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Variational inequality for a vector field on Hadamard spaces

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Abstract

Our purpose is to study the variational inequality problem for a vector field on Hadamard spaces. The existence and uniqueness of the solutions to the variational inequality problem associated with a vector field in Hadamard spaces are studied.

1 Introduction

One of the most significant theories in applied mathematics is variational inequalities theory. Variational inequality problems are powerful tools for studying optimization problems, boundary value problems of PDE's, equilibrium problems, physics issues, etc. The reader can consult with the book by Kinderlehrer and Stampacchia [9] for more details on variational inequalities theory and its applications. The existence, uniqueness, and approximation problems of the solutions to the variational inequalities have been considered by many authors, (see, for example, [7, 10, 11, 12, 13, 14, 15, 16, 18, 19] and references therein). In particular, Németh introduced the variational inequalities on Hadamard manifolds and obtained some existence theorems and uniqueness theorems in [15], (also, see [14]). Li et al. [13] extended the existence and uniqueness results of [15] to Riemannian manifolds. In 2015, Khatibzadeh and the author [10] extended the existence and uniqueness results of [15] to Hadamard spaces for the variational inequality problem associated with a nonexpansive mapping.

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In this paper, the variational inequality problem for a vector field on Hadamard spaces is formulated. By introducing the notions of monotone, strictly monotone, weakly continuous, and coercive vector fields, we study the existence and uniqueness of the solutions to the variational inequality problem associated with a vector field in Hadamard spaces.

2 Basic definitions and preliminaries

Suppose (X, d) is a metric space, $u, v \in X$ and $I = [0, d(u, v)]$. A geodesic path connecting u to v in X is an isometry $c : I \longrightarrow X$ such that $c(0) = u$, $c(d(u, v)) = v$ and $d(c(a), c(b)) = |a - b|$ for all $a, b \in I$. The image of a geodesic c is called a geodesic segment connecting u and v. When it is unique, this geodesic is denoted by $[u, v]$. We denote the unique point $z \in [u, v]$ such that $d(u, z) = td(u, v)$ and $d(v, z) = (1 - t)d(u, v)$, by $(1 - t)u \oplus tv$, where $0 \leq t \leq 1$. The metric space (X, d) is called a geodesic space if u and v are joined by a geodesic, for each $u, v \in X$. The (X, d) is said to be uniquely geodesic if there is exactly one geodesic segment connecting u and v for each $u, v \in X$. A subset K of X is called convex if $[u, v] \subseteq K$ for all $u, v \in K$.

Definition 2.1. A non-positive curvature metric space or a CAT(0) space (in honour of E. **C**artan, AD. **A**lexandrov and V.A. **T**oponogov) is a geodesic space (X, d) which comes up to the following (CN) inequality:

$$d^2(tx \oplus (1 - t)y, z) \leq td^2(x, z) + (1 - t)d^2(y, z) - t(1 - t)d^2(x, y), \qquad (2.1)$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

In particular, if x, y, z, w are points in X and $t \in [0, 1]$, then

$$d(tx \oplus (1 - t)y, z) \leq td(x, z) + (1 - t)d(y, z).$$

CAT(0) spaces are uniquely geodesic. A complete CAT(0) space is called a **Hadamard space**. We refer the reader to the standard texts such as [2, 4, 5, 6, 8] for other equivalent definitions and basic properties of CAT(0) spaces. The following are some examples of Hadamard spaces:
Hilbert spaces, Hadamard manifolds (i.e. simply connected complete Riemannian manifolds with non-positive sectional curvature which can be of infinite dimension), $\mathbb{R}$-trees as well as examples that have been built out of given Hadamard spaces such as closed convex subsets, direct products, warped products, L^2-spaces, direct limits, and Reshetnyak's gluing (see [17], Section 3).

The concept of *quasilinearization* for the CAT(0) space X introduced by Berg and Nikolaev [3]. They denoted a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and called it a *vector*. Then the quasilinearization map $\langle ., . \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ is defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \tfrac{1}{2}(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)), \qquad (a,b,c,d \in X).$$

Clearly, $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a,b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ are satisfied for all $a,b,c,d,e \in X$. Also, we have $\overrightarrow{ac} + \overrightarrow{cb} = \overrightarrow{ab}$, for all $a,b,c \in X$.

