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Fuzzy stabilities of a new hexic functional equation in various spaces

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Abstract

The advantage of various fuzzy normed spaces is to analyse impreciseness and ambiguity that arise in modelling problems. In this paper, various classical stabilities of a new hexic functional equation in different fuzzy spaces like fuzzy Banach space, Felbin's fuzzy Banach space and intuitionistic fuzzy Banach space are presented, concerning the Ulam's theory of stabilities of equations. To validate the stability results, experimental results are presented. Also, a comparative study of the results obtained in this investigation are provided.

1 Introduction

The notion of fuzzy sets was first dealt in [48]. Fuzzy sets are powerful tools to model problems with uncertainty and inexactness in many decision making problems. Recently, this theory is playing major roles in the application areas of many other disciplines and hence it is an active domain of research and a lot of growth in this field is established. This theory is to determine the fuzzy similarities with the classical set theory. Also, the fuzzy topology is mostly used as a powerful technique to deal with such problems where the application of classical theories fail.

The problem of solving classical stabilities of various functional equations is the recent trend in the research of pure mathematics. The stability theory of

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functional equations is evolved through a basic question in [45, 46] concerning the conditions of an approximate solution exists near to the exact solution of any mathematical equation. This question can be stated mathematically as: *Let* $(G_1, \cdot)$ *be a group and* $(G_2, \cdot)$ *be a metric group defined with the metric* $d(\cdot, \cdot)$. *Let* $\delta > 0$ *be a constant. Suppose a homomorphism* $h : G_1 \longrightarrow G_2$ *satisfies the inequality* $d(h(pq), h(p)h(q)) < \delta$, *for all* $p, q \in G_1$. *Does there exist a constant* $\epsilon > 0$ *and an approximate homormophism* $H : G_1 \longrightarrow G_2$ *such that* $d(H(p), h(p)) \leq \epsilon$ *for all* $p \in G_1$? The foremost response to this question is presented in [17]. The response provided in [17] is given as the proof of the following theorem:

Theorem 1.1 [17] Suppose X be a real linear space and Y be a real Banach space. Let $\epsilon > 0$ be a constant. If a mapping $a : X \longrightarrow Y$ satisfies the following inequality

$$\|a(p+q) - a(p) - a(q)\| \leq \epsilon \tag{1}$$

for all $p, q \in X$, then there exists a unique additive mapping $A : X \longrightarrow Y$ with the condition that

$$\|a(p) - A(p)\| \leq \epsilon \tag{2}$$

for all $p \in X$ and the mapping A is defined by the existence of the limit

$$A(p) = \lim_{n\to\infty} \frac{a(2^n p)}{2^n} \tag{3}$$

for all $p \in X$. In the above theorem, the exact solution $A(p)$ of the Cauchy functional equation $A(p+q) = A(p) + A(q)$ is directly derived from the inequality (1). Hence this method is called as direct method. The stability obtained using a positive constant ϵ as an upper bound is termed as Hyers-Ulam stability of ϵ-stability.

Later this stability problem is answered in a different version by considering sum of powers of norms as upper bound in [37]. This type of stability of functional equation is obtained by proving the following theorem [37].

Theorem 1.2 [37] Assume that M is a normed linear space and N is a Banach space. Suppose $0 \leq \theta < \infty$ and $0 \leq \beta < 1$. If $\phi : M \longrightarrow N$ is a mapping satisfying the inequality

$$\|\phi(p+q) - \phi(p) - \phi(q)\| \leq \theta(\|p\|^\beta + \|q\|^\beta) \tag{4}$$

for all $p, q \in M$, then a unique additive function $T : M \longrightarrow N$ exists such that

$$\|\phi(p) - T(p)\| \leq \frac{2\theta}{2 - 2^\beta}\|p\|^\beta \tag{5}$$

for all $p \in M$. Additionally, if $\phi(tp)$ is continuous in t for each fixed $t \in M$, then the function T is additive. The above type of stability is known as Hyers-Ulam-Rassias (or H-U-R) stability of functional equation. A modified stability problem was introduced in [15] by replacing the upper bounds ϵ in Theorem 1.1 and $\theta(\|p\|^\beta + \|q\|^\beta)$ in Theorem 1.2 by a general control function $\phi(p, q)$ in [15]. This type of stability is given in the subsequent theorem.

Theorem 1.3 [15] Let $(A, +)$ and $(X, \|\cdot\|)$ be an abelian group and be a Banach space, respectively. Let a function $\phi : A \times A \longrightarrow [0, \infty)$ satisfies

$$\Phi(p, q) = \sum_{k=0}^{\infty} 2^{-k} \phi(2^k p, 2^k q) < \infty \tag{6}$$

for all $p, q \in A$. Let a function $f : A \longrightarrow X$ satisfies the condition

$$\|f(p + q) - f(p) - f(q)\| \leq \phi(p, q) \tag{7}$$

for all $p, q \in A$. Then a unique mapping $T : A \longrightarrow X$ exists and satisfies

$$\|f(p) - T(p)\| \leq \frac{1}{2} \Phi(p, p) \tag{8}$$

for all $p \in A$. The above result established by Găvruta is celeberated as generalized Hyers-Ulam-Rassias stability or generalized H-U-R stability of functional equation.

There are some other methods to obtain stability results of functional equations. They are fixed point method [9, 10], hyperstability using fixed point technique [7], Brzdek's method of proving stability on restricted domains [6], Hejmej's method of obtaining stability in dislocated metric spaces [16]. Among the mentioned above methods, direct method is simple and easy to arrive at stability results. Hence this method is mostly used by many mathematicians to establish stabilities of numerous equations in various normed spaces.

A lot of fundamental ideas about functional equations and inequalities with their stability results are published as books, one can see ([3, 12, 18, 21, 27, 38, 41]). For the past five decades, this research area has attracted many mathematicians to establish stability results of a variety of functional, difference, differential and integral equations in various modern complete normed spaces in [8, 11, 22, 32, 35, 40]. One can find there are many interesting, ground breaking and noteworthy results concerning the stability results of many functional equations in various fuzzy normed spaces in ([26, 34]). Further, the approximations of homomorphisms, uniqueness theorems on functional inequalities arising addtive, quadratic and cubic mappings, stabilities of an n-dimensional mixed type addtive-quadratic functional equation and Hom-derivations in $C^\star$-algebras are dealt by many mathematicians in [19, 20, 29, 30]. Quite recently, there are many published papers on stability results and applications of rational functional equations. For more information, one can refer to [39, 42, 43, 44].

Some interesting results on the stabilities of hypergeometric differential equations and Laguerre differential equations are available in [1, 2]. Further, the stability results of trigonometric functional equations, functional equations associated to Fibonacci numbers, mean value-type functional inequalities, first order linear differential operators, Steven Butler functional equation are studied in [13, 23, 24, 25, 31, 33, 36].

In this paper, we present results pertinent to classical stabilities of a new hexic functional equation of the form

$$\begin{aligned} h(kp+lq)+h(kp-lq)+l^6\left[h\left(p+\frac{k}{l}q\right)+h\left(p-\frac{k}{l}q\right)\right] \\ = k^2l^2\left(k^2+l^2\right)\left[h(p+q)+h(p-q)\right]+2\left(k^2-l^2\right)\left(k^4-l^4\right)\left[h(p)+h(q)\right] \end{aligned} \tag{9}$$

where k, l are constants, in various abstract spaces like fuzzy Banach spaces, Felbin's type fuzzy normed spaces and intuitionistic fuzzy Banach spaces. The experimental results in support of the stabilities obtained are presented. Further, a comparative evaluation of the results obtained are discussed. It is easy to verify that the hexic mapping $h(p) = kp^6$, with k as a constant, is a solution of equation (9). In order to accomplish the results in a simplified manner, let us define a difference operator $\Delta : A \times A \longrightarrow \mathcal{F}$ as

$$\begin{aligned} \Delta h(p,q) = h(kp+lq)+h(kp-lq)+l^6\left[h\left(p+\frac{k}{l}q\right)+h\left(p-\frac{k}{l}q\right)\right] \\ - k^2l^2\left(k^2+l^2\right)\left[h(p+q)+h(p-q)\right] \\ - 2\left(k^2-l^2\right)\left(k^4-l^4\right)\left[h(p)+h(q)\right] \end{aligned}$$

for all $p, q \in A$ with A as a vector space or a normed linear space, $\mathcal{F}$ as a fuzzy Banach space or a Felbin's fuzzy Banach space or an intutionistic fuzzy Banach space, unless or otherwise specified.

2 Fuzzy stabilities of equation (9)

Firstly, we recall here some fundamental notions and definitions concerning fuzzy complete normed space, which are useful to prove our main results. **Definition 2.1** [28] *Let A be a real vector space. A function $H : A \times \mathbb{R} \longrightarrow [0, 1]$ is called a fuzzy norm on A if, for all $p, q \in A$, and all $u, v \in \mathbb{R}$:*

(N_1) $H(p, v) = 0$ *for any non-positive v,*

(N_2) *$p=0$ if and only if $H(p,v)=1$ for all positive v,*

(N_3) *$H(kp,v)=H\left(p,\frac{v}{|k|}\right)$ if $k\neq 0$,*

(N_4) *$H(p+q,u+v)\geq min\{H(p,u),H(q,v)\}$,*

(N_5) *$H(p,\cdot)$ is a nondecreasing function of $\mathbb{R}$, and $\lim\limits_{v\to\infty}H(p,v)=1$,*

(N_6) *for $p\neq 0$, $H(p,\cdot)$ is continuous on $\mathbb{R}$.*

The pair (A,H) is called a fuzzy normed vector space.

