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On the explicit geometry of a certain blowing-up of a smooth quadric

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Abstract

Using the high symmetry in the geometry of a smooth projective quadric, we construct effectively new families of smooth projective rational surfaces whose nef divisors are regular, and whose effective monoids are finitely generated by smooth projective rational curves of negative self-intersection. Furthermore, the Cox rings of these surfaces are finitely generated, the dimensions of their anticanonical complete linear systems are zero, and their nonzero nef divisors intersect positively the anticanonical ones. And in two special cases, we give efficient ways of describing any effective divisor class in terms of the given minimal generating sets for the effective monoids of these surfaces. The ground field of our varieties is algebraically closed of arbitrary characteristic.

1 Introduction

For a given finite set of points $P_1, \ldots, P_r$ of the smooth projective quadric surface $\mathbb{P}^1_k \times \mathbb{P}^1_k$, where k is an algebraically closed field of arbitrary characteristic, consider the surface X_r obtained as the blowing-up of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ at these r points and denote by $\pi_r : X_r \longrightarrow \mathbb{P}^1_k \times \mathbb{P}^1_k$ the induced morphism. One can ask the following questions: Does X_r hold finitely many irreducible reduced curves of negative self-intersection? Assuming that there are only finitely many such curves, is it possible to determine them explicitly?

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The answers, obviously, depend on the points. For the surface $\mathbb{P}^1_k \times \mathbb{P}^1_k$ is endowed naturally with a surjective morphism $\pi : \mathbb{P}^1_k \times \mathbb{P}^1_k \longrightarrow \mathbb{P}^1_k$ given by the second projection, that is $\pi(x, y) = y$ for every $x, y \in \mathbb{P}^1_k$. Recall that the fibre F_y of π over $y \in \mathbb{P}^1_k$ is equal to $\pi^{-1}(y)$ which is, of course, isomorphic to the projective line $\mathbb{P}^1_k$; similarly the horizontal line H_x of π associated to $x \in \mathbb{P}^1_k$ is equal to $\{x\} \times \mathbb{P}^1_k$. It is worth noting that two fibres (respectively, horizontal lines) of π do not intersect, and any fibre F of π intersects any horizontal line H of π exactly in one point. Thus the self-intersection of any irreducible reduced curve on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ is always nonnegative. In particular, $\mathbb{P}^1_k \times \mathbb{P}^1_k$ is not isomorphic to the projective plane $\mathbb{P}^2_k$ since any curve in $\mathbb{P}^2_k$ has positive self-intersection. Now, if r is equal to one, then X_1 holds only three irreducible reduced curves of negative self-intersection. Indeed, assuming that $P_1 = (x, y)$ for some x and $y \in \mathbb{P}^1_k$, then these negative curves are the exceptional divisor E_{P_1} of the morphism $\pi_1 : X_1 \longrightarrow \mathbb{P}^1_k \times \mathbb{P}^1_k$ and the strict transform by π_1 of the fibre F_y (respectively, the horizontal line H_x) that passes through P_1. In this special case, all these three negative curves are (-1)-curves, that is, smooth projective lines of self-intersection -1. If r is equal to two and the points P_1 and P_2 belong to the same fibre F_{12} of π, then X_2 holds only four (-1)-curves (i.e., the exceptional divisors E_{P_1}, E_{P_2} of π_2 and the strict transform by π_2 of the horizontal line H_{P_1} (respectively, H_{P_2}) that passes through P_1 (respectively, P_2)) and one (-2)-curve (i.e., a smooth projective line of self-intersection -2). The latter corresponds to the strict transform by π_2 of F_{12} that passes through P_1 and P_2. However, if r is equal to eight, it may happen that X_8 contain an infinite number of (-1)-curves (see for example [18, Theorem 1]).

In order to put the above problem in a more general setting that allows us to relate our results with the many published ones, we proceed as follows: let X be a smooth projective surface defined over k. Consider the free $\mathbb{Z}$-module $\mathrm{Div}(X)$ generated by the irreducible reduced curves on X. An element of such module is called a divisor on X, and such divisor is said to be *effective* it is a nonnegative linear combination of irreducible and reduced curves. It turns out that by intersection theory, $\mathrm{Div}(X)$ is endowed with a bilinear form which we denote by a dot. In particular, if D is a divisor, then the *self-intersection* of D means D^2. Let D_1 and D_2 be divisors on X, D_1 and D_2 are *numerically equivalent*, and we write $D_1 \equiv D_2$, if $D_1.C = D_2.C$ for every irreducible reduced curve on X. We denote $\mathrm{Div}(X)$ modulo numerical equivalence $\equiv$ by $\mathrm{NS}(X)$, this is the so-called the Néron-Severi group of X. A basic fact is that $\mathrm{NS}(X)$ is a free $\mathbb{Z}$-module of finite rank $\rho(X)$ which is usually called the Picard number of X. If X is obtained by blowing up a finite number of points $P_1, \ldots, P_r$ of $\mathbb{P}^1_k \times \mathbb{P}^1_k$, then $\mathrm{NS}(X)$ is the free $\mathbb{Z}$-module generated by the class of the pullback of a fibre F, the class of the pullback of a horizontal line H and the classes $\mathcal{E}_i$ of the blowings-up E_i of the points P_i. Thus, $\rho(X) = 2 + r$.

Moreover, we denote by $\mathrm{M}(X)$ the submomoid of $\mathrm{NS}(X)$ consisting of the classes of effective divisors on X. For more details see [32, Chapter V].

One of the main interesting problems, at least in birational geometry, coding theory, Poincaré Problem, and the so-called Harbourne-Hirschowitz Conjecture (also, known nowadays as the Segre-Harbourne-Gimigliano-Hirschowitz Conjecture), is to understand the effective monoid $\mathrm{M}(X)$ of X. Understanding $\mathrm{M}(X)$ means the following:

1. Is $\mathrm{M}(X)$ finitely generated? And if yes,
2. What is a minimal generating set for $\mathrm{M}(X)$? And,
3. Is any nef divisor on X regular? Here, a divisor D on X is said to be nef if its intersection number with any element of $\mathrm{M}(X)$ is nonnegative, and is said to be regular if $H^1(X, \mathcal{O}_X(D))$ vanishes, where $\mathcal{O}_X(D)$ is an invertible sheaf associated naturally to D, and $H^1(X, \mathcal{O}_X(D))$ is the first cohomology group of $\mathcal{O}_X(D)$. Of course as it is well-known, the latter has a structure of a finite dimensional vector space over k.

The aim of this work is to give a partial answer to the above questions when X is a surface whose minimal model is a smooth projective quadric surface. Note that the Picard number of X may be as large as one wishes. Our techniques are based mainly on intersection theory on surfaces and specializing points.

Smooth projective rational surfaces with Picard number less than or equal to ten are very well understood nowadays, see [46], [39], [31], [5], [13], [35],[36], [43],[44] and [48]. However, when the Picard number of X is larger than ten there are only few families whose effective monoids are almost understood, see for instance [3], [4], [14], [15],[16] [19], [21], [22], [37], [38], [8], [9], [23] and [24]. Below, we provide more families of smooth projective rational surfaces whose effective monoids are not only finitely generated but explicitly determined by giving the minimal generating set, see Theorem 2.1, Theorem 6.1, and Theorem 6.4. These surfaces have the property that the nefness ensures the regularity.

For any smooth projective rational surface W having the projective plane $\mathbb{P}^2_k$ as a minimal model and having an integral anticanonical divisor of nonpositive self-intersection, the finite generation of the effective monoid $\mathrm{NS}(W)$ of W ensures that the set of (-2)-curves on W is finite, and spans a $\mathbb{Z}$-submodule of $\mathrm{NS}(W)$ of rank equal to $-1+\rho(W)$, here $\rho(W)$ is the rank of the free $\mathbb{Z}$-module $\mathrm{NS}(W)$, see [26, Theorem (3.1), p. 142]. However, our surfaces X do not share this property:

The set of (-2)*-curves on* X *spans a* $\mathbb{Z}$*-submodule of* $\mathrm{NS}(X)$ *of rank less than* $-1+\rho(X)$. See Section 2 for the definition of X, and Corollary 4.4.

From the algebraic point of view, our surfaces provide new families of smooth projective rational surfaces whose Cox rings are finitely generated (see Theorem 5.7 below), and whose anticanonical complete linear systems have dimensions zero (see Corollary 3.3). In this direction, see the results obtained in [6], [25], [20], [10], [11], [12], [18],[22], [45], [49] and [40]. Here, the Cox ring of a smooth projective variety T is the $k-$algebra $\mathrm{Cox}(T)$ given by

$$\mathrm{Cox}(T) = \bigoplus_{\mathcal{L}\in \mathrm{Pic}(T)} H^0(T,\mathcal{L}),$$

where $\mathrm{Pic}(T)$ is the Picard group of T, and $H^0(T,\mathcal{L})$ is the finite dimensional $k-$vector space of global sections of $\mathcal{L}$. For more details see [6] and [34].

