1}{15}, L^* = 2\nu_2 = \frac{2}{15}, \alpha = \frac{3}{2}$.

Thus, Theorem 4.1 asserts that the problem (5.1) has at least one piecewise $\mathcal{WPAA}$ solution whenever

$$
\sum_{t>t_j} \frac{L^*}{1+|\omega|(t-t_j)^\alpha} = 0.0667 < 1.
$$

Moreover, Theorem 4.2 states that the problem (5.1) has a unique piecewise $\mathcal{WPAA}$ solution whenever

$$
CM\left\{\frac{\pi L_{G_1}|\omega|^{-\frac{1}{\alpha}}\left[1+\frac{C_k L_{h_1}}{\mu}\right]}{\alpha\sin(\frac{\pi}{\alpha})} + \sum_{t>t_j}\frac{L^*}{1+|\omega|(t-t_j)^\alpha}\right\} = 0.5826 < 1.
$$

6 Conclusion

In this manuscript, we investigate many significant results on the notion of $\mathcal{WPAA}$ functions with impulses. The utilization of this function throughout the study of many differential equations fascinated several mathematicians and substantial research works have been conducted. Recently, the application of this function throughout the study of many fractional order differential equations gained a lot more interest. Moreover, composition theorems are essential in establishing the existence and uniqueness results of $\mathcal{AA}$ class solutions to many differential equations. So we have derived a composition theorem for $\mathcal{WPAA}$ functions. This present system can also be substantially enhanced with the existence, controllability and stability results by applying various kinds of impulses.

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