Definition 2.2 [28] *Let $(\mathcal{F},H)$ be a fuzzy normed vector space. A sequence $\{p_n\}$ in $\mathcal{F}$ is said to be convergent or converges if there exists a $p\in\mathcal{F}$ such that $\lim\limits_{n\to\infty}H(p_n-p,v)=1$ for all $v>0$. In this case, p is called the limit of the sequence $\{p_n\}$, and we denote it by $H\text{-}\lim\limits_{n\to\infty}p_n=p$.*

Definition 2.3 [28] Let $(\mathcal{F},H)$ be a fuzzy normed vector space. A sequence $\{p_n\}$ in $\mathcal{F}$ is called Cauchy if for each $\epsilon>0$ and each $v>0$ there exists an $n_0\in\mathbb{N}$ such that, for all $n\geq n_0$ and all $k>0$, we have $H(p_{n+k}-p_n,v)>1-\epsilon$.

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete*, and the fuzzy normed vector space is called a *fuzzy Banach space.*

We say that a mapping $h:\mathcal{F}\longrightarrow\mathcal{F}$ between fuzzy normed vector spaces $\mathcal{F}$ and $\mathcal{F}$ is continuous at a point $p_0\in\mathcal{F}$ if, for each sequence $\{p_n\}$ converging to p_0 in $\mathcal{F}$, the sequence $|H(p_n)|$ converges to $H(p_0)$. If $h:\mathcal{F}\longrightarrow\mathcal{F}$ at each $p\in\mathcal{F}$, then $h:\mathcal{F}\longrightarrow\mathcal{F}$ is said to be *continuous* on $\mathcal{F}$.

In the following results, we prove various stabilities of equation (9) connected with Ulam stability theory in the setting of Fuzzy Banach spaces with the assumption that A to be a vector space and $(\mathcal{F},H)$ to be a fuzzy Banach space.

Theorem 2.4 *Let $\xi:A\times A\longrightarrow[0,\infty)$ be a function such that*

$$\tilde{\xi}(p,q)=\sum_{n=0}^{\infty}k^{-6n}\xi\left(k^np,k^nq\right)<\infty \tag{10}$$

for all $p,q\in A$. Let $s:A\longrightarrow\mathcal{F}$ be a mapping with $s(0)=0$ such that

$$\lim_{v\to\infty}H(\Delta s(p,q),v\xi(p,q))=1 \tag{11}$$

uniformly on $A\times A$. Then $S(p)=H\text{-}\lim\limits_{n\to\infty}\frac{s(k^np)}{k^{6n}}$ exists for each $p\in A$, and defines a hexic mapping $S:A\longrightarrow\mathcal{F}$ such that if for some $\alpha>0$

$$H(\Delta s(p,q),\xi(p,q))\geq\alpha \tag{12}$$

for all $p, q \in A$, then

$$H\left(s(p) - S(p), \frac{1}{k^6}\widetilde{\xi}(p,0)\right) \geq \alpha \tag{13}$$

for all $p \in A$. Also, the hexic mapping $S : A \longrightarrow \mathcal{F}$ is a unique mapping such that

$$\lim_{v\to\infty} H(s(p) - S(p), v\tilde{\xi}(p,0)) = 1 \tag{14}$$

uniformly on A.

Proof. For a given $\epsilon > 0$, by (11), we can find some $v_0 > 0$ such that

$$H(\Delta s(p,q), v\xi(p,q)) \geq 1 - \epsilon \tag{15}$$

for all $v \geq v_0$. Applying mathematical induction process on a positive integer n, we show that

$$H\left(k^{6n}s(p) - s\left(k^n p\right), v\sum_{m=0}^{n-1} k^{6(n-m-1)}\xi\left(k^m p, 0\right)\right) \geq 1 - \epsilon \tag{16}$$

for all $v \geq v_0$, all $p \in A$, and all $n \in \mathbb{N}$. Letting $q = 0$ in (15), we obtain

$$H(k^6 s(p) - s(kp), v\xi(p,0)) \geq 1 - \epsilon \tag{17}$$

for all $p \in A$ and all $v \geq v_0$. So we obtain (16) for $n = 1$. Assume that (16) holds for $n \in \mathbb{N}$. Then

$$\begin{aligned}
&H\left(k^{6(n+1)}s(p) - s\left(k^{n+1}p\right), v\sum_{m=0}^{n} k^{6(n-m)}\xi\left(k^m p, 0\right)\right) \\
&\quad\geq \min\Big\{H\Big(k^{6(n+1)}s(p) - k^{6n}s\left(k^n p\right), v_0\sum_{m=0}^{n-1} k^{6(n-m)}\xi\left(k^m p, 0\right)\Big), \\
&\qquad\qquad H\left(k^6 s\left(k^n p\right) - s\left(k^{n+1}p\right), v_0\xi\left(k^n p, 0\right)\right)\Big\} \\
&\quad\geq \min\{1-\epsilon, 1-\epsilon\} = 1 - \epsilon.
\end{aligned} \tag{18}$$

This completes the induction argument. Letting $v = v_0$, and replacing n and p by r and $k^n p$ in (16), respectively, we obtain

$$H\left(\frac{s\left(k^r p\right)}{k^{6r}} - \frac{s\left(k^{n+r}p\right)}{k^{6(n+r)}}, \frac{v_0}{k^{6(n+r)}}\sum_{m=0}^{r-1} k^{6(r-m-1)}\xi\left(k^{n+m}p, 0\right)\right) \geq 1 - \epsilon \tag{19}$$

for all integers $n \geq 0$, $p > 0$.

It follows from (10), and the equality

$$\sum_{m=0}^{p-1} k^{6(-n-m-1)}\xi\left(k^{n+m}p,0\right) = \frac{1}{k^6}\sum_{m=n}^{n+p-1} k^{-6m}\xi\left(k^m p,0\right) \tag{20}$$

that for a given $\delta > 0$ there is an $n_0 \in \mathbb{N}$ such that

$$\frac{v_0}{k^6}\sum_{m=n}^{n+p-1} k^{-6m}\xi\left(k^m p,0\right) < \delta \tag{21}$$

for all $n \geq n_0$, and $p > 0$. Now we deduce from (19) that

$$\begin{aligned} &H\left(\frac{s\left(k^n p\right)}{k^{6n}} - \frac{s\left(k^{n+p}p\right)}{k^{6(n+p)}}, \delta\right) \\ &\geq H\left(\frac{s\left(k^n p\right)}{k^{6n}} - \frac{s\left(k^{n+p}p\right)}{k^{6(n+p)}}, \frac{v_0}{k^{6(n+p)}}\sum_{k=0}^{p-1} k^{6(p-k-1)}\xi\left(k^{n+k}p,0\right)\right) \\ &\geq 1-\epsilon \end{aligned} \tag{22}$$

for all $n \geq n_0$, and all $p > 0$. Thus the sequence $\left\{\frac{s(k^n p)}{k^{6n}}\right\}$ is Cauchy in $\mathcal{F}$. Since $\mathcal{F}$ is a fuzzy Banach space, the sequence $\left\{\frac{s(k^n p)}{k^{6n}}\right\}$ converges to some $S(p) \in \mathcal{F}$. So we can define a mapping $S : A \longrightarrow \mathcal{F}$ by $S(p) = H\text{-}\lim_{n\to\infty} \frac{s(k^n p)}{k^{6n}}$; namely, for each $v > 0$ and $p \in A$, $\lim_{n\to\infty} H\left(\frac{s(k^n p)}{k^{6n}} - S(p), v\right) = 1$.

Let $p, q \in A$. Fix $v > 0$ and $0 < \epsilon < 1$. Since $\lim_{n\to\infty} k^{-6n}\xi\left(k^n p, k^n q\right) = 0$, there is an $n_1 > n_0$ such that $v_0\xi\left(k^n p, k^n q\right) < \frac{k^{6n}v}{k^6}$ for all $n \geq n_1$. Hence for all $n \geq n_1$, we have

$$\begin{aligned} &H(\Delta S(p,q), v) \\ &\geq \min\Big\{H\left(S(kp+lq) - k^{-6n}s\left(k^n(kp+lq)\right), \frac{v}{9}\right), \\ &\quad H\left(S(kp-lq) - k^{-6n}s\left(k^n(kp-lq)\right), \frac{v}{9}\right), \\ &\quad H\left(l^6 S\left(p+\frac{k}{l}q\right) - k^{-6n}l^6 s\left(k^n\left(p+\frac{k}{l}q\right)\right), \frac{v}{9}\right), \\ &\quad H\left(l^6 S\left(p-\frac{k}{l}q\right) - k^{-6n}l^6 s\left(k^n\left(p-\frac{k}{l}q\right)\right), \frac{v}{9}\right), \\ &\quad H\left(k^2l^2(k^2+l^2)S(p+q) - k^{-6n}k^2l^2(k^2+l^2)s\left(k^n(p+q), \frac{v}{9}\right)\right), \\ &\quad H\left(k^2l^2(k^2+l^2)S(p-q) - k^{-6n}k^2l^2(k^2+l^2)s\left(k^n(p-q), \frac{v}{9}\right)\right), \end{aligned}$$

$$
\begin{aligned}
&H\left(2(k^2-l^2)(k^4-l^4)S(p)-2k^{-6n}(k^2-l^2)(k^4-l^4)s\left(k^n(p),\frac{v}{9}\right)\right),\\
&H\left(2(k^2-l^2)(k^4-l^4)S(q)-2k^{-6n}(k^2-l^2)(k^4-l^4)s\left(k^n(q),\frac{v}{9}\right)\right),\\
&H\Big(k^{-6n}s\left(k^n(kp+lq)\right)+k^{-6n}s\left(k^n(kp-lq)\right)\\
&\quad+l^6\Big[k^{-6n}s\left(k^n\left(p+\frac{k}{l}q\right)\right)+k^{-6n}s\left(k^n\left(p-\frac{k}{l}q\right)\right)\Big]\\
&\quad-k^2l^2\left(k^2+l^2\right)[k^{-6n}s(k^n(p+q))+k^{-6n}s(k^n(p-q))]\\
&\quad-2\left(k^2-l^2\right)\left(k^4-l^4\right)[k^{-6n}s(k^np)+s(k^nq)]),\frac{v}{9}\Big)\Big\}. \qquad (23)
\end{aligned}
$$

