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Predictability and uniqueness of weak solutions of the stochastic differential equations

Ana Merkle

Abstract

Causality is a topic which receives much attention nowadays and it represents a prediction property in the context of possible reduction of available information in order to predict a given filtration. In this paper we define the concept of dependence between stochastic processes and between filtrations, named causal predictability, which is based on the Granger's definition of causality. This definition extends the ones already given in the continuous time. Then, we provide some properties of the given concept.

Finally, we apply the concept of causal predictability to the processes of the diffusion type, more precisely, to the uniqueness of weak solutions of the stochastic differential equations.

1 Introduction

After the famous paper of Granger [7] many authors considered different ways of defining causality. The study of Granger's causality has been mainly concerned the time series. We shall, instead, consider continuous time processes, since continuous time models are frequently used as a starting point in climatology, ecology, econometric practice, demography, etc. It is important to emphasize that the modern finance theory extensively uses diffusion processes and deals with the solutions of stochastic differential equations. Also, the martingale theory is widely used in finance, so it should be noted that

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the preservation of martingale property is directly connected to some of the concepts of causality, especially with self causality. Namely, in the theory of martingales the concept of self causality is equivalent to the hypothesis (H) introduced in [20]. The concept of self causality could be considered as a useful assumption in the theory of martingales and stochastic integration (see [1] and [20]).

The causality concept, introduced by Mykland [15] and generalized in this paper, can be applied to the solutions of martingale problem and to the weak solutions of stochastic differential equations (see [15] and [8]).

The connection between the causality concept and the well known notion of measurable separability of σ-fields is shown in [21] and it is applied on the Bayesian experiments. As measurable separability is very important in the parametric inference, a relationship of the concept of statistical causality with sufficient and ancillary statistic is also given in the same paper.

The goal of our paper is to give a new concept of dependence between filtrations and to consider different causality properties in continuous time models and their applications. More precisely, we consider different concepts of causality between stochastic processes and between filtrations and then relate the given concepts of causality to the weak solutions of the following stochastic differential equation (SDE)

$$dX_t = a_t(X)dt + b_t(X)dW_t, \quad X_0 = \eta$$

where W is Wiener process. Also, we consider its generalized form

$$dX_t = u_t(X)dZ_t, \quad X_0 = \eta.$$

where Z is a semimartingale.

The paper is organized as follows. After Introduction, in Section 2, we provide some of the known results that we need later. Especially, we present the causality concept introduced by Mykland [15] and give some basic properties of this concept. Also, we recall definitions of weak solutions of the SDEs.

Section 3 and 4 contain the main results. In Section 3, we introduce the concept of dependence named causal predictability between filtrations which is based on Granger's definition of causality and the causality concept given by Mykland [15]. Also, we show some major characteristics of the given concept. In Section 4, we apply this new concept to the processes of the diffusion type, more precisely, to the uniqueness of weak solutions of the Itô SDEs and the SDEs with driving semimartingales. At the end of the section, we provide a few examples that illustrates applications of the given causality concept in the various fields.

2 Preliminary notations and definitions

Suppose that $(\Omega, \mathcal{A}, \mathbf{F}, P)$ is a filtered probability space, where $(\Omega, \mathcal{A}, P)$ is a probability space and $\mathbf{F} = \{\mathcal{F}_t,\ t \in T\}$ is a "framework" filtration, that is, $\mathcal{F}_t$ is the set of all events in the model up to and including time t, which is a subfiltration of $\mathcal{A}$. The total information $\mathcal{F}_\infty$ carried by $\mathbf{F}$ is defined as $\mathcal{F}_\infty = \sigma\{\bigcup_{t\in I} \mathcal{F}_t\} \equiv \bigvee_{t\in T} \mathcal{F}_t$. We suppose that the filtration $\mathbf{F}$ satisfies the usual conditions (i.e. it is right continuous and complete, see [3]). The time index set T is equal to $\mathbb{R}_+$, unless specified otherwise. Analogous notation will be used for filtrations $\mathbf{H} = \{\mathcal{H}_t\}$, $\mathbf{G} = \{\mathcal{G}_t\}$ and $\mathbf{I} = \{\mathcal{I}_t\}$.