The Cauchy-Schwartz inequality for the space X is

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d), \qquad (a,b,c,d \in X).$$

Theorem 2.1. *[3, Corollary 3] A geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality.*

The concept of dual space of the Hadamard space X introduced by Ahmadi Kakavandi and Amini [1]. They define the map $\Theta : \mathbb{R} \times X \times X \to C(X,\mathbb{R})$ by

$$\Theta(t,a,b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \qquad (t \in \mathbb{R},\ a,b,x \in X),$$

where $C(X,\mathbb{R})$ is the space of all continuous real-valued functions on X. By the Cauchy-Schwartz inequality, $\Theta(t,a,b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t,a,b)) = |t|d(a,b)$, $\quad (t \in \mathbb{R}, \quad a,b \in X)$, where $L(\varphi) = sup\{\frac{\varphi(x)-\varphi(y)}{d(x,y)} : x,y \in X, x \neq y\}$ is the Lipschitz semi-norm for any function $\varphi : X \to \mathbb{R}$. Then, they define the pseudometric D on $\mathbb{R} \times X \times X$ by

$$D((t,a,b),(s,c,d)) = L(\Theta(t,a,b) - \Theta(s,c,d)), \quad (t,s \in \mathbb{R},\ a,b,c,d \in X).$$

Lemma 2.1. *[1, Lemma 2.1] $D((t,a,b),(s,c,d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$, for all $x,y \in X$.*

Lemma 2.1 shows there is an equivalent relation on $\mathbb{R} \times X \times X$, where the equivalence class of (t,a,b) is

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd} : D((t,a,b),(s,c,d)) = 0\}.$$

The set $X^* = \{[t\overrightarrow{ab}] : (t,a,b) \in \mathbb{R} \times X \times X\}$ is a metric space with metric $D([t\overrightarrow{ab}],[s\overrightarrow{cd}]) := D((t,a,b),(s,c,d))$, which is called the dual space of (X,d). Obviously, $[\overrightarrow{aa}] = [\overrightarrow{bb}]$ for all $a,b \in X$. We write $\mathbf{0} = [\overrightarrow{oo}]$ as the zero of the dual space where $o \in X$. Note that X^* acts on $X \times X$ by

$$\langle x^*, \overrightarrow{xy} \rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle, \quad (x^* = [t\overrightarrow{ab}] \in X^*, x,y \in X).$$

Also, we use the following notation:

$$\langle \alpha x^* + \beta y^*, \overrightarrow{xy} \rangle := \alpha\langle x^*, \overrightarrow{xy} \rangle + \beta\langle y^*, \overrightarrow{xy} \rangle, \quad (\alpha,\beta \in \mathbb{R},\ x,y \in X,\ x^*,y^* \in X^*).$$

3 Main Results

Throughout this section, let (X,d) be a Hadamard space with dual space X^*, $o \in X$ and K be a closed and convex subset of X. For all $x \in X$, set $X_x^* := \{[\alpha\overrightarrow{yx}] : \alpha \geq 0,\ y \in X\}$. Then $X^* = \bigcup_{x\in X} X_x^*$.

Definition 3.1. The operator $A : K \to X^*$ is a vector field on K if $Ax \subseteq X_x^*$ for all $x \in K$.

The variational inequality problem associated with the vector field $A : K \to X^*$ is formulated by

$$\text{Find} \quad x \in K \text{ such that } \langle Ax, \overrightarrow{xy}\rangle \geq 0, \quad \forall y \in K. \tag{3.1}$$

Theorem 3.1. *Let $A : K \to X^*$ be a vector field on K and $x \in int(K)$ (interior of K) be a solution of problem* (3.1)*, then $Ax = \mathbf{0}$.*

Proof. Let $Ax = [\alpha\overrightarrow{zx}]$, where $\alpha \geq 0$ and $z \in X$. If $\alpha = 0$, that is nothing to prove. Let $\alpha > 0$. There exists $\epsilon > 0$ such that $\{y \in X : d(x,y) < \epsilon\} \subset K$. Choose $0 < t < 1$ such that $d(x, tx \oplus (1-t)z) = (1-t)d(x,z) < \epsilon$. Thus $tx \oplus (1-t)z \in K$. Therefore, by (3.1), we have

$$\begin{aligned}
0 &\leq 2\langle Ax, \overrightarrow{x(tx \oplus (1-t)z)}\rangle \\
&= 2\alpha\langle \overrightarrow{zx}, \overrightarrow{x(tx \oplus (1-t)z)}\rangle \\
&= \alpha(d^2(z, tx \oplus (1-t)z) - d^2(x,z) - d^2(x, tx \oplus (1-t)z)) \\
&= \alpha(t^2 d^2(x,z) - d^2(x,z) - (1-t)^2 d^2(z,x)) \\
&= -2\alpha(1-t)d(x,z) \leq 0
\end{aligned}$$