The first eight terms on the right-hand side of the above inequality tend to 1 as $n\to\infty$, and the ninth term is greater than or equal to

$$
\begin{aligned}
H\Big(&s(k^n(kp+lq))+s(k^n(kp-lq))+l^6\left[s\left(k^n\left(p+\frac{k}{l}q\right)\right)+s\left(k^n\left(p-\frac{k}{l}q\right)\right)\right]\\
&-k^2l^2\left(k^2+l^2\right)[s(k^n(p+q))+s(k^n(p-q))]\\
&-2\left(k^2-l^2\right)\left(k^4-l^4\right)[s(k^np)+s(k^nq)],\frac{k^{6n}v}{9}\Big) \qquad (24)
\end{aligned}
$$

which is greater than or equal to $1-\epsilon$. Thus

$$H(\Delta S(p,q),v)\geq 1-\epsilon \qquad (25)$$

for all $v>0$. Since $H(\Delta S(p,q),v)=1$ for all $v>0$, by $H(\Delta S(p,q),v)=0$ for all $p\in A$. Thus the mapping $S:A\longrightarrow\mathcal{F}$ is hexic, that is, $\Delta S(p,q)=0$, for all $p,q\in A$. Now, let for some positive δ, and α, (22) hold. Let

$$\xi_n(p,0)=\sum_{m=0}^{n-1}k^{6(-m-1)}\xi\left(k^mp,0\right) \qquad (26)$$

for all $p,q\in A$. Let $p\in A$. By the same reasoning as in the beginning of the proof, one can deduce from (12) that

$$H\left(k^{6n}s(p)-s\left(k^np\right),\delta\sum_{m=0}^{n-1}k^{6(n-m-1)}\xi\left(k^mp,0\right)\right)\geq\alpha \qquad (27)$$

for all positive integers n. Let $v>0$. We have

$$
\begin{aligned}
&H(s(p)-S(p),\xi_n(p,0)+v)\\
&\quad\geq\min\left\{H\left(s(p)-\frac{s\left(k^np\right)}{k^{6n}},\xi_n(p,0)\right),H\left(\frac{s\left(k^np\right)}{k^{6n}}-S(p),v\right)\right\}. \qquad (28)
\end{aligned}
$$

Combining (27), and (28) and the fact that $\lim_{n\to\infty} H\left(\frac{s(k^n p)}{k^{6n}} - S(p), v\right) = 1$, we observe that

$$H(s(p) - S(p), \xi_n(p,0) + v) \geq \alpha \tag{29}$$

for large enough $n \in \mathbb{N}$. Since the function $H(s(p) - S(p), \cdot)$ is continuous, we see that $H\left(s(p) - S(p), \frac{1}{k^6}\tilde{\xi}(p,0) + v\right) \geq \alpha$. Letting $v \to 0$, we conclude that

$$H\left(s(p) - S(p), \frac{1}{k^6}\tilde{\xi}(p,0)\right) \geq \alpha. \tag{30}$$

To end the proof, it remains to prove the uniqueness assertion. Let T be another hexic mapping satisfying (14). Fix $c > 0$. Given that $\epsilon > 0$, by (14) for S and T, we can find some $v_0 > 0$ such that

$$\begin{aligned} &H\left(s(p) - S(p), v\tilde{\xi}(p,0)\right) \geq 1 - \epsilon, \\ &H\left(s(p) - T(p), v\tilde{\xi}(p,0)\right) \geq 1 - \epsilon \end{aligned} \tag{31}$$

for all $p \in A$ and all $v \geq 2v_0$. Fix some $p \in A$ and find some integer n_0 such that

$$v_0 \sum_{m=n}^{\infty} k^{-6m} \xi\left(k^m p, 0\right) < \frac{c}{2} \tag{32}$$

for all $n \geq n_0$. Since

$$\begin{aligned} \sum_{m=n}^{\infty} k^{-6m} \xi\left(k^m p, 0\right) &= \frac{1}{k^{6n}} \sum_{m=n}^{\infty} k^{-6(m-n)} \xi\left(k^{m-n}\left(k^n p\right), 0\right) \\ &= \frac{1}{k^{6n}} \sum_{m=0}^{\infty} k^{-6m} \xi\left(k^m\left(k^n p\right), 0\right) = \frac{1}{k^{6n}} \tilde{\xi}\left(k^n p, 0\right), \end{aligned} \tag{33}$$

we have

$$\begin{aligned} &H(S(p) - T(p), c) \\ &\geq \min\left\{H\left(\frac{s\left(k^n p\right)}{k^{6n}} - S(p), \frac{c}{2}\right), H\left(T(p) - \frac{s\left(k^n p\right)}{k^{6n}}, \frac{c}{2}\right)\right\} \\ &= \min\left\{H\left(s\left(k^n p\right) - S\left(k^n p\right), k^{6(n-1)} 2c\right), H\left(T\left(k^n p\right) - s\left(k^n p\right), k^{6(n-1)} 2c\right)\right\} \\ &\geq \min\Bigg\{H\left(s\left(k^n p\right) - S\left(k^n p\right), k^{6n} v_0 \sum_{m=n}^{\infty} k^{-6m} \xi\left(k^m p, k^m p\right)\right), \\ &\quad H\left(T\left(k^n p\right) - s\left(k^n p\right), k^{6n} v_0 \sum_{m=n}^{\infty} k^{-6m} \xi\left(k^m p, k^m p\right)\right)\Bigg\} \geq 1 - \epsilon. \end{aligned} \tag{34}$$

It follows that $H(S(p)-T(p),c)=1$ for all $c>0$. Thus $S(p)=T(p)$ for all $p\in A$, which completes the proof.

Corollary 2.5 *Let $s : A \longrightarrow \mathcal{F}$ be a mapping with $s(0)=0$ and $\epsilon>0$ such that*

$$\lim_{v\to\infty} H(\Delta s(p,q), v\epsilon) = 1$$

uniformly on $A\times A$. Then $S(p)=H\text{-}\lim\limits_{n\to\infty}\frac{s(k^n p)}{k^{6n}}$ exists for each $p\in A$, and defines a hexic mapping $S : A \longrightarrow \mathcal{F}$ such that if for some $\alpha>0$

$$H(\Delta s(p,q),\epsilon)\geq\alpha$$

for all $p,q\in A$, then

$$H\left(s(p)-S(p),\frac{\epsilon}{k^6-1}\right)\geq\alpha$$

for all $p\in A$. Also, the hexic mapping $S : A \longrightarrow \mathcal{F}$ is a unique mapping such that

$$\lim_{v\to\infty} H\left(s(p)-S(p),\frac{v\epsilon k^6}{k^6-1}\right)=1$$

uniformly on A.

Proof. The proof follows directly by taking $\xi(p,q)=\epsilon$ in Theorem 2.4.

Corollary 2.6 *Let $\theta\geq 0$, and let r be a real number with $0<r<6$. Let $s : A \longrightarrow \mathcal{F}$ be a mapping with $s(0)=0$ such that*

$$\lim_{v\to\infty} H\Big(\Delta s(p,q), v\theta\left(\|p\|^r+\|q\|^r\right)\Big)=1 \tag{35}$$

uniformly on $A\times A$. Then $S(p)=H\text{-}\lim\limits_{n\to\infty}\frac{s(k^n p)}{k^{6n}}$ exists for each $p\in A$, and defines a hexic mapping $S : A \longrightarrow \mathcal{F}$ such that if for some $\alpha>0$

$$H\Big(\Delta s(p,q),\theta\left(\|p\|^r+\|q\|^r\right)\Big)\geq\alpha$$

for all $p,q\in A$, then

$$H\left(s(p)-S(p),\frac{\theta}{(k^6-k^r)}\|p\|^r\right)\geq\alpha$$

for all $p\in A$. Moreover, the hexic mapping $S : A \longrightarrow \mathcal{F}$ is a unique mapping such that

$$\lim_{v\to\infty} H\left(s(p)-S(p),\frac{k^6 v\theta}{(k^6-k^r)}\|p\|^r\right)=1$$

uniformly on A.