It is said that filtration $\mathbf{G}$ is a subfiltration of $\mathbf{F}$ and written as $\mathbf{G} \subseteq \mathbf{F}$, if $\mathcal{G}_t \subseteq \mathcal{F}_t$ for each t.

We now recall the concept of conditional independence which is widely used in probability theory and statistics.

Definition 2.1. *(*compare with [2] and [18]*). Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}$ arbitrary sub-σ-algebras from $\mathcal{A}$. It is said that $\mathcal{M}$ is splitting for $\mathcal{M}_1$ and $\mathcal{M}_2$ or that $\mathcal{M}_1$ and $\mathcal{M}_2$ are conditionally independent given $\mathcal{M}$ (and written as $\mathcal{M}_1 \perp \mathcal{M}_2 \mid \mathcal{M}$) if*

$$E[X_1 X_2 \mid \mathcal{M}] = E[X_1 \mid \mathcal{M}] E[X_2 \mid \mathcal{M}].$$

where X_i denotes a nonnegative $\mathcal{M}_i$ measurable random variable, $i = 1, 2$.

The basic properties of this concept are given in [5] and [6].

In the papers [4] and [5] it is shown how conditional independence can serve as a basis for the general probabilistic theory of causality for both processes and single events. In this paper, we also consider different concepts of causality involving conditional independence. Motivated by Granger causality, Mykland introduces the following definition.

Definition 2.2. *([15]) Let $\mathbf{F} = \{\mathcal{F}_t\}$, $\mathbf{G} = \{\mathcal{G}_t\}$ and $\mathbf{H} = \{\mathcal{H}_t\}$ $t \in T$, be filtrations on the same probability space. It is said that $\mathbf{G}$ is a cause of $\mathbf{H}$ within $\mathbf{F}$ relative to P (and written as $\mathbf{H} \mid\!< \mathbf{G}; \mathbf{F}; P$) if $\mathbf{H} \subseteq \mathbf{F}$, $\mathbf{G} \subseteq \mathbf{F}$ and if $\mathcal{H}_\infty$ is conditionally independent of $\mathcal{F}_t$ given $\mathcal{G}_t$ for each t, i.e.*

$$\mathcal{H}_\infty \perp \mathcal{F}_t \mid \mathcal{G}_t, \tag{1}$$

that is

$$\forall A \in \mathcal{H}_\infty, \quad P(A|\mathcal{F}_t) = P(A|\mathcal{G}_t).$$

If there is no doubt about P, we omit "relative to P".

It is easy to see that (1) may be formulated as

$$\mathcal{H}_u \perp \mathcal{F}_t \mid \mathcal{G}_t \ \text{ for all } \ t, u \in T.$$

Intuitively, (1) means that, for arbitrary t, information about $\mathcal{H}_\infty$ provided by $\mathcal{F}_t$ is not "bigger" than that provided by $\mathcal{G}_t$.

If $\mathbf{G}$ and $\mathbf{F}$ are such that $\mathbf{G} \mid\!< \mathbf{G}; \mathbf{F}$, we shall say that $\mathbf{G}$ is its own cause within $\mathbf{F}$, or, that $\mathbf{G}$ is self caused within $\mathbf{F}$. It should be mentioned that the notion of subordination (as introduced in [18]) is equivalent to the notion of being one's own cause or self causality, as defined here. The concept of self causality could be considered as a useful assumption in the theory of martingales and stochastic integration ([1], [20]). It is interesting to notice that there is a strong connection between the preservation of the martingale property and the causality concept. It is well known that the martingale property remains valid if the filtration decreases. But, if the filtration increases the preservation of martingale property is directly connected to the concept of self causality. Namely, in the theory of martingales the concept of self causality is equivalent to the hypothesis (H) introduced in [1]: if $\mathbf{G} \subseteq \mathbf{F}$, every $\mathbf{G}$-martingale is a $\mathbf{F}$-martingale, that is, $\mathbf{G}$ is immersed in $\mathbf{F}$.