This paper is structured as follows. In Section 1, we principally stated some of our main results (see Theorems 2.1, and 2.2, and Corollary 2.3). Section 2 (respectively, Section 3) deals with the proof of the finiteness of the set of (-1)-curves (respectively, of (-2)-curves) on X. Section 4 gives the fact that the nefness of divisors implies their regularities. Section 5 states Lemma 5.5, an ingredient that we needed for showing the finite generation of the Cox rings of our surfaces. Section 6 deals with our efficient computational aspect for the effective monoids of some surfaces obtained as two specializations in our construction, in fact these special surfaces are constructed by following either a horizontal section (see Theorem 6.1), or a fibre (see Theorem 6.4); in particular by means of examples, it illustrates that the Riemann-Roch Theorem for surfaces may not guaranty the effectiveness of some divisors, however our results do prove the effectiveness of such divisors (see Example, p. 17). Also, it extends, e.g., the results obtained in [47, Theorem 1, p. 420; and Theorem 2, p. 424], [48, Theorems 1 and 2, p.120], [41], [42] and [17].

2 Notation and preliminary results

In order to state our results, we need to fix some notation. We denote by Σ_0 a smooth projective quadric surface, and we remind the reader that it is the rational ruled surface defined by the locally free sheaf $\mathcal{O}_{\mathbb{P}^1_k}\oplus\mathcal{O}_{\mathbb{P}^1_k}$ of rank two on the projective line $\mathbb{P}^1_k$. It is well-known that the set $\{\mathcal{C}_0^m,\mathcal{F}^m\}$ is a minimal set of generators of $\mathrm{NS}(\Sigma_0)$ as $\mathbb{Z}-$Module, where $\mathrm{NS}(\Sigma_0)$ is the Néron-Severi group of Σ_0 (that is, the quotient group of the abelian group of divisors on Σ_0 modulo numerical equivalence), $\mathcal{C}_0^m$ is the class of a section C_0^m of Σ_0 and $\mathcal{F}^m$ is the class of a fibre f^m of Σ_0. The intersection form on Σ_0 is given by the three equalities $(\mathcal{C}_0^m)^2=0$, $(\mathcal{F}^m)^2=0$, and $\mathcal{C}_0^m.\mathcal{F}^m=1$, for more details, see for example [32, Section 2, pp. 369-383], and [48, Theorem 1, p. 120]. On the other hand, for a fixed positive integer r, we denote the blowing-up of Σ_0 at r closed points (not necessarily ordinary points) of such a surface by Y_0. Henceforth, there is a natural projective birational morphism π between Y_0 and Σ_0. A minimal set of generators of $\mathrm{NS}(Y_0)$ as $\mathbb{Z}-$Module is the set $\{\mathcal{C}_0, \mathcal{F}, -\mathcal{E}_1, -\mathcal{E}_2, \ldots, -\mathcal{E}_r\}$, where $\mathcal{C}_0$ is the class of the total transform of ${C_0}^m$ by π, $\mathcal{F}$ is the class of the total transform of a fibre f^m of Σ_0 by π, and $\mathcal{E}_j$ is the class of the exceptional divisor corresponding to the j^{th} point blown-up for every $j \in \{1, 2, \ldots, r\}$. The intersection form on Y_0 is given by the following equalities: $\mathcal{C}_0^2 = 0$, $\mathcal{F}^2 = 0$, $\mathcal{C}_0.\mathcal{F} = 1$, $\mathcal{C}_0.\mathcal{E}_j = 0$, $\mathcal{F}.\mathcal{E}_j = 0$, $\mathcal{E}_j^2 = -1$ for every $j \in \{1, 2, \ldots, r\}$, and $\mathcal{E}_i.\mathcal{E}_j = 0$ for $i, j \in \{1, 2, \ldots, r\}$ such that $i \neq j$.

Let p be a closed point of Σ_0 and s a nonnegative integer. A constellation $\mathfrak{C}$ with origin p means the following: $\mathfrak{C} = \bigcup_{i=0}^{s} \mathfrak{C}_i$, here $\mathfrak{C}_0$ is the set consisting of only one element which is the point p, $\mathfrak{C}_1$ is a nonempty finite subset of the exceptional divisor of the blowing-up of Σ_0 at p and by induction $\mathfrak{C}_{i+1}$ is a nonempty finite subset of the exceptional locus of the blowing-up of the i^{th} blowing-up of Σ_0 (which includes $\mathfrak{C}_i$) at the points of $\mathfrak{C}_i$, for every $i = 0, \ldots, s-1$. We refer to s as the rank of the constellation. Moreover, $\mathfrak{C}$ will be called a chain if $\mathfrak{C}_i$ consists of only one element, for every $i = 0, \ldots, s$. The union of two constellations with different origins will be called a constellation with two origins. For more information about the constellations see [37, Subsection 2.2, p. 1222], and [2].

Using the above notation, we proceed to construct our family of smooth projective rational surfaces (see the Figure 1 for a simple illustration). Let $a, b, d, a', b', d', \rho_1, \rho_2, \ldots, \rho_d, \rho'_1, \rho'_2, \ldots, \rho'_{d'}$ be nonnegative integers, and let O and O' be two closed points of Σ_0. Consider V (respectively, H) the unique fibre (respectively, the unique horizontal smooth projective rational curve) in Σ_0 that passes through O, in particular V (respectively, H) is numerically equivalent to f^m (respectively, to C_0^m). Similarly, consider V' (respectively, H') the unique fibre (respectively, the unique horizontal smooth projective rational curve) in Σ_0 that passes through O'. To avoid special cases, we assume throughout that $V \neq V'$, $H \neq H'$, and a, b, a', and b' are larger than one, and d, and d' are positive. To these data, we shall associate the surface X obtained as the blowing-up of the quadric surface Σ_0 at its closed zero-dimensional subscheme $Z = \mathfrak{C} \bigcup \mathfrak{C}'$, where $\mathfrak{C}$ (respectively, $\mathfrak{C}'$) is the constellation with origin O (respectively, O') such that:

1. $\mathfrak{C} = \{O, A_1, \ldots, A_a, B_1, \ldots, B_b, D_1, D_1^1, \ldots, D_1^{\rho_1}, D_2, D_2^1, \ldots, D_2^{\rho_2}, \ldots, D_d, D_d^1, \ldots, D_d^{\rho_d}\}$, where
   (a) A_1 is the point of the first neighborhood of O determined uniquely by the intersection of the strict transform of V via the blowing-up of

Σ_0 at O and the corresponding exceptional divisor E_O, and for each $j = 2, \ldots, a$, A_j is the point given uniquely by the intersection of the first infinitesimal neighborhood of A_{j-1} and the strict transform of V.

(b) B_1 is the point of the first neighborhood of O determined uniquely by the intersection of the strict transform of H via the blowing-up of Σ_0 at O and the corresponding exceptional divisor E_O, and for each $j = 2, \ldots, b$, B_j is the point given uniquely by the intersection of the first infinitesimal neighborhood of B_{j-1} and the strict transform of H.

(c) For $t = 1, \ldots, d$, D_t is a general point of the first neighborhood of O, in particular it is other than A_1 and B_1 with $D_t \neq D_{t'}$ if $t \neq t'$, and for each $j = 1, \ldots, \rho_t$, the point D_t^j is the point which is uniquely given by the intersection of the first infinitesimal neighborhood of D_t^{j-1} and the strict transform of E_O with the convention $D_t^0 = D_t$. And

2. $\mathfrak{C}' = \{O', A'_1, \ldots, A'_{a'}, B'_1, \ldots, B'_{b'}, D'_1, {D'_1}^1, \ldots, {D'_1}^{\rho'_1}, D'_2, {D'_2}^1, \ldots, {D'_2}^{\rho'_2}, \ldots, D'_{d'}, {D'_{d'}}^1, \ldots, {D'_{d'}}^{\rho'_{d'}}\}$, where

(a) A'_1 is the point of the first neighborhood of O' determined uniquely by the intersection of the strict transform of V' via the blowing-up of Σ_0 at O' and the corresponding exceptional divisor $E_{O'}$, and for each $j = 2, \ldots, a$, A'_j is the point given uniquely by the intersection of the first infinitesimal neighborhood of A'_{j-1} and the strict transform of V'.

(b) B'_1 is the point of the first neighborhood of O' determined uniquely by the intersection of the strict transform of H' via the blowing-up of Σ_0 at O' and the corresponding exceptional divisor $E_{O'}$, and for each $j = 2, \ldots, b$, B'_j is the point given uniquely by the intersection of the first infinitesimal neighborhood of B'_{j-1} and the strict transform of H'.

(c) For $t = 1, \ldots, d'$, D'_t is a general point of the first neighborhood of O', in particular it is other than A'_1 and B'_1 with $D'_t \neq D'_{t'}$ if $t \neq t'$, and for each $j = 1, \ldots, \rho'_t$, the point ${D'_t}^j$ is the point which is uniquely determined by the intersection of the first infinitesimal neighborhood of ${D'_t}^{j-1}$ and the strict transform of $E_{O'}$ with the convention ${D'_t}^0 = D'_t$.