Proof. The proof is obtained by defining $\xi(p,q)=\theta\left(\|p\|^r+\|q\|^r\right)$, and following with similar arguments as in Theorem 2.4.

The proof of following theorem is similar to Theorem 2.4 since this is dual result of Theorem 2.4, Hence we present only the main part of the proof.

Theorem 2.7 *Let $\xi : A\times A \longrightarrow [0,\infty)$ be a function such that*

$$\tilde{\xi}(p,q)=\sum_{n=1}^{\infty} k^{6n}\xi\left(\frac{p}{k^n},\frac{q}{k^n}\right)<\infty$$

for all $p,q\in A$. Let $s : A\longrightarrow \mathcal{F}$ be a mapping satisfying (11)*, and $s(0)=0$. Then $S(p)=H\text{-}\lim_{n\to\infty} k^{6n}s\left(\frac{p}{k^n}\right)$ exists for each $p\in A$, and defines a hexic mapping $S : A\longrightarrow \mathcal{F}$ such that if for some $\alpha>0$*

$$H\Big(\Delta s(p,q),\xi(p,q)\Big)\geq\alpha$$

for all $p,q\in A$, then

$$H\left(s(p)-S(p),k^6\tilde{\xi}\left(\frac{p}{k},0\right)\right)\geq\alpha$$

for all $p\in A$. Moreover, the hexic mapping $S : A\longrightarrow \mathcal{F}$ is a unique mapping such that

$$\lim_{v\to\infty} H\left(s(p)-S(p),v\tilde{\xi}\left(\frac{p}{k},0\right)\right)=1$$

uniformly on A. Proof. For a given $\epsilon>0$, by (11), we can find some $v_0>0$ such that (15) holds for all $v\geq v_0$. By the application of induction technique on an integer $n>0$, we have

$$H\left(\frac{1}{k^{6n}}s(p)-s\left(\frac{p}{k^n}\right),v\sum_{m=0}^{n-1}\frac{1}{k^{6(n-m-1)}}\xi\left(\frac{p}{k^m},0\right)\right)\geq 1-\epsilon \tag{36}$$

for all $v\geq v_0$, all $p\in A$, and all $n\in\mathbb{N}$. Now, replacing (p,q) by $\left(\frac{p}{k},0\right)$ in (15), we obtain

$$H\left(\frac{1}{k^6}s(p)-s\left(\frac{p}{k}\right),v\xi\left(\frac{p}{k},0\right)\right)\geq 1-\epsilon \tag{37}$$

for all $p\in A$ and all $v\geq v_0$. The rest of the proof is obtained by similar arguments as in Theorem 2.4.

Corollary 2.8 *Let $\theta\geq 0$, and let r be a real number with $r>6$. Let $s : A\longrightarrow \mathcal{F}$ be a mapping satisfying* (35)*, and $s(0)=0$. Then $S(p)=N\text{-}\lim_{n\to\infty} k^{6n}s\left(\frac{p}{k^n}\right)$ exists for each $p\in A$, and defines a hexic mapping $S : A\longrightarrow \mathcal{A}$ such that if for some $\alpha>0$*

$$H\Big(\Delta s(p,q),\theta\left(\|p\|^r+\|q\|^r\right)\Big)\geq\alpha$$

for all $p, q \in A$*, then*

$$H\left(s(p) - S(p), \frac{\theta}{(k^r - k^6)} \|p\|^r\right) \geq \alpha$$

for all $p \in A$*. Moreover, the hexic mapping* $S : A \longrightarrow \mathcal{F}$ *is a unique mapping such that*

$$\lim_{v \to \infty} H\left(s(p) - S(p), \frac{k^6 v\theta}{(k^r - k^6)} \|p\|^r\right) = 1$$

uniformly on A*.*
Proof. The proof directly follows by considering $\xi(p, q) = \theta\left(\|p\|\,|^r + \|q\|^r\right)$ in Theorem 2.7 with similar arguments.

3 Felbin's fuzzy stabilities of equation (9)

The preliminary notions of fuzzy number, fuzzy set and various concepts concerning Felbin's fuzzy complete normed space are discussed in [14]. The following lemma plays a major role in obtaining our desired results and hence we present its statement. The proof is available in [47].
Lemma 3.1 [47] *Let* $(\mathcal{F}, \|\cdot\|, L, R)$ *be a fuzzy normed linear space, and suppose that*
$(R-1)$ $R(a, b) \leq max(a, b)$,
$(R-2)$ $\forall \alpha \in (0, 1], \exists \beta \in (0, \alpha]$ *such that* $R(\beta, y) \leq \alpha$ *for all* $y \in (0, \alpha)$,
$(R-3)$ $\lim_{a \to 0^+} R(a, a) = 0$.
Then $(R-1) \Rightarrow (R-2) \Rightarrow (R-3)$ *but not conversely.*

In the upcoming results, various fundamental stabilities of equation (9) associated with Ulam stability theory are obtained in the framework of Felbin's type fuzzy complete normed spaces.
Theorem 3.2 *Let* A *be a vector space and* $(\mathcal{F}, \|\cdot\|, L, R)$ *be a fuzzy Banach space satisfying* $(R-2)$*. Let* $s : A \longrightarrow \mathcal{F}$ *be a mapping for which there exists a function* $\xi : A \times A \longrightarrow \mathcal{F}^*(\mathbb{R})$ *such that*

$$\sum_{i=0}^{\infty} \frac{\left(\xi\left(k^i p, k^i q\right)\right)_{\alpha}^{+}}{k^{6i}} < \infty \tag{38}$$

for all $p, q \in A$*.*

$$\|\Delta s(p, q)\| \preceq \xi(p, q) \tag{39}$$

for all $p, q \in A$ *and all* $\alpha \in (0, 1]$*. Then there exists a unique hexic mapping* $S : A \longrightarrow \mathcal{F}$ *such that* $\forall \alpha \in (0, 1], \exists \beta \in (0, \alpha]$,

$$\|s(p) - S(p)\|_{\alpha}^{+} \leq \frac{1}{k^{12}} \sum_{i=0}^{\infty} \frac{\left(\xi(k^i p, 0)\right)_{\beta}^{+}}{k^{6i}} \quad \textit{for all} \quad p \in A. \tag{40}$$

Proof. Setting $q = 0$ in (39), we get

$$\left\| s(kp) - k^6 s(p) \right\| \preceq \xi(p, 0) \tag{41}$$

for all $p \in A$. Replacing p by $k^n p$ and then multiplying by $\frac{1}{k^{6(n+1)}}$ in (41) in the fuzzy scalar multiplication sense, we obtain

$$\left\| \frac{s(k^{n+1}p)}{k^{6(n+1)}} - \frac{s(k^n p)}{k^{6n}} \right\| \preceq \frac{1}{k^{6(n+1)}} \odot \xi(k^n p, 0) \tag{42}$$

for all $p \in A$ and all non-negative integers $n \in \mathbb{N}$. By Lemma 3.1 and inequality (42), we conclude that for all $\alpha \in (0, 1]$ there exist $\beta \in (0, \alpha]$ such that

$$\left\| \frac{s(k^{n+1}p)}{k^{6(n+1)}} - \frac{s(k^m p)}{k^{6m}} \right\|_{\alpha}^{+} \preceq \frac{1}{k^6} \sum_{i=m}^{n} \frac{1}{k^{6i}} \big(\xi(k^i p, 0)\big)_{\beta}^{+} \tag{43}$$

for all $p \in A$ and all non-negative integers m and n with $n \geq m$. Now (38) and (43) imply that $\{\frac{s(k^n p)}{k^{6n}}\}$ is a fuzzy Cauchy sequence in $\mathcal{F}$ for all $p \in A$. Since $\mathcal{F}$ is a fuzzy Banach space, the sequence $\{\frac{s(k^n p)}{k^{6n}}\}$ converges for all $p \in A$. Hence we can define a mapping $S : A \longrightarrow \mathcal{F}$ by

$$S(p) = \lim_{n \to \infty} \frac{s(k^n p)}{k^{6n}}$$

for all $p \in A$. Letting $m = 0$ and passing the limit $n \to \infty$ in (43), by continuity of $\| \cdot \|_{\alpha}^{+}$, we have

$$\| s(p) - S(p) \|_{\alpha}^{+} \preceq \frac{1}{k^6} \sum_{i=0}^{\infty} \frac{\big(\xi(k^i p, 0)\big)_{\beta}^{+}}{k^{6i}} \tag{44}$$

for all $p \in A$. Therefore, we obtain (40). Next, we show that S is hexic and unique. Applying (38), (39) and the continuity of $\| \cdot \|_{\alpha}^{+}$, we have

$$\| \Delta S(p, q) \|_{\alpha}^{+} \preceq \lim_{n \to \infty} \frac{\big(\xi(k^n p, k^n q)\big)_{\alpha}^{+}}{k^{6n}} = 0$$

for all $p, q \in A$. Therefore, the mapping $S : A \longrightarrow \mathcal{F}$ is hexic. To prove the uniqueness of S, let $T : A \longrightarrow \mathcal{F}$ be another hexic mapping satisfying (40). By Lemma 3.1, we have

$$\begin{aligned} \| S(p) - T(p) \| &\preceq \lim_{n \to \infty} \frac{1}{k^{6(n+1)}} \sum_{i=0}^{\infty} \frac{\big(\xi(k^{n+i} p, 0)\big)_{\beta}^{+}}{k^{6i}} \\ &\preceq \lim_{n \to \infty} \frac{1}{k^6} \sum_{i=n}^{\infty} \frac{\big(\xi(k^i p, 0)\big)_{\beta}^{+}}{k^{6i}} = 0 \end{aligned}$$

for all $p \in A$ which implies $S = T$. Hence, S is unique, which completes the proof.