If $\mathbf{G}$ and $\mathbf{F}$ are such that $\mathbf{G} \mid\!< \mathbf{G}; \mathbf{G} \bigvee \mathbf{F}$ (where $\mathbf{G} \bigvee \mathbf{F}$ is a family determined by $(\mathcal{G} \bigvee \mathcal{F})_t = \mathcal{G}_t \bigvee \mathcal{F}_t$), we shall say that $\mathbf{F}$ does not cause $\mathbf{G}$. It is clear that the interpretation of Granger's causality is now that $\mathbf{F}$ does not cause $\mathbf{G}$ if $\mathbf{G} \mid\!< \mathbf{G}; \mathbf{G} \bigvee \mathbf{F}$ (see [15]).

It should be mentioned that Definition 2.2 of causality is equivalent to the definition of strong global noncausality, given by Florens and Fougères [4].

A family of σ-algebras induced by a stochastic process $X = \{X_t, t \in T\}$ is given by $\mathbf{F}^X = \{\mathcal{F}_t^X, t \in T\}$, where

$$\mathcal{F}_t^X = \sigma\{X_u, u \in T, u \leq t\},$$

being the smallest σ-algebra with respect to which the random variables $X_u, u \leq t$ are measurable.

The process $X = \{X_t\}$ is adapted to $\mathbf{F} = \{\mathcal{F}_t\}$ or $\mathbf{F}$-adapted if $\mathcal{F}_t^X \subseteq \mathcal{F}_t$ for each t . The notation $(X_t, \mathcal{F}_t)$ means that process $X = \{X_t\}$ is $\mathbf{F}$-adapted.

A family of σ-algebras may be induced by several processes, e.g. $\mathbf{F}^{X,Y} = \{\mathcal{F}_t^{X,Y}, t \in T\}$, where

$$\mathcal{F}_t^{X,Y} = \mathcal{F}_t^X \bigvee \mathcal{F}_t^Y, t \in T.$$

Definition 2.2 can be applied to stochastic processes. It will be said that stochastic processes are in a certain relationship if and only if the corresponding induced filtrations are in that relationship. Specially, an $\mathbf{F}$-adapted stochastic process $X = \{X_t\}$ is its own cause within $\mathbf{F}$ if $\mathbf{F}^X = \{\mathcal{F}_t^X\}$ is its own cause within $\mathbf{F}$, i.e. if $\mathbf{F}^X \mid\!< \mathbf{F}^X; \mathbf{F}; P$ holds (see, for example, [21]).

We consider a weak solution of a SDE

$$\begin{aligned} dX_t &= a_t(X)dt + b_t(X)dW_t \\ X_0 &= \eta, \end{aligned} \tag{2}$$

where X is a continuous d-dimensional process and W a d-dimensional Wiener process, which is well defined when the following elements are given: the dimension d (of X and W), functionals a_t (d-dimensional vector) and b_t($d \times d$ matrix) and d-dimensional distribution F_η function (see [12], [13], [17]).

Definition 2.3. *([12]) The object $(\Omega, \mathcal{A}, \mathbf{F}, P, W, X)$ is said to be a weak solution of SDE (2) if*

i) $(\Omega, \mathcal{A}, \mathbf{F}, P)$ *is a filtered probability system with time axis of the form $T = [0, t_0]$, with F_t right continuous and with F and F_0 complete,*

ii) $(W_t, \mathcal{F}_t)$ *is a d-dimensional Wiener process,*

iii) $\{X_t\}$ *is a continuous adapted process,*

iv) $P(X_0 \leq x) = F_\eta(x)$,

v) $\int_0^{t_0} |a_s(X)ds| < \infty$ *and* $\int_0^{t_0} |b_s(X)|^2 ds < \infty$ *a.s.,*

vi) $X_t = X_0 + \int_0^t a_s(X)ds + \int_0^t b_s(X)dW_s$ *a.s.*

Definition 2.4. *([12]) For given d, a_t, b_t and F_η, a weak solution of (2) is weakly unique if for any two solutions $(\Omega^i, \mathcal{A}^i, \mathbf{F}^i, P^i, W^i, X^i)$, i=1,2, of the system, the induced measures μ_{X^1} and μ_{X^2} coincide (where $\mu_X(B) = P(X(\omega) \in B)$).*

Now, let us consider the following stochastic differential equation

$$\begin{aligned} dX_t &= u_t(X)dZ_t \\ X_0 &= \eta. \end{aligned} \tag{3}$$

where $Z = \{Z_t, t \in T\}$ is a $m-$dimenisonal semimartingale and coefficient u_t is a $(n \times m)-$dimensional predictable functional (see [8] and [9] for more details). Jacod and Memin [9] have studied the existence and the uniqueness of solutions of equation (3), which was generalized by Lebedev [11]. We adopt the approach developed by Mykland [15] of considering a probability measure P on $(\Omega, \mathcal{A})$ such that the given process X is a solution with respect to the given process Z, which is typically a sort of the problem related to the so-called martingale problem. The following definitions connected with the solution of equation (3) are taken from [16].