Hence, $-2\alpha(1-t)d(x,z) = 0$ which implies $d(x,z) = 0$. Therefore

$$Ax = [\alpha\overrightarrow{xx}] = \mathbf{0}.$$

□

Definition 3.2. The vector field $A : K \to X^*$ is

- monotone if $0 \leq \langle Ax - Ay, \overrightarrow{yx}\rangle, \quad \forall x, y \in K,$
- strictly monotone if $0 < \langle Ax - Ay, \overrightarrow{yx}\rangle, \quad \forall x, y \in K.$

Definition 3.3. The vector field $A : K \to X^*$ is weakly continuous if

$$\langle A(tz \oplus (1-t)x), \overrightarrow{zx}\rangle \to \langle Ax, \overrightarrow{zx}\rangle, \quad \forall z, x \in K \text{ as } t \to 0.$$

Example 3.1. Let $T : K \to X$ be a nonexpansive mapping (i.e. $d(Tx, Ty) \le d(x, y)$, for all $x, y \in K$). Define $A : K \to X^*$ with $Ax = [\overrightarrow{(Tx)x}]$. Then

$$\begin{aligned}
\langle \overrightarrow{(Tx)x} - \overrightarrow{(Ty)y}, \overrightarrow{yx} \rangle &= \langle \overrightarrow{(Tx)(Ty)} + \overrightarrow{(Ty)y} + \overrightarrow{yx} - \overrightarrow{(Ty)y}, \overrightarrow{yx} \rangle \\
&= \langle \overrightarrow{(Tx)(Ty)}, \overrightarrow{yx} \rangle + \langle \overrightarrow{yx}, \overrightarrow{yx} \rangle \\
&\ge d^2(x, y) - d(Tx, Ty)d(x, y) \\
&= d(x, y)(d(x, y) - d(Tx, Ty)) \ge 0,
\end{aligned}$$

which shows A is a monotone vector field. Moreover, we have

$$d(T(tz \oplus (1-t)x), y) \to d(Tx, y), \quad \forall x, y, z \in K \ \text{ as } \ t \to 0,$$

which implies A is weakly continuous.

Proposition 3.2. *Let $A : K \to X^*$ be a weakly continuous and monotone vector field. Then x is a solution of the problem* (3.1) *if and only if*

$$\langle Ay, \overrightarrow{xy} \rangle \ge 0, \ \forall y \in K.$$

Proof. If x is a solution of the problem (3.1) then, by monotonicity, we have

$$0 \le \langle Ay - Ax, \overrightarrow{xy} \rangle, \quad \forall y \in K,$$

which implies

$$0 \le \langle Ax, \overrightarrow{xy} \rangle \le \langle Ay, \overrightarrow{xy} \rangle, \quad \forall y \in K.$$

That is desired result.
Now, suppose $\langle Ay, \overrightarrow{xy} \rangle \ge 0, \quad \forall y \in K$. Thus, if $z \in K$, we get

$$0 \le \langle A(tz \oplus (1-t)x), \overrightarrow{x(tz \oplus (1-t)x)} \rangle.$$

Moreover, there exists $\alpha \ge 0$ and $w \in X$ such that

$$A(tz \oplus (1-t)x) = [\alpha \overrightarrow{w(tz \oplus (1-t)x)}].$$

Hence

$$\begin{aligned}
0 &\le 2\alpha \langle \overrightarrow{w(tz \oplus (1-t)x)}, \overrightarrow{x(tz \oplus (1-t)x)} \rangle \\
&= \alpha(d^2(w, tz \oplus (1-t)x) - d^2(x, w) + d^2(x, tz \oplus (1-t)x)) \\
&\le \alpha(td^2(w, z) + (1-t)d^2(w, x) - t(1-t)d^2(x, z) + t^2 d^2(z, x) - d^2(x, w)) \\
&= t\alpha(d^2(w, z) - d^2(w, x) - d^2(x, z)) + 2\alpha t^2 d^2(z, x) \\
&= 2t\alpha \langle \overrightarrow{wx}, \overrightarrow{xz} \rangle + 2\alpha t^2 d^2(z, x),
\end{aligned}$$