Corollary 3.3 *Let A be a vector space and $(\mathcal{F}, \|\cdot\|, L, R)$ be a fuzzy Banach space satisfying $(R-2)$. Let $\epsilon \geq 0$ be a constant. Let $s : A \longrightarrow \mathcal{F}$ be a mapping satisfying*

$$\|\Delta s(p,q)\| \preceq \epsilon \tag{45}$$

for all $p, q \in A$ and all $\alpha \in (0,1]$. Then there exists a unique hexic mapping $S : A \longrightarrow \mathcal{F}$ such that $\forall \alpha \in (0,1], \exists \beta \in (0,\alpha]$,

$$\|s(p) - S(p)\|_\alpha^+ \preceq \frac{\epsilon}{k^2(k^6-1)} \quad \text{for all} \quad p \in A. \tag{46}$$

Proof. The proof is obtained by replacing $\xi(p,q) = \epsilon$ in Theorem 3.2 and proceeding with the similar arguments.

Corollary 3.4 *Let μ be non-negative fuzzy real number, and x, y be non-negative real numbers such that $\mu = x + y < 6$. Let A be a linear space and $(\mathcal{F}, \|\cdot\|, L, R)$ be a fuzzy Banach space satisfying $(R-1)$. Suppose that $s : A \longrightarrow \mathcal{F}$ be a mapping satisfies the inequality*

$$\|\Delta s(p,q)\| \preceq \mu \otimes \left(\|p\|_A^x \oplus \|q\|_A^y\right)$$

for all $p, q \in A$. Then there exists a unique hexic mapping $S : A \longrightarrow \mathcal{F}$ such that $\forall \alpha \in (0,1], \exists\ \beta \in (0,\alpha]$,

$$\|s(p) - S(p)\|_\alpha^+ \preceq \frac{\mu_\beta^+}{2(k^6 - k^\mu)} \left(\|p\|_\beta^+\right)^\mu$$

for all $p \in A$.

Proof. Taking $\xi(p,q) = \mu \otimes \left(\|p\|_A^x \oplus \|q\|_A^y\right)$ in Theorem 3.2, the proof follows.

Theorem 3.5 *Let A be a linear space and $(\mathcal{F}, \|\cdot\|, L, R)$ be a fuzzy Banach space satisfying $(R-2)$. Let $s : A \longrightarrow \mathcal{F}$ be a mapping for which there exists a function $\xi : A \times A \longrightarrow \mathcal{F}^*(\mathbb{R})$ such that*

$$\sum_{i=1}^{\infty} k^{6i} \left(\xi\left(\frac{p}{k^i}, \frac{q}{k^i}\right)\right)_\alpha^+ < \infty \tag{47}$$

for all $p, q \in A$ and s satisfies

$$\|\Delta s(p,q)\| \preceq \xi(p,q) \tag{48}$$

for all $p, q \in A$ and all $\alpha \in (0,1]$. Then there exists a unique hexic mapping $S : A \longrightarrow \mathcal{F}$ such that $\forall \alpha \in (0,1], \exists \beta \in (0,\alpha]$,

$$\|s(p) - S(p)\|_\alpha^+ \preceq \frac{1}{k^6} \sum_{i=1}^{\infty} k^{6i} \left(\xi\left(\frac{p}{k^i}, 0\right)\right)_\beta^+ \tag{49}$$

for all $p \in A$.
Proof. Setting $y = 0$ in (48), we get

$$\left\| s(kp) - k^6 s(p) \right\| \preceq \xi(p, 0) \tag{50}$$

for all $p \in A$. Replacing p by $\frac{p}{k^{n+1}}$ and multiplying both sides by k^{6n} in (50) in the fuzzy scalar multiplication sense, we obtain

$$\left\| k^{6n} s\left(\frac{p}{k^n}\right) - k^{6(n+1)} s\left(\frac{p}{k^{n+1}}\right) \right\| \preceq k^{6n} \odot \xi\left(\frac{p}{k^{n+1}}, 0\right) \tag{51}$$

for all $p \in A$. By Lemma 3.1 and inequality (51), we conclude that for all $\alpha \in (0, 1]$ there exist $\beta \in (0, \alpha]$ such that

$$\left\| k^{6(n+1)} s\left(\frac{p}{k^{n+1}}\right) - k^{6m} s\left(\frac{p}{k^m}\right) \right\|_\alpha^+ \preceq \sum_{i=m}^{n} k^{6i} \left(\xi\left(\frac{p}{k^{i+1}}, 0\right)\right)_\beta^+ \tag{52}$$

for all $p \in A$ and all non-negative integers m and n with $n \geq m$. Now (47) and (52) imply that $\{k^{6n} s\left(\frac{p}{k^n}\right)\}$ is a fuzzy Cauchy sequence in $\mathcal{F}$ for all $p \in A$. Since $\mathcal{F}$ is a fuzzy Banach space, the sequence $\{k^{6n} s\left(\frac{p}{k^n}\right)\}$ converges for all $p \in A$. The rest of this proof is similar to Theorem 3.2.
Corollary 3.6 *Let μ be non-negative fuzzy real number, and x, y be non-negative real numbers such that $\mu = x + y > 6$. Let A be a linear space and $(\mathcal{F}, \|\cdot\|, L, R)$ be a fuzzy Banach space satisfying $(R-1)$. Suppose that $s : A \longrightarrow \mathcal{F}$ be a mapping satisfies the inequality*

$$\|\Delta s(p, q)\| \preceq \mu \otimes \left(\|p\|_A^x \oplus \|q\|_A^y\right)$$

for all $p, q \in A$. Then there exists a unique hexic mapping $S : A \longrightarrow \mathcal{F}$ such that $\forall \alpha \in (0, 1], \exists\, \beta \in (0, \alpha]$,

$$\|s(p) - S(p)\|_\alpha^+ \preceq \frac{\mu_\beta^+}{2(k^{mu} - k^6)} \left(\|p\|_\beta^+\right)^\mu$$

for all $p \in A$.
Proof. By taking $\xi(p, q) = \mu \otimes \left(\|p\|_A^x \oplus \|q\|_A^y\right)$ in Theorem 3.5, the proof follows.

4 Intuitionistic fuzzy stabilities of equation (9)

The elementary definitions and other conceptions relevant to intuitionistic fuzzy normed spaces are discussed in [4]. The following theorems provide the classical stabilities of equation (9) pertinent to Ulam stability theory in the structure of intuitionistic fuzzy Banach spaces.

Theorem 4.1 *Let A be a linear space and $(\mathcal{F}, \mu, v)$ be an intuitionistic fuzzy Banach space. Let $\xi : A \times A \to [0, \infty)$ be a control function such that*

$$\tilde{\xi}(p, q) = \sum_{n=0}^{\infty} k^{-6n} \xi\left(k^{6n} p, k^{6n} q\right) < \infty \tag{53}$$

for all $p, q \in A$. Let $s : A \longrightarrow \mathcal{F}$ be a mapping satisfying the following with respect to ξ

$$\lim_{t\to\infty} \mu(\Delta s(p, q), t\xi(p, q)) = 1,$$

and

$$\lim_{t\to\infty} \nu(\Delta s(p, q), t\xi(p, q)) = 0 \tag{54}$$

for all $p, q \in A$. Then $S(p) = (\mu, \nu)\text{-}\lim_{n\to\infty} \frac{s(k^n p)}{k^{6n}}$ for each $p \in A$ exists and defines a hexic mapping $S : A \longrightarrow \mathcal{F}$ such that if for some $\alpha > 0$ and all $p, q \in A$,

$$\mu(\Delta s(p, q), \xi(p, q)) > \alpha,$$

and

$$\nu(\Delta s(p, q), \xi(p, q)) < 1 - \alpha, \tag{55}$$

then $\mu\left(S(p) - s(p), \tilde{\xi}(p, 0)\right) > \alpha$, and $\nu\left(S(p) - s(p), \tilde{\xi}(p, 0)\right) < 1 - \alpha$.