Definition 2.5. *([16]) For the stochastic differential equation (3),* $(\Omega, \mathcal{A}, \mathbf{F}, P, Z, X)$ *is a regular weak solution if*

i) $\mu(A) = P(Z \in A)$ *coincides with a predetermined measure on the function space where* Z *takes values;*

ii) X *and* Z *satisfy (3);*

iii) Z *is its own cause within* $\mathbf{F} = \{\mathcal{F}_t\}$ *relative to* P*, i.e.* $\mathbf{F}^Z \mid\!< \mathbf{F}^Z; \mathbf{F}; P$ *holds.*

Definition 2.6. *([16]) A regular solution is weakly unique if for every regular solution* $(\Omega, \mathcal{A}, \mathbf{F}, P, Z, X)$ *of equation (3) there is no measure* Q *on* $\mathcal{F}_\infty^{X,Z}$ *such that* $(\Omega, \mathcal{F}_\infty^{X,Z}, \mathbf{F}^{X,Z}, Q, Z, X)$ *is a regular solution of (3).*

Definition 2.7. *([16]) An extremal regular weak solution* $(\Omega, \mathcal{A}, \mathbf{F}, P, Z, X)$ *of (3) is a regular weak solution such that there are measures* Q_1 *and* Q_2 *on* $\mathcal{F}_\infty^{X,Z}$ *satisfying* $P = a_1 Q_1 + a_2 Q_2, a_1, a_2 > 0$ *on* $\mathcal{F}_\infty^{X,Z}$ *and such that* $(\Omega, \mathcal{F}_\infty^{X,Z}, \mathbf{F}^{X,Z}, Q_i, Z, X)$ *is a regular weak solution, then* $Q_1 = Q_2 = P$ *on* $\mathcal{F}_\infty^{X,Z}$*.*

3 Causal predictability between filtrations

In this section, we consider a generalization of causality concept given in Definition 2.2. More precisely, we develop a concept named causal predictability between filtrations which is based on the causality concept from Section 2. This concept is shown to be connected to a generalization of the notion of weak uniqueness of weak solutions of the stochastic differential equations. The relation that we want to show between the filtrations $\mathbf{F}^X$ and $\mathbf{F}^{X,W}$ in a weakly unique solution of stochastic differential equation (2) is that for any filtration $\mathbf{F}$ and for any probability measure P satisfying that (W_t, F_t) is a Wiener process relative to P we must have that

$$\mathbf{F}^X \mid\!< \mathbf{F}^{X,W}; \mathbf{F}; P.$$

However, there is no probability space that can contain all possible $\mathbf{F}$, so we have to use isomorphisms between probability spaces.

A probability space $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{P})$ is an extension of $(\Omega, \mathcal{A}, P)$ if there is a measurable function

$$f : (\hat{\Omega}, \hat{\mathcal{A}}) \longrightarrow (\Omega, \mathcal{A})$$

such that $\hat{P} f^{-1} = P$ on $\mathcal{A}$ (see [8]).

For filtrations $\mathbf{G} = \{\mathcal{G}_t\}$, $\mathbf{H} = \{\mathcal{H}_t\}$ and $\mathbf{I} = \{\mathcal{I}_t\}$, $t \in T$, on the same probability space $(\Omega, \mathcal{A}, P)$, we have

$$\hat{\mathcal{G}}_t = \{f^{-1}(A) : A \in \mathcal{G}_t\}, \quad \hat{\mathcal{H}}_t = \{f^{-1}(A) : A \in \mathcal{H}_t\}, \quad \hat{\mathcal{I}}_t = \{f^{-1}(A) : A \in \mathcal{I}_t\}.$$

Note that there may be filtrations on the probability space $(\hat{\Omega}, \hat{\mathcal{A}})$ which are not defined as an inverse image of some filtration on the original probability space $(\Omega, \mathcal{A})$. Also, note that $\hat{\mathcal{A}} \neq f^{-1}(\mathcal{A})$.