So, the Néron-Severi group $\mathrm{NS}(X)$ of X is a free $\mathbb{Z}$-module of rank $\rho(X) = 4 + a + a' + b + b' + d + d' + \sum_{i=1}^{i=d} \rho_i + \sum_{j=1}^{j=d'} \rho'_j$. Moreover, it has naturally

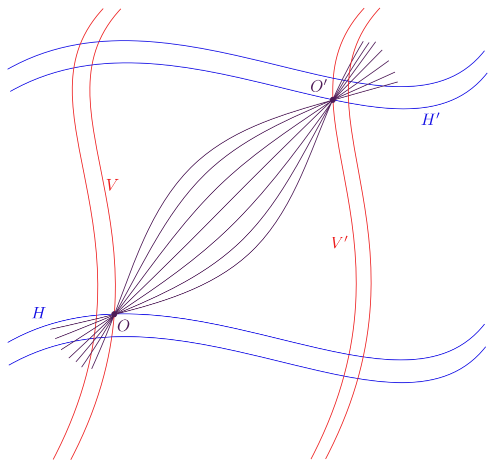

Figure 1: The curves V, H, V' and H' are used in the construction of X as well as the points O and O'.

the following integral basis:

$$\Big(\mathcal{C}_0; \mathcal{F}; -\mathcal{E}_O, -\mathcal{E}_{A_1}, \ldots, -\mathcal{E}_{A_a}, -\mathcal{E}_{B_1}, \ldots, -\mathcal{E}_{B_b}, -\mathcal{E}_{D_1}, \ldots, -\mathcal{E}_{D_1^{\rho_1}}, \ldots,$$
$$-\mathcal{E}_{D_d}, \ldots, -\mathcal{E}_{D_d^{\rho_d}}; -\mathcal{E}_{O'}, -\mathcal{E}_{A'_1}, \ldots, -\mathcal{E}_{A'_{a'}}, -\mathcal{E}_{B'_1}, \ldots, -\mathcal{E}_{B'_{b'}}, -\mathcal{E}_{D'_1}, \ldots,$$
$$-\mathcal{E}_{{D'_1}^{\rho'_1}}, \ldots, -\mathcal{E}_{D'_{d'}}, \ldots, -\mathcal{E}_{{D'_{d'}}^{\rho'_d}}\Big)$$

which is defined by:

- $\mathcal{C}_0$ is the class of a general section C_0^m of Σ_0,
- $\mathcal{F}$ is the class of a general fibre f^m of Σ_0,
- $\mathcal{E}_O$ (respectively, $\mathcal{E}_{O'}$) is the class of the exceptional divisor corresponding to the point O (respectively, O').
- $\mathcal{E}_{A_i}$ (respectively, $\mathcal{E}_{A'_j}$) is the class of the exceptional divisor corresponding to the point A_i (respectively, A'_j) for every $i = 1, \ldots, a$ (respectively, $j = 1, \ldots, a'$).
- $\mathcal{E}_{B_i}$ (respectively, $\mathcal{E}_{B'_j}$) is the class of the exceptional divisor corresponding to the point B_i (respectively, B'_j) for every $i = 1, \ldots, b$ (respectively, $j = 1, \ldots, b'$).
- $\mathcal{E}_{D_t^u}$ is the class of the exceptional divisor corresponding to the point D_t^u, for every $t = 1, \ldots, d$ and $u = 0, \ldots, \rho_t$.

- $\mathcal{E}_{D'^u_t}$ is the class of the exceptional divisor corresponding to the point D'^u_t, for every $t = 1, \dots, d'$ and $u = 0, \dots, \rho'_t$.

Any class of a divisor on X can be then represented by a tuple with integer entries such as:

$$(x; y; c, \lambda_1, \dots, \lambda_a, \mu_1, \dots, \mu_b, \delta_1, \dots, \delta_1{}^{\rho_1}, \dots, \delta_d, \dots, \delta_d{}^{\rho_d};$$
$$c', \lambda'_1, \dots, \lambda'_{a'}, \mu'_1, \dots, \mu'_{b'}, \delta'_1, \dots, \delta'_1{}^{\rho'_1}, \dots, \delta'_{d'}, \dots, \delta'_{d'}{}^{\rho'_d}).$$

As a consequence of our results below, we succeed to give explicitly the minimal generating set for the effective monoid $\mathrm{M}(X)$ of X (Figure 2 gives all the irreducible components of the reduced anticanonical divisor (they form a polygon), and also other negative curves), in particular every generator is of negative self-intersection, as the following theorem shows:

Theorem 2.1. *With notation as above. The effective monoid* $\mathrm{M}(X)$ *of* X *is finitely generated by smooth projective rational curves of negative self-intersection. More precisely,* $\mathrm{M}(X)$ *is generated by the following elements:*

1. $\mathcal{E}_O - \mathcal{E}_{A_1} - \mathcal{E}_{B_1} - \sum_{i=1}^{i=d} \sum_{j=0}^{j=\rho_i} \mathcal{E}_{D_i^j}$, *and*
2. $\mathcal{F} - \mathcal{E}_O - \sum_{i=1}^{i=a} \mathcal{E}_{A_i}$, *and*
3. $\mathcal{C}_0 - \mathcal{E}_O - \sum_{i=1}^{i=b} \mathcal{E}_{B_i}$, *and*
4. $\mathcal{E}_{O'} - \mathcal{E}_{A'_1} - \mathcal{E}_{B'_1} - \sum_{i=1}^{i=d'} \sum_{j=0}^{j=\rho'_i} \mathcal{E}_{D'^j_i}$, *and*
5. $\mathcal{F} - \mathcal{E}_{O'} - \sum_{i=1}^{i=a'} \mathcal{E}_{A'_i}$, *and*
6. $\mathcal{C}_0 - \mathcal{E}_{O'} - \sum_{i=1}^{i=b'} \mathcal{E}_{B'_i}$, *and*
7. $\mathcal{E}_{A_a}, \mathcal{E}_{B_b}, \mathcal{E}_{A'_{a'}}, \mathcal{E}_{B'_{b'}}$, *and*
8. $\mathcal{E}_{D_i^{\rho_i}}, \mathcal{E}_{D'^{\rho'_j}_j}$ *for any* $i \in \{1, \dots, d\}$ *and any* $j \in \{1, \dots, d'\}$, *and*
9. $\mathcal{C}_0 + \mathcal{F} - \mathcal{E}_O - \mathcal{E}_{O'} - \mathcal{E}_{D'_t}$ *for any* $t \in \{1, \dots, d'\}$, *and*
10. $\mathcal{C}_0 + \mathcal{F} - \mathcal{E}_O - \mathcal{E}_{O'} - \mathcal{E}_{D_t}$, *for any* $t \in \{1, \dots, d\}$, *and*
11. $\mathcal{E}_{A_i} - \mathcal{E}_{A_{i+1}}$, *for any* $i \in \{1, \dots, a-1\}$, *and*
12. $\mathcal{E}_{B_j} - \mathcal{E}_{B_j}$, *for any* $j \in \{1, \dots, b-1\}$, *and*
13. $\mathcal{E}_{A'_i} - \mathcal{E}_{A'_{i+1}}$, *for any* $i \in \{1, \dots, a'-1\}$, *and*

14. $\mathcal{E}_{B'_j} - \mathcal{E}_{B'_{j+1}}$ *for any* $j \in \{1, \ldots, b'-1\}$, *and*

15. $\mathcal{E}_{D_t^u} - \mathcal{E}_{D_t^{u+1}}$, *for any* $t \in \{1, \ldots, d\}$ *and for any* $u \in \{0, \ldots, \rho_t - 1\}$, *and*

16. $\mathcal{E}_{{D'}_t^u} - \mathcal{E}_{{D'}_t^{u+1}}$, *for any* $t \in \{1, \ldots, d'\}$ *and any* $u \in \{0, \ldots, \rho'_t - 1\}$.