Proof. Given $\epsilon > 0$. By (54), we can find some $t_0 > 0$ such that

$$\mu(\Delta s(p, q), t\xi(p, q)) \geq 1 - \epsilon,$$

and

$$\nu(\Delta s(p, q), t\xi(p, q)) \leq \epsilon \tag{56}$$

for all $p, q \in A$, and all $t \geq t_0$. By induction on n, we shall show that

$$\mu\left(s\left(k^n p\right) - k^{6n} s(p), t \sum_{m=0}^{n-1} k^{n-m-1} \xi\left(k^m p, k^m p\right)\right) \geq 1 - \epsilon,$$

and (57)

$$\nu\left(s\left(k^n p\right) - k^{6n} s(p), t \sum_{m=0}^{n-1} k^{n-m-1} \xi\left(k^m p, k^m p\right)\right) \leq \epsilon,$$

for all $p \in A$, $t \geq t_0$, and all positive integers n. Putting $q = 0$ in (56), we obtain (57) for $n = 1$. Let (57) hold for some positive integer n. Then

$$\begin{aligned}
\mu &\left(s\left(k^{n+1}p\right) - k^{6(n+1)}s(p), t\sum_{m=0}^{n} k^{6(n-m)}\xi\left(k^m p, 0\right)\right) \\
&\geq \mu\left(s\left(k^{n+1}p\right) - k^{6n}s\left(k^n p\right), t_0\xi\left(k^n p, 0\right)\right) \\
&\qquad \star\, \mu\left(k^{6n}s\left(k^n p\right) - k^{6(n+1)}s(p), t_0\sum_{m=0}^{n-1} k^{6(n-m)}\xi\left(k^m p, 0\right)\right) \\
&\geq (1-\epsilon)\star(1-\epsilon) = 1-\epsilon,
\end{aligned}$$

and

$$\begin{aligned}
\nu &\left(s\left(k^{n+1}p\right) - k^{6(n+1)}s(p), t\sum_{m=0}^{n} k^{6(n-m)}\xi\left(k^m p, 0\right)\right) \\
&\leq \nu\left(s\left(k^{n+1}p\right) - k^{6n}s\left(k^n p\right), t_0\xi\left(k^n p, 0\right)\right) \\
&\qquad \Diamond\nu\left(k^{6n}s\left(k^n p\right) - k^{6(n+1)}s(p), t_0\sum_{m=0}^{n-1} k^{6(n-m)}\xi\left(k^m p, 0\right)\right) \leq \epsilon\Diamond\epsilon = \epsilon.
\end{aligned}$$

This completes the induction argument. Let $t = t_0$ and put $n = x$ then by replacing p with $k^n p$ in (57), we obtain

$$\mu\left(\frac{s\left(k^{n+x}p\right)}{k^{6(n+x)}} - \frac{s\left(k^n p\right)}{k^{6n}}, \frac{t_0}{k^{6(n+x)}}\sum_{m=0}^{x-1} k^{6(x-m-1)}\xi\left(k^{n+m}p, 0\right)\right) \geq 1-\epsilon,$$

and

$$\nu\left(\frac{s\left(k^{n+x}p\right)}{k^{6(n+x)}} - \frac{s\left(k^n p\right)}{k^{6n}}, \frac{t_0}{k^{6(n+x)}}\sum_{m=0}^{x-1} k^{6(x-m-1)}\xi\left(k^{n+m}p, 0\right)\right) \leq \epsilon, \qquad (58)$$

for all integers $n \geq 0$ and $x > 0$. The convergence of (53), and

$$\sum_{m=0}^{x-1} k^{-6(n+m+1)}\xi\left(k^{n+m}p, 0\right) = \frac{1}{2}\sum_{m=n}^{n+x-1} k^{-6m}\xi\left(k^m p, 0\right)$$

imply that for given $\delta > 0$ there is $n_0 \in \mathbb{N}$ such that

$$\frac{t_0}{2}\sum_{m=n}^{n+x-1} k^{-6m}\xi\left(k^m p, 0\right) < \delta,$$

for all $n \geq n_0$ and all $x > 0$. Now we deduce that from (58) that

$$\begin{aligned}
&\mu\left(\frac{s\left(k^{n+x}p\right)}{k^{6(n+x)}} - \frac{s\left(k^{n}p\right)}{k^{6n}}, \delta\right) \\
&\quad \geq \mu\left(\frac{s\left(k^{n+x}p\right)}{k^{6(n+x)}} - \frac{s\left(k^{n}p\right)}{k^{6n}}, \frac{t_0}{k^{6(n+x)}} \sum_{m=0}^{x-1} k^{6(x-m-1)} \xi\left(k^{n+m}p, 0\right)\right) \\
&\quad \geq 1 - \epsilon,
\end{aligned}$$

and

$$\begin{aligned}
&\nu\left(\frac{s\left(k^{n+x}p\right)}{k^{6(n+x)}} - \frac{s\left(k^{n}p\right)}{k^{6n}}, \delta\right) \\
&\quad \leq \nu\left(\frac{s\left(k^{n+x}p\right)}{k^{6(n+x)}} - \frac{s\left(k^{n}p\right)}{k^{6n}}, \frac{t_0}{k^{6(n+x)}} \sum_{m=0}^{x-1} k^{6(x-m-1)} \xi\left(k^{n+m}p, 0\right)\right) \\
&\quad \leq \epsilon,
\end{aligned}$$

for all $n \geq n_0$ and all $p > 0$. Hence $\left\{\frac{s(k^n p)}{k^{6n}}\right\}$ is a Cauchy sequence in $\mathcal{F}$. Since $\mathcal{Y}$ is an intuitionistic fuzzy Banach space, $\left\{\frac{s(k^n p)}{k^{6n}}\right\}$ converges to some $S(p) \in \mathcal{F}$. Hence, we can define a mapping $S : A \longrightarrow \mathcal{F}$ such that $S(p) = (\mu, \nu)\text{-}\lim_{n\to\infty} \frac{s(k^n p)}{k^{6n}}$, namely, for each $t > 0$, and $p \in A$,

$$\mu\left(S(p) - \frac{s\left(k^{n}p\right)}{k^{6n}}, t\right) = 1, \qquad \text{and} \qquad \nu\left(S(p) - \frac{s\left(k^{n}p\right)}{k^{6n}}, t\right) = 0.$$

Now, let $p, q \in A$. Choose any fixed value of $t > 0$, and $\epsilon \in (0, 1)$. Since $\lim_{n\to\infty} k^{-6n} \xi\left(k^n p, k^n q\right) = 0$, there exists $n_1 > n_0$ such that $t_0 \xi\left(k^n p, k^n q\right) < \frac{k^{6n} t}{k}$ for all $n \geq n_1$. Hence for each $n \geq n_1$, we have

$$\begin{aligned}
&\mu(\Delta S(p, q), t) \\
&\quad \geq \mu\left(S(kp + lq) - \frac{s\left(k^n(kp + lq)\right)}{k^{6n}}, \frac{t}{9}\right) \\
&\qquad \star \mu\left(S(kp - lq) - \frac{s\left(k^n(kp - lq)\right)}{k^{6n}}, \frac{t}{9}\right) \\
&\qquad \star \mu\left(l^6 S\left(p + \frac{k}{l}q\right) - l^6 \frac{s\left(k^{6n}\left(p + \frac{k}{l}q\right)\right)}{k^{6n}}, \frac{t}{9}\right) \\
&\qquad \star \mu\left(l^6 S\left(p - \frac{k}{l}q\right) - l^6 \frac{s\left(k^{6n}\left(p - \frac{k}{l}q\right)\right)}{k^{6n}}, \frac{t}{9}\right)
\end{aligned}$$

$$\star\, \mu\left(k^2l2^2(k^2+l^2)S(p+q) - k^2l^2\frac{s\left(k^{6n}(p+q)\right)}{k^{6n}}, \frac{t}{9}\right)$$
$$\star\, \mu\left(k^2l2^2(k^2+l^2)S(p-q) - k^2l^2\frac{s\left(k^{6n}(p-q)\right)}{k^{6n}}, \frac{t}{9}\right)$$
$$\star\, \mu\left(2(k^2-l^2)(k^4-l^4)S(p) - 2(k^2-l^2)(k^4-l^4)\frac{s\left(k^{6n}(p)\right)}{k^{6n}}, \frac{t}{9}\right)$$
$$\star\, \mu\left(2(k^2-l^2)(k^4-l^4)S(q) - 2(k^2-l^2)(k^4-l^4)\frac{s\left(k^{6n}(q)\right)}{k^{6n}}, \frac{t}{9}\right)$$
$$\star\, \mu\left(\Delta s(k^np, k^nq), \frac{k^{6n}t}{9}\right), \tag{59}$$

and also

$$\mu\left(\Delta s(k^np, k^nq), \frac{k^{6n}t}{9}\right) \geq \mu\left(\Delta s(k^np, k^nq), t\xi\left(k^np, k^nq\right)\right). \tag{60}$$

Letting $n \to \infty$ in (59) and then using (56) and (60), we obtain

$$\mu(\Delta S(p,q), t) \geq 1 - \epsilon$$

for all $t > 0$ and $\epsilon \in (0,1)$. Similarly, we obtain

$$\nu(\Delta S(p,q), t) \leq \epsilon$$

for alll $t > 0$ and $\epsilon \in (0,1)$. It follows that $\mu(\Delta S(p,q), t) = 1$, and $\nu(\Delta S(p,q), t) = 0$, for all $t > 0$. Therefore S satisfies (9).