Motivated by the previous considerations, through the following definition we introduce the main concept used in the paper.

Definition 3.1. *Let $\mathbf{G} = \{\mathcal{G}_t\}$, $\mathbf{H} = \{\mathcal{H}_t\}$ and $\mathbf{I} = \{\mathcal{I}_t\}$, $t \in T$, be filtrations on the same probability space $(\Omega, \mathcal{A}, P)$ with a common time axis T. It is said that $\mathbf{I}$ is causally predictable by $\mathbf{H}$ relative to $\mathbf{G}$ if*

$$\mathbf{H} \subseteq \mathbf{G} \tag{4}$$

and

$$\mathbf{H} \mid\!< \mathbf{G}; \mathbf{G}'; P, \tag{5}$$

where

$$\mathbf{G}' = \mathbf{G} \vee \mathbf{I} \tag{6}$$

and if

any extension $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{P})$ of the probability space $(\Omega, \mathcal{G}'_\infty, P)$ (with $\hat{P} f^{-1} = P$) containing filtrations $\hat{\mathbf{F}}, \hat{\mathbf{G}}', \hat{\mathbf{G}}, \hat{\mathbf{H}}$ and $\hat{\mathbf{I}}$ satisfying

a)

$$\hat{\mathbf{G}}' \subseteq \hat{\mathbf{F}}; \tag{7}$$

b) for every $A \in \mathcal{H}_\infty$ and $t \in T$

$$\left.\begin{array}{l} g(A) = P(A \mid \mathcal{G}'_t)\,(P\text{-}a.s.) \\ g \text{ is } \mathcal{G}'_t\text{-measurable} \end{array}\right\} \Rightarrow g \circ f(A) = \hat{P}(f^{-1}(A) \mid \hat{\mathcal{G}}'_t)\,(\hat{P}\text{-}a.s.);$$

c)

$$\hat{\mathbf{H}} \mid\!< \hat{\mathbf{G}}; \hat{\mathbf{F}}; \hat{P};$$

must also satisfy

$$\hat{\mathbf{I}} \mid\!< \hat{\mathbf{G}}'; \hat{\mathbf{F}}; \hat{P}. \tag{8}$$

If $\mathbf{G} = \mathbf{H}$ in the above definition, we say that $\mathbf{I}$ is causally predictable by $\mathbf{H}$.

It should be noted that causal predictability is a notion of dependence.

Now, to get more familiar with the concept of causal predictability, we are going to give some of the main properties of the introduced concept.

Lemma 3.1. *Let* $\mathbf{G} = \{\mathcal{G}_t\}$, $\mathbf{H} = \{\mathcal{H}_t\}$ *and* $\mathbf{I} = \{\mathcal{I}_t\}$, $t \in T$, *be filtrations on the same probability space* $(\Omega, \mathcal{A}, P)$. *If conditions a) and b) of Definition 3.1 of causal predictability are satisfied for* $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{P})$, $\hat{\mathbf{G}}', \hat{\mathbf{G}}, \hat{\mathbf{H}}$ *and* $\hat{\mathbf{I}}$, *and if (5) is satisfied, then*

$$\hat{\mathbf{H}} \mid\!< \hat{\mathbf{G}}; \hat{\mathbf{G}}'; \hat{P}. \tag{9}$$

Proof. Let $A \in \mathcal{H}_\infty$ and $t \in T$. By (5), there is a $\mathcal{G}_t$-measurable g satisfying

$$g = P(A|\mathcal{G}'_t).$$

By condition b) and the definition of $\hat{\mathcal{G}}_t$, $\hat{P}(f^{-1}(A)|\hat{\mathcal{G}}'_t)$ can be chosen to be $\hat{\mathcal{G}}_t-$measurable. Now, (9) follows by the definition of $\hat{\mathcal{H}}_\infty$. □