Proof. Since the Picard number of the smooth projective anticanonical rational surface X is larger than two (see Proposition 3.2 below for the fact that X is rational and anticanonical), $\mathrm{M}(X)$ is generated by the classes of the negative integral curves on X and by $-\mathcal{K}_X$, where $\mathcal{K}_X$ is the class of a canonical divisor K_X on X, see [38, Lemma 4.1, p. 108]. Now, let E be an integral curve on X of negative self-intersection, and let $\mathcal{E}$ be its class in $\mathrm{NS}(X)$. If the intersection number $\mathcal{E}.\mathcal{K}_X$ is positive, then E is a fixed component of the anticanonical complete linear system of X, hence $\mathcal{E}$ is equal to $\mathcal{E}_O - \mathcal{E}_{A_1} - \mathcal{E}_{B_1} - \sum_{i=1}^{i=d} \sum_{j=0}^{j=\rho_i} \mathcal{E}_{D_i^j}$, $\mathcal{F} - \mathcal{E}_O - \sum_{i=1}^{i=a} \mathcal{E}_{A_i}$, $\mathcal{C}_0 - \mathcal{E}_O - \sum_{i=1}^{i=b} \mathcal{E}_{B_i}$, $\mathcal{E}_{O'} - \mathcal{E}_{A'_1} - \mathcal{E}_{B'_1} - \sum_{i=1}^{i=d'} \sum_{j=0}^{j=\rho'_i} \mathcal{E}_{{D'}_i^j}$, $\mathcal{F} - \mathcal{E}_{O'} - \sum_{i=1}^{i=a'} \mathcal{E}_{A'_i}$, or $\mathcal{C}_0 - \mathcal{E}_{O'} - \sum_{i=1}^{i=b'} \mathcal{E}_{{B'}_i}$. However, if the intersection number $\mathcal{E}.\mathcal{K}_X$ is negative, then E is a (-1)-curve (that is, a smooth projective rational curve of self-intersection -1), and $\mathcal{E}$ is equal to one of the list in Proposition 3.4. Lastly, if $\mathcal{E}$ is orthogonal to $\mathcal{K}_X$, then E is a (-2)-curve (i.e., a smooth rational curve of self-intersection -2), and $\mathcal{E}$ is equal to one of the list in Proposition 4.2. Conversely, every element in the list of the theorem is the class of a smooth projective rational curve of negative self-intersection. From the fact that $-\mathcal{K}_X$ is a linear combination of negative integral curves with positive integers, and these curves are in the list of the theorem, the result follows. So, we are done. $\square$

We are able to determine the dimension of complete linear systems associated to nef divisors on X, this can be seen using the following theorem:

Theorem 2.2. *With notation as above. Every nef divisor on X is regular.*

Proof. Let D be a nef divisor on X. If the intersection number $D.K_X$ is equal to zero, then Proposition 5.1 ensures that D is equal to zero. Hence, it is regular since X is rational. If $D.K_X$ is larger than one, then the regularity of D is given by [28, Teorem III.1 (a), and (b), p. 1197]. $\square$

Corollary 2.3. *With notation as above. Let D be a nef divisor on X. The dimension of the complete linear system $|D|$ associated to D is equal to*

$$\frac{1}{2}\left(D^2 - D.K_X\right),$$

where K_X denotes a canonical divisor on X.

Proof. The equality follows from the Riemann-Roch Theorem and the facts that D is regular by the last theorem, and $K_X - D$ is not effective by [27, Proposition 4]. □

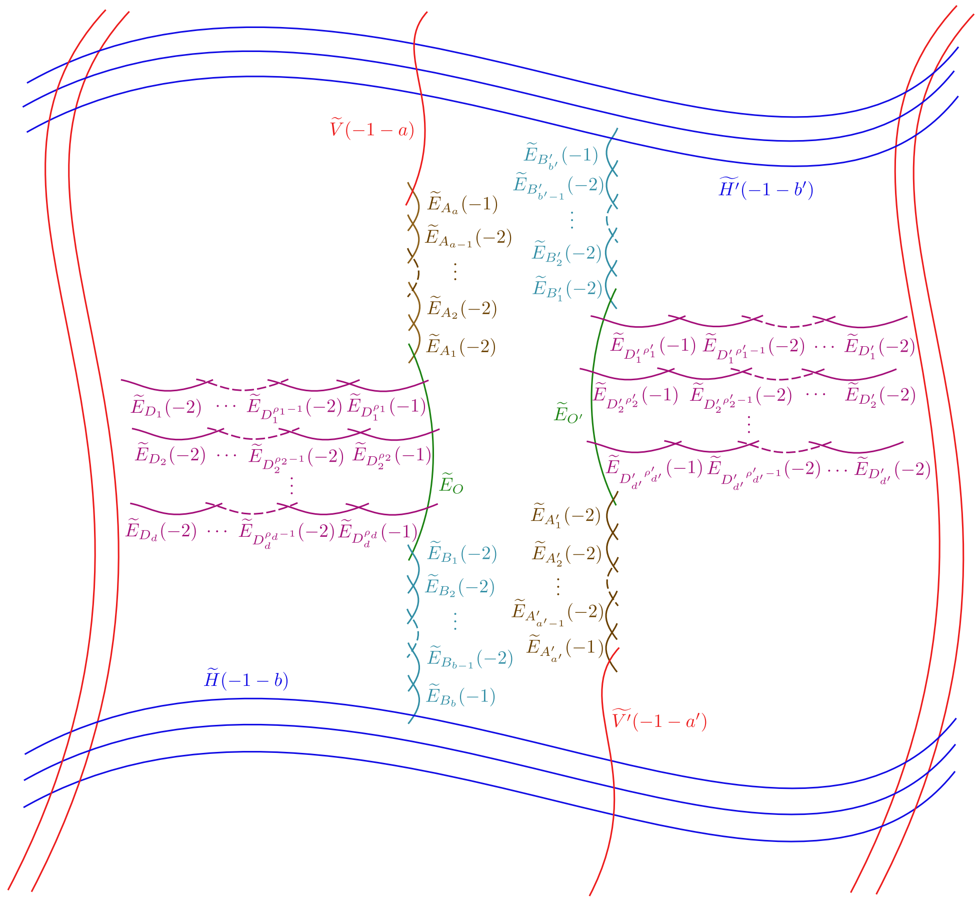

Figure 2: All the negative curves on X are in this figure except the ones which are coming from some diagonals.

3 The set of (-1)-curves on X

Definition 3.1. With notation as above. A curve E on X is a (-1)-curve if $E^2 = E.K_X = -1$, where K_X denotes a canonical divisor on X.

Here, we determine explicitly the set of (-1)-curves on X. In particular, we deduce that its cardinality is equal to $2(2+d+d')$.

To this end, we show first that X is anticanonical, that is, supporting an effective anticanonical divisor.

Proposition 3.2. *With notation as above, X is rational and anticanonical.*

Proof. By construction X is birationally equivalent to Σ_0, thus X is rational. On the other hand, the class $\mathcal{K}_X$ of a canonical divisor on X in $\mathrm{NS}(X)$ is given by:

$$-\left(\mathcal{E}_O - \mathcal{E}_{A_1} - \mathcal{E}_{B_1} - \sum_{i=1}^{i=d}\sum_{j=0}^{j=\rho_i} \mathcal{E}_{D_i^j}\right) - \left(\mathcal{F} - \mathcal{E}_O - \sum_{i=1}^{i=a} \mathcal{E}_{A_i}\right) -$$

$$\sum_{i=1}^{i=a-1} (\mathcal{E}_{A_i} - \mathcal{E}_{A_{i+1}}) - \mathcal{E}_{A_a} - \left(\mathcal{C}_0 - \mathcal{E}_O - \sum_{i=1}^{i=b} \mathcal{E}_{B_i}\right) - \sum_{i=1}^{i=b-1} (\mathcal{E}_{B_i} - \mathcal{E}_{B_{i+1}}) - \mathcal{E}_{B_b} -$$

$$\left(\mathcal{E}_{O'} - \mathcal{E}_{A'_1} - \mathcal{E}_{B'_1} - \sum_{i=1}^{i=d'}\sum_{j=0}^{j=\rho'_i} \mathcal{E}_{D'^j_i}\right) - \left(\mathcal{F} - \mathcal{E}_{O'} - \sum_{i=1}^{i=a'} \mathcal{E}_{A'_i}\right) -$$

$$\sum_{i=1}^{i=a'-1} (\mathcal{E}_{A'_i} - \mathcal{E}_{A'_{i+1}}) - \mathcal{E}_{A'_{a'}} - \left(\mathcal{C}_0 - \mathcal{E}_{O'} - \sum_{i=1}^{i=b'} \mathcal{E}_{B'_i}\right) - \sum_{i=1}^{i=b'-1} (\mathcal{E}_{B'_i} - \mathcal{E}_{B'_{i+1}}) - \mathcal{E}_{B'_{b'}}.$$

Therefore, X is anticanonical. $\square$

This implies that the anticanonical complete linear system of X is of dimension zero:

Corollary 3.3. *With notation as above, let K_X denotes a canonical divisor on X. The anticanonical complete linear system $|-K_X|$ of X is of dimension zero. Its unique element is reduced, and has $(6+a+b+a'+b')$ irreducible components, four of them are (-1)-curves.*

Now we handle the problem of determining the set of (-1)-curves on X.