which follows
$$0 \leq \langle [\alpha \overrightarrow{wx}], \overrightarrow{xz} \rangle + td^2(z, x).$$
Now, $t \to 0$ in the last inequality, we get
$$0 \leq \langle [\alpha \overrightarrow{wx}], \overrightarrow{xz} \rangle.$$
Therefore, it is enough that we show $\langle [\alpha \overrightarrow{wx}], \overrightarrow{xz} \rangle = \langle Ax, \overrightarrow{xz} \rangle$. By weakly continuity of A, we have
$$\lim_{t \to 0} (\langle A(tz \oplus (1-t)x), \overrightarrow{zx} \rangle) = \langle Ax, \overrightarrow{zx} \rangle.$$
On the other hand,
$$\begin{aligned}
&2\langle A(tz \oplus (1-t)x), \overrightarrow{zx} \rangle = 2\alpha \langle \overrightarrow{w(tz \oplus (1-t)x)}, \overrightarrow{zx} \rangle \\
&= \alpha(d^2(w, x) - d^2(w, z) + d^2(tz \oplus (1-t)x, z) - d^2(tz \oplus (1-t)x, x)) \\
&= \alpha(d^2(w, x) - d^2(w, z) + (1-t)^2 d^2(x, z) - t^2 d^2(z, x)),
\end{aligned}$$
which, as $t \to 0$, is converges to
$$\alpha(d^2(w, x) - d^2(w, z) + d^2(x, z)) = 2\langle [\alpha \overrightarrow{wx}], \overrightarrow{zx} \rangle.$$
Hence, we obtain $\langle [\alpha \overrightarrow{wx}], \overrightarrow{xz} \rangle = \langle Ax, \overrightarrow{xz} \rangle$. This completes the proof. □

Remark 3.1. By Proposition 3.2, if $A : K \to X^*$ is a weakly continuous and monotone vector field, then the set of the solution of the problem (3.1) is closed.

Generally, if K is not bounded then the problem (3.1) does not always admit a solution. For this, let $X = K = \mathbb{R}$ and $Ax = \overrightarrow{(Tx)x}$, then the variational inequality
$$\text{Find} \quad x \in K : (x - Tx)(y - x) \geq 0, \quad \forall y \in K,$$
has no solution for $Tx = x + 1$.
We do not know whether the problem (3.1) has a solution if K is bounded. Set $K_R = K \cap B_R(o)$, where $B_R(o) := \{y \in K : d(o, y) \leq R\}$. Clearly, the problem
$$x_R \in K_R : \langle Ax_R, \overrightarrow{x_R y} \rangle \geq 0, \quad \forall y \in K_R, \tag{3.2}$$
admits a solution whenever $K_R \neq \emptyset$ and the following condition is satisfied,
$$\Lambda : \text{Variational inequality (3.1) is solvable, when } K \text{ is bounded.}$$

The condition Λ is satisfied for $Ax = [\overrightarrow{(Tx)x}]$, where $T : K \to X$ is a nonexpansive mapping, (see [10, Theorem 3.1]). Also, the condition Λ is satisfied for continuous vector fields in Hadamard manifolds, (see [15]).

Theorem 3.3. *Suppose $A : K \to X^*$ is a vector field. Then, the problem* (3.1) *admits a solution if and only if there exist $R > 0$ and a solution $x_R \in K_R$ of problem* (3.2) *such that $d(o, x_R) < R$.*

Proof. It is clear that if there exists a solution x of the problem (3.1), then x is a solution of the problem (3.2) whenever $d(o, x) < R$. Suppose now that a solution $x_R \in K_R$ of the problem (3.2) satisfies $d(o, x_R) < R$ and $y \in K$. Let $Ax_R = [\alpha\overrightarrow{z_Rx_R}]$. We can choose a $0 < t < 1$ such that $d(o, (1-t)x_R \oplus ty) \le R$. Thus $(1 - t)x_R \oplus ty \in K_R$. The problem (3.2) implies

$$\begin{aligned}
0 &\le 2\alpha\langle\overrightarrow{z_Rx_R}, \overrightarrow{x_R((1-t)x_R \oplus ty)}\rangle \\
&= \alpha(d^2(z_R, (1-t)x_R \oplus ty) - d^2(z_R, x_R) - d^2(x_R, (1-t)x_R \oplus ty)) \\
&\le \alpha((1-t)d^2(z_R, x_R) + td^2(z_R, y) - t(1-t)d^2(x_R, y) - d^2(z_R, x_R) - t^2d^2(x_R, y)) \\
&= 2t\alpha\langle\overrightarrow{z_Rx_R}, \overrightarrow{x_Ry}\rangle.
\end{aligned}$$