Lastly, suppose that for some α, (55) holds, and

$$\xi_n(p,q) = \sum_{m=0}^{n-1} k^{-6(m+1)}\xi\left(k^mp, k^mq\right),$$

for all $p, q \in A$. By a similar argument as in the beginning of the proof one can deduce from (55)

$$\mu\left(s\left(k^np\right) - k^{6n}s(p), \sum_{m=0}^{n-1} k^{6(n-m-1)}\xi\left(k^mp, 0\right)\right) \geq \alpha,$$

and (61)

$$\nu\left(s\left(k^np\right) - k^{6n}s(p), \sum_{m=0}^{n-1} k^{6(n-m-1)}\xi\left(k^mp, 0\right)\right) \leq 1 - \alpha,$$

for all positive integers n. For $\ell > 0$, we have

$$\begin{aligned}&\mu(s(p) - S(p), \xi_n(p,0) + \ell)\\ &\qquad \geq \mu\left(s(p) - \frac{s\left(k^n p\right)}{k^{6n}}, \xi_n(p,0)\right) \star \mu\left(\frac{s\left(k^n p\right)}{k^{6n}} - S(p), \ell\right),\end{aligned}$$

and

$$\begin{aligned}&\nu(s(p) - S(p), \xi_n(p,0) + \ell)\\ &\qquad \leq \nu\left(s(p) - \frac{s\left(k^n p\right)}{k^{6n}}, \xi_n(p,0)\right) \Diamond \nu\left(\frac{s\left(k^n p\right)}{k^{6n}} - S(p), \ell\right). \qquad (62)\end{aligned}$$

Combining (61) with (62), and using the fact that

$$\lim_{n\to\infty} \mu\left(\frac{s\left(k^n p\right)}{k^{6n}} - S(p), \ell\right) = 1,$$

and

$$\lim_{n\to\infty} \nu\left(\frac{s\left(k^n p\right)}{k^{6n}} - S(p), \ell\right) = 0,$$

we obtain $\mu(s(p)-S(p), \xi_n(p,0)+\ell) \geq \alpha$, and $\nu(s(p)-S(p), \xi_n(p,0)+\ell) \leq 1-\alpha$, for sufficiently large n. From the (upper semi) continuity of real functions $\mu(s(p) - S(p), \cdot)$, and $\nu(s(p) - S(p), \cdot)$, we see that

$$\mu\left(s(p) - S(p), \tilde{\xi}(p,0) + \ell\right) \geq \alpha,$$

and

$$v\left(s(p) - S(p), \tilde{\xi}(p,0) + \ell\right) \leq 1 - \alpha.$$

Taking the limit $\ell \to \infty$, we obtain $\mu\left(s(p) - S(p), \tilde{\xi}(p,0)\right) \geq \alpha$, and

$$v\left(s(p) - S(p), \tilde{\xi}(p,0)\right) \leq 1 - \alpha.$$

Hence the proof.

Corollary 4.2 *Let A be a linear space and $(\mathcal{F}, \mu, v)$ be an intuitionistic fuzzy Banach space. Let $s : A \longrightarrow \mathcal{F}$ be a mapping satisfying the following*

$$\lim_{t\to\infty} \mu(\Delta s(p,q), t\epsilon) = 1, \quad \textit{and} \quad \lim_{t\to\infty} \nu(\Delta s(p,q), t\epsilon) = 0 \qquad (63)$$

for all $p, q \in A$. Then $S(p) = (\mu, \nu)\text{-}\lim_{n\to\infty} \frac{s(k^n p)}{k^{6n}}$ for each $p \in A$ exists and defines a hexic mapping $S : A \longrightarrow \mathcal{F}$ such that if for some $\alpha > 0$ and all $p, q \in A$,

$$\mu(\Delta s(p,q), \epsilon) > \alpha, \quad \textit{and} \quad \nu(\Delta s(p,q), \epsilon) < 1 - \alpha, \qquad (64)$$

then $\mu\left(S(p)-s(p), \frac{\epsilon k^6}{k^6-1}\right) > \alpha,$ *and* $\nu\left(S(p)-s(p), \frac{\epsilon k^6}{k^6-1}\right) < 1-\alpha.$

Theorem 4.3 *Let A be a linear space and $(\mathcal{F}, \mu, \nu)$ be an intuitionistic fuzzy Banach space. Let $\xi : A \times A \longrightarrow [0,\infty)$ be a control function satisfying (53). Let $s : A \longrightarrow \mathcal{F}$ be a mapping satisfying (54) with respect to ξ. Then there is a unique hexic mapping $S : A \longrightarrow \mathcal{F}$ such that*

$$\lim_{n\to\infty} \mu\left(s(p)-S(p), t\tilde{\xi}\left(\frac{p}{k},0\right)\right) = 1,$$

and (65)

$$\lim_{n\to\infty} \nu\left(s(p)-S(p), t\tilde{\xi}\left(\frac{p}{k},0\right)\right) = 0$$

uniformly in A.

Proof. The existence of uniform limit (65) immediately follows from Theorem 4.1. It remains to prove the uniqueness assertion. Let T be another hexic mapping satisfying (65). Choose $t > 0$. Given $\epsilon > 0$, there is some $t_0 > 0$ such that from (65), we have

$$\mu\left(s(p)-S(p), t\tilde{\xi}\left(\frac{p}{k},0\right)\right) \geq 1-\epsilon, \quad \text{and} \quad \nu\left(s(p)-S(p), t\tilde{\xi}\left(\frac{p}{k},0\right)\right) \leq \epsilon$$

for all $p \in A$, and all $t \geq t_0$. For some $p \in A$, we can find some integer n_0 such that

$$t_0 \sum_{m=n}^{\infty} k^{6m} \xi\left(\frac{p}{k^m},0\right) < \frac{t}{2},$$

for all $n \geq n_0$. Since

$$\sum_{m=n}^{\infty} k^{6m} \xi\left(\frac{p}{k^m},0\right) = k^{6n} \sum_{m=n}^{\infty} k^{6(m-n)} \xi\left(k^{(n-m)}\frac{p}{k^n},0\right) = k^{6n}\tilde{\xi}\left(\frac{p}{k^n},0\right),$$

we have

$$\begin{aligned}
\mu(s(p)-S(p),t) &\geq \mu\left(k^{6n}s\left(\frac{p}{k^n}\right)-S(p),\frac{t}{2}\right) \star \mu\left(S(p)-k^{6n}s\left(\frac{p}{k^n}\right),\frac{t}{2}\right) \\
&= \mu\left(s\left(\frac{p}{k^n}\right)-S\left(\frac{p}{k^n}\right), k^{6(n-1)}t\right) \star \mu\left(S\left(\frac{p}{k^n}\right)-s\left(\frac{p}{k^n}\right), k^{6(n-1)}t\right) \\
&\geq \mu\left(s\left(\frac{p}{k^n}\right)-S\left(\frac{p}{k^n}\right), k^{6n}t_0\sum_{m=n}^{\infty} k^{6m}\xi\left(\frac{p}{k^m},0\right)\right) \\
&\quad \star \mu\left(S\left(\frac{p}{k^n}\right)-s\left(\frac{p}{k^n}\right), k^{6n}t_0\sum_{m=n}^{\infty} k^{6m}\xi\left(\frac{p}{k^m},0\right)\right) \\
&= \mu\left(s\left(\frac{p}{k^n}\right)-S\left(\frac{p}{k^n}\right), t_0\tilde{\xi}\left(\frac{p}{k^n},0\right)\right) \\
&\quad \star \mu\left(S\left(\frac{p}{k^n}\right)-s\left(\frac{p}{k^n}\right), t_0\tilde{\xi}\left(\frac{p}{k^n},0\right)\right) \geq 1-\epsilon,
\end{aligned}$$

and similarly

$$\begin{aligned}
\nu(S(p)-T(p),t) &\leq \nu\left(\frac{s\left(k^{n}p\right)}{k^{6n}}-S(p),\frac{t}{2}\right)\Diamond\nu\left(T(p)-\frac{s\left(k^{n}p\right)}{k^{6n}},\frac{t}{2}\right)\\
&\leq \nu\left(s\left(\frac{p}{k^{n}}\right)-S\left(\frac{p}{k^{n}}\right),\frac{1}{k^{6n}}t_0\sum_{m=n}^{\infty}k^{6m}\xi\left(\frac{p}{k^{m}},0\right)\right)\\
&\qquad\Diamond\nu\left(T\left(\frac{p}{k^{n}}\right)-s\left(\frac{p}{k^{n}}\right),\frac{1}{k^{6n}}t_0\sum_{m=n}^{\infty}k^{6m}\xi\left(\frac{p}{k^{m}},0\right)\right)\\
&= \nu\left(s\left(\frac{p}{k^{n}}\right)-S\left(\frac{p}{k^{n}}\right),t_0\tilde{\xi}\left(\frac{p}{k^{n}},0\right)\right)\\
&\qquad\Diamond\nu\left(T\left(\frac{p}{k^{n}}\right)-s\left(\frac{p}{k^{n}}\right),t_0\tilde{\xi}\left(\frac{p}{k^{n}},0\right)\right)\leq\epsilon.
\end{aligned}$$

It follows that $\mu(S(p)-T(p),t)=1$ and $\nu(S(p)-T(p),t)=0$ for all $t>0$. Hence $S(p)=T(p)$ for all $p\in A$, which completes the proof.

Corollary 4.4 *Let A be a normed linear space, and $(\mathcal{F},\mu,v)$ be an intuitionistic fuzzy Banach space. Let $s:A\longrightarrow\mathcal{F}$ be a function such that for all $\theta\geq 0$, $0\leq r\leq 1$*

$$\lim_{t\to\infty}\mu\left(\Delta s(p,q),t\theta\left(\|p\|^{r}+\|q\|^{r}\right)\right)=1,$$

uniformly in $A\times A$. Then there exists a unique hexic mapping $S:A\longrightarrow\mathcal{F}$ such that

$$\lim_{t\to\infty}\mu\left(S(p)-s(p),\frac{t\theta\|p\|^{r}}{|1-k^{r-6}|}\right)=1,$$

and

$$\lim_{t\to\infty}\nu\left(S(p)-s(p),\frac{t\theta\|p\||^{r}}{|1-k^{r-6}|}\right)=0,$$

uniformly in A.