Proposition 3.4. *With notation as above. Let E be an integral curve on X. If E is a (-1)-curve, then the class $\mathcal{E}$ of E in $\mathrm{NS}(X)$ is one of the following:*

1. $\mathcal{E}_{A_a}, \mathcal{E}_{B_b}, \mathcal{E}_{A'_{a'}}, \mathcal{E}_{B'_{b'}}$
2. $\mathcal{E}_{D_i^{\rho_i}}, \mathcal{E}_{D'^{\rho'_j}_j}$ *where* $i \in \{1, \dots, d\}$ *and* $j \in \{1, \dots, d'\}$,
3. $\mathcal{C}_0 + \mathcal{F} - \mathcal{E}_O - \mathcal{E}_{O'} - \mathcal{E}_{D'_t}$ *where* $t \in \{1, \dots, d'\}$,

4. $\mathcal{C}_0 + \mathcal{F} - \mathcal{E}_O - \mathcal{E}_{O'} - \mathcal{E}_{D_t}$ *where* $t \in \{1, \dots, d\}$.

Proof. $\mathcal{E}$ is determined in $\mathrm{NS}(X)$ by some tuple with integer entries such as

$$(x; y; c, \lambda_1, \dots, \lambda_a, \mu_1, \dots, \mu_b, \delta_1, \dots, {\delta_1}^{\rho_1}, \dots, \delta_d, \dots, {\delta_d}^{\rho_d};$$

$$c', \lambda'_1, \dots, \lambda'_{a'}, \mu'_1, \dots, \mu'_{b'}, \delta'_1, \dots, {\delta'_1}^{\rho'_1}, \dots, \delta'_{d'}, \dots, {\delta'_{d'}}^{\rho'_d}).$$

From the equation $\mathcal{E}.\mathcal{K}_X = -1$, we infer the following equality:

$$\left(c - \lambda_1 - \mu_1 - \sum_{i=1}^{i=d} \sum_{j=0}^{j=\rho_i} \delta_i^j\right) + \left(x - c - \sum_{i=1}^{i=a} \lambda_i\right) +$$

$$\sum_{i=1}^{i=a-1} (\lambda_i - \lambda_{i+1}) + \lambda_a + \left(y - c - \sum_{i=1}^{i=b} \mu_i\right) + \sum_{i=1}^{i=b-1} (\mu_i - \mu_{i+1}) + \mu_b +$$

$$\left(c' - \lambda'_1 - \mu'_1 - \sum_{i=1}^{i=d'} \sum_{j=0}^{j=\rho'_i} {\delta'}_i^j\right) + \left(x - c' - \sum_{i=1}^{i=a'} \lambda'_i\right) +$$

$$\sum_{i=1}^{i=a'-1} (\lambda'_i - \lambda'_{i+1}) + \lambda'_{a'} + \left(y - c' - \sum_{i=1}^{i=b'} \mu'_i\right) + \sum_{i=1}^{i=b'-1} (\mu'_i - \mu'_{i+1}) + \mu'_{b'} = 1.$$

In the last equality, every member of such sum is a nonnegative integer. For E is a (-1)-curve and we may assume that it is not equal to $E_{A_a}, E_{B_b}, E_{A'_{a'}}$ and $E_{B'_{b'}}$.
By symmetry, we need only to check the three following cases:

Case 1:

$$\left(c - \sum_{i=1}^{i=d} \sum_{j=0}^{j=\rho_i} \delta_i^j\right) = 1, \text{ and}$$

$$\left(c' - \sum_{i=1}^{i=d'} \sum_{j=0}^{j=\rho'_i} {\delta'}_i^j\right) = 0, \text{ and}$$

$$x = y = c = c', \text{ and}$$

$$\lambda_i = 0, \text{ for every } \mathrm{i} \in \{1, \dots, \mathrm{a}\}, \text{ and}$$

$$\mu_j = 0, \text{ for every } \mathrm{j} \in \{1, \dots, \mathrm{b}\}, \text{ and}$$

$$\lambda'_{i'} = 0, \text{ for every } \mathrm{i}' \in \{1, \dots, \mathrm{a}'\}, \text{ and}$$

$$\mu'_{j'} = 0, \text{ for every } \mathrm{j}' \in \{1, \dots, \mathrm{b}'\}.$$

Case 2:

$$\left(c - \sum_{i=1}^{i=d} \sum_{j=0}^{j=\rho_i} \delta_i^j\right) = 0, \text{ and}$$

$$\left(c' - \sum_{i=1}^{i=d'} \sum_{j=0}^{j=\rho'_i} \delta'^j_i\right) = 0, \text{ and}$$

$$x - c = 1, \text{ and}$$

$$x = y = c = c', \text{ and}$$

$$\lambda_i = 0, \text{ for every } \text{ i} \in \{1, \ldots, \text{a}\}, \text{ and}$$

$$\mu_j = 0, \text{ for every } \text{ j} \in \{1, \ldots, \text{b}\}, \text{ and}$$

$$\lambda'_{i'} = 0, \text{ for every } \text{ i}' \in \{1, \ldots, \text{a}'\}, \text{ and}$$

$$\mu'_{j'} = 0, \text{ for every } \text{ j}' \in \{1, \ldots, \text{b}'\}.$$

Case 3:

There exists $\alpha \in \{1, \ldots, a\}$ such that $\lambda_i = 1$ for every $i = 1, \ldots, \alpha$, and $\lambda_j = 0$ for every $j = \alpha + 1, \ldots, a$, and

$$\left(c - 1 - \sum_{i=1}^{i=d} \sum_{j=0}^{j=\rho_i} \delta_i^j\right) = 0, \text{ and}$$

$$x - c - \alpha = 0, \text{ and}$$

$$x = y = c = c', \text{ and}$$

$$\left(c' - \sum_{i=1}^{i=d'} \sum_{j=0}^{j=\rho'_i} \delta'^j_i\right) = 0, \text{ and}$$

$$\mu_j = 0, \text{ for every } \text{ j} \in \{1, \ldots, \text{b}\}, \text{ and}$$

$$\lambda'_{i'} = 0, \text{ for every } \text{ i}' \in \{1, \ldots, \text{a}'\}, \text{ and}$$

$$\mu'_{j'} = 0, \text{ for every } \text{ j}' \in \{1, \ldots, \text{b}'\}.$$

Assume that we are in the Case 1. The equation $\mathcal{E}^2 = -1$ implies that

$$1 = \left(\sum_{i=1}^{i=d}\sum_{j=0}^{j=\rho_i} {\delta_i^j}^2\right) + \left(\sum_{i=1}^{i=d'}\sum_{j=0}^{j=\rho'_i} {\delta'^j_i}^2\right).$$

Hence, either
$\left(\sum_{i=1}^{i=d}\sum_{j=0}^{j=\rho_i} {\delta_i^j}^2\right) = 1$ and $\left(\sum_{i=1}^{i=d'}\sum_{j=0}^{j=\rho'_i} {\delta'^j_i}^2\right) = 0$, or
$\left(\sum_{i=1}^{i=d}\sum_{j=0}^{j=\rho_i} {\delta_i^j}^2\right) = 0$ and $\left(\sum_{i=1}^{i=d'}\sum_{j=0}^{j=\rho'_i} {\delta'^j_i}^2\right) = 1$.

The first possibility would prove the existence of $t \in \{1, \ldots, d\}$ such that $\delta_t = 1$, $\delta_i^j = 0$ for every $i = 1, \ldots, d$ and for every $j = 0, \ldots, \rho_i$ with $i \neq t$, and $\delta_t^l = 0$ for every $l = 1, \ldots, \rho_t$. Thus, it gives rise to the equalities $2 = c = c' = 0$. Consequently we are in the second case, so there exists $t \in \{1, \ldots, d'\}$ such that $\delta'_t = 1$, and ${\delta'}_i^j = 0$ for every $i = 0, \ldots, d'$ and for every $j = 0, \ldots, \rho'_i$ with $i \neq t$, and ${\delta'}_t^l = 0$ for every $l = 1, \ldots, \rho'_t$. Therefore, $1 = c' = x = y = c$. So, $\mathcal{E} = \mathcal{C}_0 + \mathcal{F} - \mathcal{E}_O - \mathcal{E}_{O'} - \mathcal{E}_{D'_t}$ for some $t \in \{1, \ldots, d'\}$. And by symmetry, we get also $\mathcal{E} = \mathcal{C}_0 + \mathcal{F} - \mathcal{E}_O - \mathcal{E}_{O'} - \mathcal{E}_{D_t}$ for some $t \in \{1, \ldots, d\}$. Note that for every $l \in \{1, \ldots, d'\}$, $\mathcal{C}_0+\mathcal{F}-\mathcal{E}_O-\mathcal{E}_{O'}-\mathcal{E}_{D'_l}$ is the class of a (-1)-curve on X, and for every $l \in \{1, \ldots, d\}$, $\mathcal{C}_0+\mathcal{F}-\mathcal{E}_O-\mathcal{E}_{O'}-\mathcal{E}_{D_l}$ is the class of a (-1)-curve on X, So we have at least $(d+d')$ exceptional curves of the first kind.