Lemma 3.2. *Condition c) of the definition of causal predictability can be replaced by*

c′) $$\hat{\mathbf{H}} \mid\!< \hat{\mathbf{G}}'; \hat{\mathbf{F}}; \hat{P}.$$

Proof. According to Proposition 2.2 in [15], from (6) and (7) it follows that $\hat{\mathbf{H}} \mid\!< \hat{\mathbf{G}}; \hat{\mathbf{F}}; \hat{P}$ is equivalent with

$$\hat{\mathbf{H}} \mid\!< \hat{\mathbf{G}}; \hat{\mathbf{G}}'; \hat{P} \quad \text{and} \quad \hat{\mathbf{H}} \mid\!< \hat{\mathbf{G}}'; \hat{\mathbf{F}}; \hat{P}.$$

The result is proved. □

Corollary 3.3. *The following statements are equivalent:*

i) $\mathbf{I}$ *is causally predictable by* $\mathbf{H}$ *relative to* $\mathbf{G}$.

ii) $\mathbf{I}$ *is causally predictable by* $\mathbf{H}$ *relative to* $\mathbf{G}'$ *and conditions* $(4) - (5)$ *are satisfied.*

Proof. Replace *c*) by *c*′) in Definition 3.1. The only difference between causal predictability relative to $\mathbf{G}$ and relative to $\mathbf{G}'$ are (4) and (5). □

In the following result we prove the invariance of causal predictability under stochastic equivalence.

Proposition 3.4. *Let* $\mathbf{G}$, $\tilde{\mathbf{G}}$, $\mathbf{H}$, $\tilde{\mathbf{H}}$, $\mathbf{I}$ *and* $\tilde{\mathbf{I}}$ *be filtrations in a probability space* $(\Omega, \mathcal{A}, P)$ *such that*

$$\mathbf{G} = \tilde{\mathbf{G}},\ \mathbf{H} = \tilde{\mathbf{H}},\ \mathbf{I} = \tilde{\mathbf{I}}\ \textit{a.s.}$$

and

$$\tilde{\mathbf{H}} \subseteq \tilde{\mathbf{G}}.$$

If $\mathbf{I}$ *is causally predictable by* $\mathbf{H}$ *relative to* $\mathbf{G}$, *then* $\tilde{\mathbf{I}}$ *is causally predictable by* $\tilde{\mathbf{H}}$ *relative to* $\tilde{\mathbf{G}}$.

Proof. According to Theorem 4.4. from [14], this result follows from the invariance of causality, in the sense of Definition 2.2, under stochastic equivalence. □

4 Some applications

In the following result we show how the theory of causal predictability can be applied to the weak uniqueness of weak solution of the stochastic differential equation (2)

$$dX_t = a_t(X)dt + b_t(X)dW_t, \quad X_0 = \eta.$$

Theorem 4.1. *Assume that SDE (2) has a weak solution. If for every weak solution* $(\Omega, \mathcal{A}, \mathbf{F}, P, W, X)$ *of (2),* $\mathbf{F}^X$ *is causally predictable by* $\mathbf{F}^W$ *relative to* $\mathbf{F}^{X,W}$*, then the solution is weakly unique.*

Proof. If $\mathbf{F}^X$ is causally predictable by $\mathbf{F}^W$ relative to $\mathbf{F}^{X,W}$, from Definition 3.1 of causal predictability for $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{P}) = (\Omega, \mathcal{A}, P)$, it follows that X is entirely caused by itself and by W within $\mathbf{F}$, i.e. $\mathbf{F}^X \ltimes \mathbf{F}^{X,W}; \mathbf{F}; P$ holds.

Now, let $(\Omega^i, \mathcal{A}^i, \mathbf{F}^i, P^i, W^i, X^i)$, $i = 1, 2$, be two weak solutions of (2) and without loosing generality suppose that $\Omega^1 \cap \Omega^2 = \emptyset$. Set

$$\begin{aligned}
\Omega &= \Omega^1 \cup \Omega^2 \\
\mathcal{A} &= \left\{A \cup B : A \in \mathcal{A}^1,\ B \in \mathcal{A}^2\right\} \\
\mathcal{F}_t &= \left\{A \cup B : A \in \mathcal{F}_t^1,\ B \in \mathcal{F}_t^2\right\} \\
P(A \cup B) &= \frac{1}{2}[P(A) + P(B)],\ \text{for } A \in \mathcal{A}^1,\ B \in \mathcal{A}^2
\end{aligned}$$

$$W_t(\omega) = \begin{cases} W_t^1(\omega) \text{ for } \omega \in \Omega^1 \\ W_t^2(\omega) \text{ for } \omega \in \Omega^2 \end{cases},$$

$$X_t(\omega) = \begin{cases} X_t^1(\omega) \text{ for } \omega \in \Omega^1 \\ X_t^2(\omega) \text{ for } \omega \in \Omega^2 \end{cases},$$