Consequently, $\langle Ax_R, \overrightarrow{x_Ry}\rangle \ge 0$, which implies x_R is a solution of the problem (3.1). ☐

Let $A : K \to X^*$ be a vector field. We say A is coercive if there exists $x_0 \in K$ such that

$$\frac{\langle Ax, \overrightarrow{x_0x}\rangle - \langle Ax_0, \overrightarrow{x_0x}\rangle}{d(x, x_0)} \to \infty, \qquad \textit{as} \quad d(x, o) \to \infty, \quad x \in K. \tag{3.3}$$

Theorem 3.4. *Suppose $A : K \to X^*$ is a vector field, the condition Λ is satisfied and A is coercive at x_0. Then, the problem* (3.1) *has a solution.*

Proof. Let $Ax_0 = [\alpha_0\overrightarrow{z_0x_0}]$. By coerciveness condition, choose $H > \alpha_0 d(x_0, z_0)$ and $R > d(x_0, o)$ such that

$$\langle Ax, \overrightarrow{x_0x}\rangle - \langle Ax_0, \overrightarrow{x_0x}\rangle \ge Hd(x_0, x), \qquad \text{for} \quad d(x, o) \ge R, \quad x \in K.$$

Then,

$$\begin{aligned}
\langle Ax, \overrightarrow{x_0x}\rangle &\ge \alpha_0\langle\overrightarrow{z_0x_0}, \overrightarrow{x_0x}\rangle + Hd(x_0, x) \\
&\ge -\alpha_0 d(x_0, z_0)d(x_0, x) + Hd(x_0, x) \\
&\ge (-\alpha_0 d(x_0, z_0) + H)(d(x, o) - d(x_0, o)) \\
&> 0.
\end{aligned}$$

Now, if $x_R \in K_R$ is a solution of the problem (3.2), then

$$\langle Ax_R, \overrightarrow{x_0 x_R} \rangle = -\langle Ax_R, \overrightarrow{x_R x_0} \rangle \leq 0,$$

which implies $d(x_R, o) < R$. Hence, by Theorem 3.3, the problem (3.1) has a solution. □

Generally, the solution of the variational inequality problem (3.1) is not unique. In the following theorem, we present a natural condition that ensures uniqueness.

Theorem 3.5. *Let the vector field $A : K \to X^*$ be strictly monotone. Then, the solution of the variational inequality* (3.1) *is unique.*

Proof. By contradiction, suppose $x, z \in K$ are two distinct solutions of the problem (3.1). Then,

(i) $\langle Ax, \overrightarrow{xy} \rangle \geq 0, \quad \forall y \in K,$

(ii) $\langle Az, \overrightarrow{zy} \rangle \geq 0, \quad \forall y \in K.$

Set $y = z$ in (i) and $y = x$ in (ii). Summing (i) with (ii), we obtain

$$\langle Ax - Az, \overrightarrow{xz} \rangle \geq 0,$$

which is contradiction with strictly monotonicity of A. □

4 Conclusion

The existence and uniqueness of the solutions of the variational inequalities on Hilbert spaces, Banach spaces, Hadamard manifolds, and Riemannian manifolds are investigated, (see [9, 11, 12, 13, 15, 18, 19] and the references therein). In this paper, the existence and uniqueness of the solution of the variational inequality for a vector field on complete CAT(0) metric spaces are discussed. In general, the variational inequality problem on the subset K of the Riemannian manifold M does not have a solution, even if the set K is totally convex and compact and the vector field is continuous, (see [13, Example 4.1]). However, it seems that the existence and uniqueness results of the solution of the variational inequality are satisfied locally on subset A of the generic Remainnian manifold M. Therefore, it is valuable to extend the results to other spaces. Extending the results to CAT(κ) spaces for $\kappa > 0$ or other spaces, and designing algorithms to approximate the solution of the variational inequality problem can be considered as future works.

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