Assume that we are in the Case 2. It would imply that $1 = x - c = 0$. So, this case does not occur.
Assume that we are in the Case 3, it would imply that the positive integer α is equal to zero. □

Corollary 3.5. *With notation as above. The number of (-1)-curves on X is equal to $2(2 + d + d')$.*

4 The set of (-2)-curves on X

Definition 4.1. With notation as above. A curve N on X is a (-2)-curve if $E^2 = -2$ and $E.K_X = 0$, where K_X denotes a canonical divisor on X.

In this section, we determine the set of (-2)-curves on X. We obtain the surprising feature on the geometry of X which states that the set of integral curves on X orthogonal to a canonical divisor on X is equal to the set of (-2)-curves on X. Such feature holds also for any smooth projective rational surface having a canonical divisor of positive self-intersection, e.g. Del Pezzo surfaces. In particular, every integral curve orthogonal to a canonical divisor on X is smooth and rational.

Proposition 4.2. *With notation as above. Let N be an integral curve on X. If N is a (-2)-curve, then the class $\mathcal{N}$ of N in $\mathrm{NS}(X)$ is one of the following:*

1. $\mathcal{E}_{A_i} - \mathcal{E}_{A_{i+1}}$, *where* $i \in \{1, \dots, a-1\}$,
2. $\mathcal{E}_{B_j} - \mathcal{E}_{B_{j+1}}$, *where* $j \in \{1, \dots, b-1\}$,
3. $\mathcal{E}_{A'_i} - \mathcal{E}_{A'_{i+1}}$, *where* $i \in \{1, \dots, a'-1\}$,
4. $\mathcal{E}_{B'_j} - \mathcal{E}_{B'_{j+1}}$ *where* $j \in \{1, \dots, b'-1\}$,
5. $\mathcal{E}_{D_t^u} - \mathcal{E}_{D_t^{u+1}}$, *where* $t \in \{1, \dots, d\}$ *and* $u \in \{0, \dots, \rho_t - 1\}$,
6. $\mathcal{E}_{D'^u_t} - \mathcal{E}_{D'^{u+1}_t}$, *where* $t \in \{1, \dots, d'\}$ *and* $u \in \{0, \dots, \rho'_t - 1\}$.

Proof. $\mathcal{N}$ is given in $\mathrm{NS}(X)$ by some tuple with integer entries of the form

$$(x; y; c, \lambda_1, \dots, \lambda_a, \mu_1, \dots, \mu_b, \delta_1, \dots, {\delta_1}^{\rho_1}, \dots, \delta_d, \dots, {\delta_d}^{\rho_d};$$

$$c', \lambda'_1, \dots, \lambda'_{a'}, \mu'_1, \dots, \mu'_{b'}, \delta'_1, \dots, {\delta'_1}^{\rho'_1}, \dots, \delta'_{d'}, \dots, {\delta'_{d'}}^{\rho'_d}).$$

We assume that N is not a component of $-K_X$ and is not one of the listed curves in the items 5. and 6. of the proposition. From the fact that the integer $\mathcal{N}.\mathcal{K}_X$ is zero, we obtain the following equalities:

$$\lambda_i = 0, \text{ for every } \mathrm{i}, \dots, \mathrm{a}, \text{ and}$$
$$\mu_j = 0, \text{ for every } \mathrm{j} = 1, \dots, \mathrm{b}, \text{ and}$$
$$\lambda'_{i'} = 0, \text{ for every } \mathrm{i}' = 1, \dots, \mathrm{a}', \text{ and}$$
$$\mu'_{j'} = 0, \text{ for every } \mathrm{j}' = 1, \dots, \mathrm{b}', \text{ and}$$
$$x = y = c = c', \; and$$
$$c - \sum_{i=1}^{i=d} \sum_{j=0}^{j=\rho_i} \delta_i^j = 0, \; and$$
$$c' - \sum_{i=1}^{i=d'} \sum_{j=0}^{j=\rho'_i} {\delta'}_i^j = 0.$$

Hence, from the equality $\mathcal{N}^2 = -2$, we obtain the equality:

$$\left(\sum_{i=1}^{i=d} \sum_{j=0}^{j=\rho_i} {\delta_i^j}^2 \right) + \left(\sum_{i=1}^{i=d'} \sum_{j=0}^{j=\rho'_i} {{\delta'}_i^j}^2 \right) = 2.$$

Then, there are only three possibilities:
either $\sum_{i=1}^{i=d}\sum_{j=0}^{j=\rho_i} {\delta_i^j}^2 = 2$ and $\sum_{i=1}^{i=d'}\sum_{j=0}^{j=\rho'_i} {\delta'^j_i}^2 = 0$, or
$\sum_{i=1}^{i=d}\sum_{j=0}^{j=\rho_i} {\delta_i^j}^2 = 1$ and $\sum_{i=1}^{i=d'}\sum_{j=0}^{j=\rho'_i} {\delta'^j_i}^2 = 1$, or
$\sum_{i=1}^{i=d}\sum_{j=0}^{j=\rho_i} {\delta_i^j}^2 = 0$ and $\sum_{i=1}^{i=d'}\sum_{j=0}^{j=\rho'_i} {\delta'^j_i}^2 = 2$.
By symmetry, we may only look at the two situations:
$\sum_{i=1}^{i=d}\sum_{j=0}^{j=\rho_i} {\delta_i^j}^2 = 2$ and $\sum_{i=1}^{i=d'}\sum_{j=0}^{j=\rho'_i} {\delta'^j_i}^2 = 0$, and
$\sum_{i=1}^{i=d}\sum_{j=0}^{j=\rho_i} {\delta_i^j}^2 = 1$ and $\sum_{i=1}^{i=d'}\sum_{j=0}^{j=\rho'_i} {\delta'^j_i}^2 = 1$.
In the first case, we would obtain the existence of $t \in \{1,\dots,d\}$ such that $\delta_t = \delta_t^1 = 1$, $\delta_i^j = 0$ for every $i = 1,\dots,d$ and for every $j = 0,\dots,\rho_i$ with $i \neq t$, and $\delta_t^l = 0$ for every $l = 2,\dots,\rho_t$. This would imply that $2 = c = c' = 0$. In the second case, there exist $t \in \{1,\dots,d\}$ and $t' \in \{1,\dots,d'\}$ such that $\delta_t = \delta_{t'} = 1$ and $\delta_i^j = 0$ for every $i = 1,\dots,d$ and for every $j = 0,\dots,\rho_i$ with $i \neq t$ and $\delta_t^l = 0$ for every $l = 2,\dots,\rho_t$ and ${\delta'}_i^j = 0$ for every $i = 1,\dots,d'$ and for every $j = 0,\dots,\rho'_i$ with $i \neq t'$, and ${\delta'}_t^l = 0$ for every $l = 2,\dots,\rho'_t$. That is, $\mathcal{N} = \mathcal{C}_0 + \mathcal{F} - \mathcal{E}_O - \mathcal{E}_{D_t} - \mathcal{E}_{O'} - \mathcal{E}_{D'_{t'}}$ for some $t \in \{1,\dots,d\}$ and $t' \in \{1,\dots,d'\}$. These possibilities do not occur since $D_1,\dots,D_d$, and $D'_1,\dots,D'_{d'}$ are general. □

Corollary 4.3. *With notation as above. The number of* (-2)*-curves on* X *is equal to* $a + b + a' + b' - 4 + \sum_{i=1}^{i=d}\rho_i + \sum_{j=1}^{j=d'}\rho'_j$.

Corollary 4.4. *With notation as above. The set of* (-2)*-curves spans a* $\mathbb{Z}$*-submodule of* $\mathrm{NS}(X)$ *of rank less than* $-1 + \rho(X)$.

Proof. It is a consequence of the fact that the (-2)-curves on X are linearly independent in $\mathrm{NS}(X)$. □

Remark 4.5. Assume that d is larger than or equal to d'. Even allowing the points D'_1, ..., $D'_{d'}$ to be in special positions on the smooth projective rational curve $\left(\mathcal{E}_{O'} - \mathcal{E}_{A'_1} - \mathcal{E}_{B'_1}\right)$, we may increase the number of (-2)-curves by d'. In this case, the (-2)-curves on X will still generate a $\mathbb{Z}$-submodule of $\mathrm{NS}(X)$ of rank less than $-1 + \rho(X)$. Indeed, it has corank equal to $8 + d$.

5 Regularity of nef divisors, Cox ring

The aim of this section is three-fold: to prove that every integral curve orthogonal to a canonical divisor on X is a (-2)-curve, every nef divisor on X is not only regular, but also has a higher multiple whose complete linear system is base loci free.