It is easy to see that $(\Omega, \mathcal{A}, \mathcal{F}_t, P, W_t, X_t)$ is a weak solution of (2).
Set

$$j(\omega) = \begin{cases} 1, \text{ for } \omega \in \Omega^1 \\ 2, \text{ for } \omega \in \Omega^2 \end{cases},$$

X_0 and j are independent. It follows that $\mathcal{F}_\infty^X$ is conditionally independent of $\mathcal{F}_0$ given $\mathcal{F}_0^X$ (using $W_0 = 0$), since X is entirely caused by itself and by W within $\mathbf{F}$. From Theorem 2.5 in [14] it follows that $\mathcal{F}_\infty^X$ is independent of j, which implies that $P(X \in A|j)$ is constant for $A \in B(C^d)$ P-a.s. Now, as

$$P(X \in A|j)(\omega) = P(X^i \in A)$$

holds for almost all $\omega \in \Omega^i$ (P-a.s. and P^i-a.s.), we have

$$P(X^1 \in A) = P(X^2 \in A),$$

which shows that the solution is weakly unique. □

Now it is easy to prove the following result.

Proposition 4.2. *Assume that SDE (2) has a weak solution. If for every weak solution* $(\Omega, \mathcal{A}, \mathbf{F}, P, W, X)$ *of (2),* X *is its own cause within* $\mathbf{F}$*, then the solution is weakly unique.*

Proof. Suppose that X is its own cause within $\mathbf{F}$, i.e. that $\mathbf{F}^X \mid\!< \mathbf{F}^X; \mathbf{F}$ holds. From Definition 2.2 of causality and since $\mathbf{F}^X \subseteq \mathbf{F}^{X,W} \subseteq \mathbf{F}$, we have

$$\forall A \in \mathcal{F}^X_\infty, \quad P(A|\mathcal{F}^X_t) = P(A|\mathcal{F}_t) = P(A|\mathcal{F}^{X,W}_t).$$

It follows that X is caused by itself and by W within $\mathbf{F}$, i.e. $\mathbf{F}^X \mid\!< \mathbf{F}^{X,W}; \mathbf{F}; P$. Now, following the procedure from the proof of Theorem 4.1 we conclude that the solution is weakly unique. □

The following example illustrates the usefulness of the causality concept presented in this paper.

Example 4.1. *Considering the concrete SDE of the form (2), the well known result of Theorem 4.12 in [12] gives conditions for a weak uniqueness of weak solutions. Now, we have new conditions in terms of causal predictability from Definition 3.1. Namely, from Theorem 4.3 it follows that for weakly uniqueness of the given weak solution it is enough to check whether for every weak solution* $(\Omega, \mathcal{A}, \mathbf{F}, P, W, X)$ *of (2),* $\mathbf{F}^X$ *is causally predictable by* $\mathbf{F}^W$ *relative to* $\mathbf{F}^{X,W}$*.*

Now, we give another example of the application of causal predictability considering SDE (3)

$$dX_t = u_t(X) dZ_t, \quad X_0 = \eta,$$

which generalizes the diffusion equation (2). The definition of weak solution of SDE (3) in terms of causality (in the sense of Definition 2.2) is given in Definition 2.3. Now, we prove the following result which gives conditions for a solution of stochastic differential equation (3) to be weakly unique in terms of causal predictability.

Theorem 4.3. *Assume that SDE (3) has a weak solution. If for every weak solution* $(\Omega, \mathcal{A}, \mathbf{F}, P, Z, X)$ *of (3),* $\mathbf{F}^{X,Z}$ *is causally predictable by* $\mathbf{F}^X$ *relative to* $\mathbf{F}^{X,Z}$*, then the solution is weakly unique.*

Proof. Suppose that for the weak solution $(\Omega, \mathcal{A}, \mathbf{F}, P, Z, X)$ of (3) we have that $\mathbf{F}^{X,Z}$ is causally predictable by $\mathbf{F}^X$ relative to $\mathbf{F}^{X,Z}$.