Proposition 5.1. *With notation as above. Let D be a nef divisor on X. Then D may be considered as an effective divisor, and the following assertions are equivalents:*

1. *The divisor D is equal to zero.*
2. *The integer $D.K_X$ is equal to zero, here K_X denotes a canonical divisor on X.*

Proof. D may be considered as an effective divisor comes from the fact that X is an anticanonical rational surface, D is nef, and the Riemann-Roch theorem applied to D. Now, the class $\mathcal{D}$ of D in $\mathrm{NS}(X)$ is given by some tuple with integer entries of the form

$$(x; y; c, \lambda_1, \ldots, \lambda_a, \mu_1, \ldots, \mu_b, \delta_1, \ldots, {\delta_1}^{\rho_1}, \ldots, \delta_d, \ldots, {\delta_d}^{\rho_d};$$

$$c', \lambda'_1, \ldots, \lambda'_{a'}, \mu'_1, \ldots, \mu'_{b'}, \delta'_1, \ldots, {\delta'_1}^{\rho'_1}, \ldots, \delta'_{d'}, \ldots, {\delta'_{d'}}^{\rho'_d}).$$

From the equation $\mathcal{E}.\mathcal{K}_X = 0$ and the nefness of D, we obtain the following equalities:

$$\lambda_i = 0, \text{ for every } \mathrm{i} = \ldots, \mathrm{a}, \text{ and}$$
$$\mu_j = 0, \text{ for every } \mathrm{j} = 1, \ldots, \mathrm{b}, \text{ and}$$
$$\lambda'_{i'} = 0, \text{ for every } \mathrm{i}' = 1, \ldots, \mathrm{a}', \text{ and}$$
$$\mu'_{j'} = 0, \text{ for every } \mathrm{j}' = 1, \ldots, \mathrm{b}', \text{ and}$$
$$x = y = c = c', \text{ and}$$
$$c - \sum_{i=1}^{i=d} \sum_{j=0}^{j=\rho_i} \delta_i^j = 0, \text{ and}$$
$$c' - \sum_{i=1}^{i=d'} \sum_{j=0}^{j=\rho'_i} {\delta'}_i^j = 0.$$

Hence, again from the nefness of D, we obtain the equality:

$$\left(\sum_{i=1}^{i=d} \sum_{j=0}^{j=\rho_i} {\delta_i^j}^2 \right) + \left(\sum_{i=1}^{i=d'} \sum_{j=0}^{j=\rho'_i} {{\delta'}_i^j}^2 \right) = 0.$$

Then, we conclude that $\mathcal{D} = \mathcal{O}_X$, and consequently, $D = 0$. So, we are done. $\square$

As immediate consequences, the following two results occur:

Corollary 5.2. *With notation as above. Let* Γ *be an integral curve on* X*. If* Γ *is orthogonal to a canonical divisor on* X*, then the self-intersection of* Γ *is negative.*

Proof. Assume that the self-intersection of Γ is nonnegative, then Γ would be a nef divisor, and by the last proposition, Γ would be equal to zero. □

Proposition 5.3. *With notation as above. Let* Γ *be an integral curve on* X*. The following assertions are equivalent:*

1. Γ *is orthogonal to a canonical divisor on* X*.*
2. Γ *is a* (-2)*-curve.*

Proof. It is enough to prove that the orthogonality of Γ to a canonical divisor implies the statement that Γ is a (-2)-curve. This is straightforward from the last corollary and the adjunction formula, see [29, Theorem 1.1.2 (Adjunction)]. □

Next, we deduce the emptiness of the fixed loci of any higher multiple of any nef divisor on X:

Corollary 5.4. *Let* D *be a nef divisor on* X*. The complete linear system* $|rD|$ *is base point free for every integer* r *larger than one.*

Proof. Let D be a nef divisor on X. If D is equal to zero, then the complete linear system $|rD|$ is obviously base point free for every positive integer. So, we may assume that D is nonzero. From Proposition 5.1, we deduce that the intersection number $-K_X.D$ is positive. Therefore, the intersection number $(-K_X).(rD)$ is larger than or equal to r. Thus, using [28, Theorem III.1. (a)], we get the information that the complete linear system $|rD|$ is base point free. So, we are done. □

The following result characterizes the finite generation of the Cox ring of any smooth projective surface having a finitely generated effective monoid:

Lemma 5.5. *Let* S *be a smooth projective surface defined over an algebraically closed field of arbitrary characteristic such that the effective monoid* $\mathrm{M}(S)$ *of* S *is finitely generated. The following assertions are equivalent:*

1. $\mathrm{Cox}(S)$ *is finitely generated.*
2. *Every nef divisor on* S *has a positive multiple whose complete linear system is base point free.*

Proof. This is a consequence of [10, Theorem 21] when the effective monoid $\mathrm{M}(S)$ of S is finitely generated. □

Remark 5.6. The property 2. in the last lemma can be stated equivalently as: Every nef divisor on S has a positive multiple whose complete linear system is fixed component free.

Theorem 5.7. *The Cox ring of X is finitely generated.*

Proof. By using Theorem 2.1, Lemma 5.5, and Corollary 5.4. □

6 Explicit computational description of the effective monoid in some cases

In this section, we present two special cases of our construction which gave rise to the surfaces studied in Theorem 2.1, and offer efficient proofs. These two cases are obtained by reducing the constellation to special chains and allowing some data to be zero.

In the first case, we select a horizontal section in the ruling of Σ_0:

Theorem 6.1. *Let p be a closed point of Σ_0 and let $\mathfrak{C}$ be a chain with origin p and rank r. If the element of $\mathfrak{C}_i$ belongs to the i^{th}-strict transform of C_0^m for every $i \in \{1, \ldots, r\}$, then the effective monoid $\mathrm{M}(Y_0)$ of Y_0 is generated by $\mathcal{C}_0 - \sum_{i=1}^{r} \mathcal{E}_i$, $\mathcal{F} - \mathcal{E}_1$, $\mathcal{E}_1 - \mathcal{E}_2$, $\mathcal{E}_2 - \mathcal{E}_3$, $\ldots$, $\mathcal{E}_{r-1} - \mathcal{E}_r$, and $\mathcal{E}_r$. Here, Y_0 is the blowing-up of Σ_0 at the r points of $\mathfrak{C}$, $\mathcal{C}_0$ is the class of the total transform of ${C_0}^m$, $\mathcal{F}$ is the class of the total transform of a fibre f^m of Σ_0, and $\mathcal{E}_j$ is the class of the exceptional divisor corresponding to the j^{th} point blown-up for every $j \in \{1, 2, \ldots, r\}$.*

Proof. It is clear that $\mathbb{Z}_+(\mathcal{C}_0 - \sum_{i=1}^{r} \mathcal{E}_i) + \mathbb{Z}_+(\mathcal{F} - \mathcal{E}_1) + \sum_{i=1}^{r-1} \mathbb{Z}_+(\mathcal{E}_i - \mathcal{E}_{i+1}) + \mathbb{Z}_+\mathcal{E}_r$ is contained in $M(Y_0)$, since $\mathcal{C}_0 - \sum_{i=1}^{r} \mathcal{E}_i$, $\mathcal{F} - \mathcal{E}_1$, $\mathcal{E}_r$, and $\mathcal{E}_j - \mathcal{E}_{j+1}$ are classes of prime divisors, for every $j \in \{1, 2, \ldots, r-1\}$. Conversely, let z be an element of $\mathrm{NS}(Y_0)$ such that z is effective, so $z = a\mathcal{C}_0 + b\mathcal{F} - \sum_{i=1}^{r} c_i\mathcal{E}_i$ for some integers $a, b, c_1, c_2, \ldots, c_r$, and one may assume without loss of generality that z is irreducible, and not belonging to the set $\{\mathcal{C}_0 - \sum_{i=1}^{r} \mathcal{E}_i, \mathcal{E}_1 - \mathcal{E}_2, \mathcal{E}_2 - \mathcal{E}_3, \ldots, \mathcal{E}_{r-1} - \mathcal{E}_r, \mathcal{E}_r\}$. It follows that z can be written as:

$$a(\mathcal{C}_0 - \textstyle\sum_{i=1}^{r} \mathcal{E}_i) + b(\mathcal{F} - \mathcal{E}_1) + (a + b - c_1)(\mathcal{E}_1 - \mathcal{E}_2) +$$
$$(2a + b - c_1 - c_2)(\mathcal{E}_2 - \mathcal{E}_3) + \cdots + ((r-1)a + b - \textstyle\sum_{i=1}^{r-1} c_i)(\mathcal{E}_{r-1} - \mathcal{E}_r) +$$
$$(ra + b - \textstyle\sum_{i=1}^{r} c_i)\mathcal{E}_r.$$

It is worth noting that the integers a, b and $b - \sum_{i=1}^{r} c_i$ are nonnegative since $\mathcal{C}_0$ and $\mathcal{F}$ are nef, and z is different from $\mathcal{C}_0 - \sum_{i=1}^{r} \mathcal{E}_i$. So, we are done. □

Example 6.2. With the notation of the previous theorem, the following elements of $\mathrm{NS}(Y_0)$ are effective:

1. Let x be the element $\mathcal{C}_0 + 3r\mathcal{F} - \mathcal{E}_1 - \cdots - \mathcal{E}_{r-1} - r\mathcal{E}_r$. From our computation in the proof of Theorem 6.1, it is clear that x is effective. Indeed, one may write this element as
$$(\mathcal{C}_0 - \sum_{i=1}^{r} \mathcal{E}_i) + 3r\big((\mathcal{F} - \mathcal{E}_1) + \sum_{i=1}^{r-1}(\mathcal{E}_i - \mathcal{E}_{i+1})\big) + (2r+1)\mathcal{E}_r.$$
However, it is impossible from the natural computation using Riemann-Roch theorem to confirm the effectiveness of x. Indeed, such computation give us using the fact that $h^2(Y_0, x) = 0$ (this comes from the nefness of $\mathcal{F}$ and the Serre duality) the following equality:
$$h^0(Y_0, x) - h^1(Y_0, x) = 2 + 6r - (r-1) - \frac{r(r+1)}{2},$$
and this number in general is negative for every $r \geq 11$.

2. Let y be the element $\mathcal{C}_0 + r^2\mathcal{F} - r\mathcal{E}_1 - \cdots - r\mathcal{E}_r$. According to the proof given in Theorem 6.1, one may write this element as
$$(\mathcal{C}_0 - \sum_{i=1}^{r} \mathcal{E}_i) + r^2(\mathcal{F} - \mathcal{E}_1) + \sum_{i=1}^{r-1}(r^2 + i(1-r))(\mathcal{E}_i - \mathcal{E}_{i+1}) + r\mathcal{E}_r,$$
and we get the fact that it is an effective element. Nevertheless, the natural computation using Riemann-Roch theorem gives no information about the effectiveness of y. In fact, with in hand Serre duality and the nefness of $\mathcal{F}$, that computation gives the equality
$$h^0(Y_0, y) - h^1(Y_0, y) = r^2 + 2 - \frac{r(r+1)(2r+1)}{12} - \frac{r(r+1)}{4},$$
and the latter number is negative for every r greater than 3.

3. Let z be $3\mathcal{C}_0 + \frac{r(r+1)}{2}\mathcal{F} - \mathcal{E}_1 - 2\mathcal{E}_2 - \cdots - r\mathcal{E}_r$. By the proof of Theorem 6.1, one may write this element as
$$3(\mathcal{C}_0 - \sum_{i=1}^{r} \mathcal{E}_i) + \frac{r(r+1)}{2}(\mathcal{F} - \mathcal{E}_1) + \sum_{i=1}^{r-1}(3i + \frac{r(r+1) - i(i+1)}{2})(\mathcal{E}_i - \mathcal{E}_{i+1}) + 3r\mathcal{E}_r.$$
Observe that if one use the natural computation from the Riemann-Roch theorem, one obtains that
$$h^0(Y_0, z) - h^1(Y_0, z) = 4 + 2r(r+1) - \sum_{i=1}^{r} \frac{i(i+1)}{2}.$$

Here, $h^2(Y_0, z) = 0$ because the nefness of $\mathcal{F}$ and Serre duality. Now, if r is greater than 10, the integer $4 + 2r(r+1) - \sum_{i=1}^{r} \frac{i(i+1)}{2}$ is no longer positive.

Remark 6.3. It is worth noting that the effective classes of the elements x, y, and z in the above example are not regular. Here, an invertible sheaf $\mathcal{G}$ on a smooth projective surface W is regular if $h^1(W, \mathcal{G})$ is equal to zero.

In the second case, we select a fibre in the ruling of Σ_0:

Theorem 6.4. *Let p be a closed point of Σ_0 and let $\mathfrak{C}$ be a chain with origin p and rank r. If the element of $\mathfrak{C}_i$ belongs to the i^{th}-strict transform of f^m for every $i \in \{1, \ldots, r\}$, then the effective monoid* $\mathrm{M}(Z_0)$ *is equal to* $\mathbb{Z}_+(\mathcal{C}_0 - \mathcal{E}_1) + \mathbb{Z}_+(\mathcal{F} - \sum_{i=1}^{r} \mathcal{E}_i) + \sum_{i=1}^{r-1} \mathbb{Z}_+(\mathcal{E}_i - \mathcal{E}_{i+1}) + \mathbb{Z}_+\mathcal{E}_r$*. Here, Z_0 is the blowing-up of Σ_0 at the r points of $\mathfrak{C}$, $\mathcal{C}_0$ is the class of the total transform of ${C_0}^m$, $\mathcal{F}$ is the class of the total transform of a fibre f^m of Σ_0, and $\mathcal{E}_j$ is the class of the exceptional divisor corresponding to the j^{th} point blown-up for every $j \in \{1, 2, \ldots, r\}$.*

Proof. It is enough to prove that $\mathbb{Z}_+(\mathcal{C}_0-\mathcal{E}_1)+\mathbb{Z}_+(\mathcal{F}-\sum_{i=1}^{r} \mathcal{E}_i)+\sum_{i=1}^{r-1} \mathbb{Z}_+(\mathcal{E}_i-\mathcal{E}_{i+1})+\mathbb{Z}_+\mathcal{E}_r$ contains $M(Z_0)$, for $\mathcal{C}_0 - \mathcal{E}_1$, $\mathcal{F} - \sum_{i=1}^{r} \mathcal{E}_i$, $\mathcal{E}_r$, and $\mathcal{E}_j - \mathcal{E}_{j+1}$ are classes of prime divisors, for every $j \in \{1, 2, \ldots, r-1\}$. Let z be an element of $\mathrm{M}(Z_0)$, so $z = a\mathcal{C}_0 + b\mathcal{F} - \sum_{i=1}^{r} c_i\mathcal{E}_j$ for some integers $a, b, c_1, c_2, \ldots, c_r$, and one may assume that z is irreducible, and not belonging to the set $\{\mathcal{F}_0 - \sum_{i=1}^{r} \mathcal{E}_i, \mathcal{E}_1 - \mathcal{E}_2, \mathcal{E}_2 - \mathcal{E}_3, \ldots, \mathcal{E}_{r-1} - \mathcal{E}_r, \mathcal{E}_r\}$. It turns out that z can be written as:

$$a(\mathcal{C}_0 - \mathcal{E}_1) + b(\mathcal{F} - \textstyle\sum_{i=1}^{r} \mathcal{E}_i) + (b + a - c_1)(\mathcal{E}_1 - \mathcal{E}_2)+$$
$$(2b + a - c_1 - c_2)(\mathcal{E}_2 - \mathcal{E}_3) + \cdots + ((r-1)b + a - \textstyle\sum_{i=1}^{r-1} c_i)(\mathcal{E}_{r-1} - \mathcal{E}_r)+$$
$$(rb + a - \textstyle\sum_{i=1}^{r} c_i)\mathcal{E}_r.$$

Noting that the integers a, b and $a - \sum_{i=1}^{r} c_i$ are nonnegative (since $\mathcal{C}_0$ and $\mathcal{F}$ are nef, and z is different from $\mathcal{F}_0 - \sum_{i=1}^{r} \mathcal{E}_i$), we conclude the proof. □

7 Discussion

The effective monoid of an algebraic variety is an interesting object very close to the cone of curves of the variety. Finite generation of the cone of curves and of the Cox ring are interesting matters in the minimal model program which was very investigated in the last decade. The minimal model program is addressed to higher dimensional varieties but not much is known about finite generation of Cox rings and cone of curves of rational surfaces. The main available result which states that such finite generation holds is a criterion when the surface X is rational smooth and its anticanonical Iitaca dimension is 1 ([1, Theorem 4.2, p. 5259]). The subject of our study is a new family F of rational projective surfaces defined over an algebraically closed field of arbitrary characteristic. Surfaces in F are obtained after blowing-ups of a projective quadric surface Σ_0 at a constellation of infinitely near points. This constellation contains two closed points O and O' in Σ_0 and finitely many infinitely near points which are a) those that follow the strict transforms of the unique fibre (respectively, horizontal smooth projective rational curve) in Σ_0 going through O and O' and b) infinitely near points which are successively proximate to a fixed set of points at the exceptional divisors E_O and $E_{O'}$ obtained by blowing-ups O and O'. Our main interest was to give an explicit set of generators of the effective monoid $\mathrm{M}(X)$ of the surfaces X in F, proving that for those surfaces $\mathrm{M}(X)$ is finitely generated. In this direction we proved that the Cox ring of surfaces as above is finitely generated. Moreover, we were able to prove another interesting result concerning the set of other generators for the effective monoid for special subfamilies of F. The obtained result permits to decide about effectiveness of divisors for which this property cannot be deduced from Riemann-Roch Theorem. Further studies on the finite generation of Cox rings of smooth projective rational surfaces which are different from ours can be found in [1], [7], [10], [11], [22], [33], [45] and [49]. We add some few comments: Theorem 6.4 may be obtained directly from Theorem 6.1 by changing the fibration of Σ_0. We include it in order to make the decomposition of every effective divisor ready to use without any effort.

The proofs of Theorems 6.1 and 6.4 give efficient ways of decomposing any effective divisor class in the Néron-Severi group of those families of surfaces Y_0 and Z_0 with respect to the lists given in these theorems.

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