We will first show that P is extremal on $\mathcal{F}_\infty^{X,Z}$. Suppose that P is not extremal, i.e. $P = a_1Q_1 + a_2Q_2, a_1, a_2 > 0$, so that Q_1 and Q_2 coincide on $\mathcal{F}_{-\infty}^{X,Z}$, but $Q_1 \neq Q_2$. Set the extension of probability space $(\Omega, \mathcal{F}_\infty^{X,Z}, P)$ in the definition of causal predictability as follows

$$\hat{\Omega} = \Omega \times \{p, q\}$$

$$\hat{\mathcal{F}}_t = \mathcal{F}_t^{X,Z} \times \{\phi, \{p\}, \{q\}, \{p, q\}\}$$

and

$$\hat{P}(A \times \{p\} \cup B \times \{q\}) = \frac{1}{2}\left(Q_1(A) + Q_2(B)\right).$$

It is straightforward to check that the conditions in the definition of causal predictability are satisfied, but that (8) is not, i.e.

$$\hat{\mathbf{F}}^{X,Z} \mid\!< \hat{\mathbf{F}}^{X,Z}; \hat{\mathbf{F}}, \hat{P}$$

does not hold, which contradicts the assumption that $\mathbf{F}^{X,Z}$ is causally predictable by $\mathbf{F}^X$ relative to $\mathbf{F}^{X,Z}$. Hence, P is extremal on every weak solution. Now, from [15] it follows that the solution is weakly unique. □

Remark 1. *It should be noted that the results from Theorem 4.1, Proposition 4.2 and Theorem 4.3 remain valid even when we consider different ways of defining causal predictability as given in Lemma 3.1 and Lemma 3.2.*

The concept of causality is widely used in financial mathematics and the default risk modeling which was extensively studied in numerous recent papers. Here, a special attention is paid to the hypothesis (H), i.e. the concept of self causality. When hypothesis (H) is satisfied, all the contingent claims are hedgeable.

In the following example we will establish a sufficient condition for the martingale hazard process Λ to determine the conditional survival probability of τ given the σ-field $\mathbf{F}$ in terms of causality.

Example 4.2. *Assume that* $\mathbf{G}$ *is its own cause within* $\mathbf{F}$*, i.e. that* $\mathbf{G} \mid\!< \mathbf{G}; \mathbf{F}$ *holds and that the* $\mathbf{G}$ *-martingale hazard process* Λ *of* τ *is continuous. If the process* V *given by the formula*

$$V_t = E(e^{\Lambda_t - \Lambda_s} | \mathcal{F}_t), \quad \forall t \in [0, s]$$

is continuous at τ*, that is,* $\Delta V_{s\wedge\tau} = V_{s\wedge\tau} - V_{(s\wedge\tau)-} = 0$*, then for any* $t \leq s$ *we have*

$$P(\tau > s | \mathcal{F}_t) = I_{\{\tau > t\}} E(e^{\Lambda_t - \Lambda_s} | \mathcal{G}_t).$$

It is well known that the theory of optimal transport in continuous time and stochastic analysis are powerfully connected. Now, we give an example illustrating the connection with the causality concept from Definition 3.1.

Example 4.3. *A pair* (X, Y) *of continuous processes is a causal coupling if* $\mathbf{F^Y}$ *is causally predictable by* $\mathbf{F^X}$.

In the future work, it might be interesting to deal with the case of progressive causal predictability, i.e. with the case when $\{\mathcal{J}_t, t \in \tau \cap (-\infty, u)\}$ is causally predictable by $\{\mathcal{H}_t, t \in \tau \cap (-\infty, u)\}$ for stopping time τ and all $u \in \mathbb{R}$.

Also, it might be interesting to see how the theory of causal predictability can be applied to stochastic filtering and control theory (see, for example, [19]).

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Ana MERKLE,
Department of Probability and Mathematical Statistics,
University of Belgrade, Faculty of Mathematics
Studentski trg 16, 11158 Belgrade, Serbia.
Email: ana_merkle@matf.bg.ac.rs