Proof. The proof follows immediately from Theorems 4.1, and 4.3 by considering the control function $\xi(p,q)=\theta\left(\|p\|^{r}+\|q\|^{r}\right)$ for some $\theta>0$.

5 Experimental results

In this section, we discuss about the graphs of exact solution and approximate solution of equation (9). It is easy to check that the hexic function $S(p)=p^6$ is an exact solution to equation (9). For experimental purpose, we considered another function $s(p)=p^6\ln(|p|)+p^6$ which is not a hexic function. These two functions are plotted on a graph using geogebra and it is observed that the graphs of the two functions $S(p)$ and $s(p)$ coincide with each other at many points. This indicates that $s(p)$ is an approximate solution to equation (9).

Value of the variable p	**Exact solution** $S(p)$	**Approximate solution** $s(p)$	**Difference** $\lvert S(p) - s(p)\rvert$
-1	1.0000	1.0000	0.00000
-0.9	0.5314	0.4754	0.05599
-0.8	0.2621	0.2036	0.05850
-0.7	0.1176	0.0757	0.04196
-0.6	0.0467	0.0228	0.02383
-0.5	0.0156	0.0048	0.01083
-0.4	0.0041	0.0003	0.00375
-0.3	0.0007	-0.0001	0.00088
-0.2	0.0001	0.0000	0.00010
-0.1	0.0000	0.0000	0.00000
0.1	0.0000	0.0000	0.00000
0.2	0.0001	0.0000	0.00010
0.3	0.0007	-0.0001	0.00088
0.4	0.0041	0.0003	0.00375
0.5	0.0156	0.0048	0.01083
0.6	0.0467	0.0228	0.02383
0.7	0.1176	0.0757	0.04196
0.8	0.2621	0.2036	0.05850
0.9	0.5314	0.4754	0.05599
1	1.0000	1.0000	0.00000

The graphs of the functions $S(p)$ (in green colour) and $s(p)$ (in red colour) are shown in Figure 1.

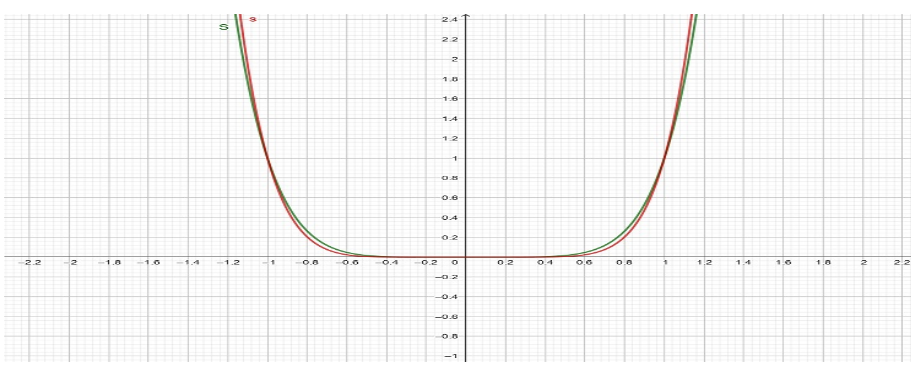

Figure 1: Graphs of $S(p)$ and $s(p)$

Also, the difference between the values of $S(p)$ and $s(p)$ are shown in the above Table 5. From the table, we observe that the differences between the functional values of $S(p)$ and $s(p)$ are small and the maximum of the calculated difference values is 0.54441. Hence, from the experimental analysis, we infer that there exists an approximate solution to equation 9.

6 Comparative evaluation of the results

We have dealt with a new hexic functional equation (9) in this study. We have proved that its stability results holds good in various fuzzy settings namely fuzzy Banach space, Felbin's type fuzzy Banach space and intuitionistic fuzzy Banach space. The main results obtained are summarized as follows:

Corollary No.	**Fuzzy setting**	**Stability result**
Corollary 2.5	Fuzzy Banach space	$H\left(s(p)-S(p), \frac{\epsilon}{k^6-1}\right) \geq \alpha$
Corollary 3.3	Felbin's type Banach space	$\|s(p)-S(p)\|_\alpha^+ \preceq \frac{\epsilon}{k^2(k^6-1)}$
Corollary 4.2	Intuitionistic fuzzy Banach space	$\mu\left(S(p)-s(p), \frac{\epsilon k^6}{k^6-1}\right) > \alpha$, $\nu\left(S(p)-s(p), \frac{\epsilon k^6}{k^6-1}\right) < 1-\alpha$

for all $p \in A$. From the compared results in the table, we observe that the approximate solution $s(p)$ is very close to the exact solution $S(p)$ in the setting of Felbin's type Banach space since the upper bound $\frac{\epsilon}{k^2(k^6-1)}$ is less when compared when the upper bounds in fuzzy Banach space and intuitionistic fuzzy Banach space. The stability results concerning Hyers-Ulam stability involved with a positive constant ϵ in the upper bound are obtained in Corollaries 2.5, 3.3 and 4.2.

7 Discussion on the investigation

In this investigation, we have proved the stability results of equation (9) in fuzzy Banach spaces, Felbin's type fuzzy Banach spaces and intuitionistic fuzzy Banach spaces. A fuzzy norm on a real vector space is defined on a real vector space to produce fuzzy normed space. A left fuzzy norm and a right fuzzy norm are defined to construct Felbin's fuzzy normed space. In the case of intuitionistic fuzzy normed spaces, the continuous t-norm and t-conorm are defined on $[0,1]\times[0,1]$. Then a vector space becomes intuitionistic fuzzy normed space with these t-norm and t-conorm, and fuzzy sets μ and ν. Using these different notions and conditions in various fuzzy settings, we have obtained the stability results in different fuzzy spaces. In the recent research work in other disciplines, fuzzy Banach space, Felbin's type Banach space and intuitionistic fuzzy Banach space play a major role in solving problems involving uncertainities.

8 Conclusion

Using the notions and definitions introduced in [5], we analysed critically the fuzzy conditions with different norms to investigate stability results of equation (9). At the end of this study, we arrived at valid stability results in various fuzzy settings using direct method. There are various stability results of different types of functional, differential and integral equations obtained by many mathematicians in the setting of many normed spaces like Banach spaces, paranormed spaces, quasi-β-normed spaces, non-Archimedean spaces, matrix normed spaces, dislocated metric spaces, p-Banach spaces, Menger probabilistic normed spaces, etc. In this work, we have proved that the stability results of equation (9) hold good in fuzzy spaces like fuzzy Banach space, Felbin's fuzzy Banach space and intutionistic fuzzy Banach space. These stability results obtained in various fuzzy settings could be implemented to solve problems with inexactness and vagueness. We do hope these results would pave a different direction to analyze approximate solutions to a given equation whenever uncertainty occurs.

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References

[1] M. R. Abdollahpour and M.T. Rassias, *Hyers-Ulam stability of hypergeometric differential equations*, Aequa. Math. **93** (4) (2019), 691–698.

[2] M. R. Abdollahpour, R. Aghayari and M. T. Rassias, *Hyers-Ulam stability of associated Laguerre differential equations in a subclass of analytic functions*, J. Math. Anal. Appl. **437** (2016), 605– 612.

[3] J. Aczel and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, **31**, 1989.

[4] A. Alotaibi, M. Mursaleen, H. Dutta and S. A. Mohiuddine, *On the Ulam stability of Cauchy functional equation in IFN-spaces*, Appl. Math. Inf. Sci. **8**(3) (2014),1135–1143.

[5] T. Bag and S. K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **3** (2003), 687–705.

[6] J. Brzdek, *On a method of proving the Hyers-Ulam stability of functional equations on restricted domains*, Aust. J. Math. Anal. Appl. **6** (1), 4 (2009), 10 pages.

[7] J. Brzdek, J. Chudziak and Z. Pales, *A fixed point approach to stability of functional equations*, Nonlinear Anal. **74** (17) (2011), 6728–6732.

[8] D. G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc. **57** (1951), 223–237.

[9] L. Cadariu and V. Radu, *Fixed points and the stability of quadratic functional equations*, An. Univ. Timisoara, Ser. Mat. Inform. **41** (2003), 25–48.

[10] L. Cadariu and V. Radu, *On the stability of the Cauchy functional equation: a fixed point apporach*, Grazer Math. Ber, **346** (2004), 43–52.

[11] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg, **62** (1992), 59–64.

[12] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, 2002.

[13] E. Elqorachi and M. T. Rassias, *Generalized Hyers-UlamsStability of trigonometric functional equations*, Mathematics, **6** (5):83 (2018), 1–11.

[14] C. Felbin, *Finite dimensional fuzzy normed linear spaces*, Fuzzy Sets and Syst. **48** (1992), 239–248.

[15] P. Gavruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mapppings,* J. Math. Anal. Appl. **184** (1994), 431–436.

[16] B. Hejmej, *Stability of functional equations in dislocated quasi-metric spaces*, Annales mathematicae Silesianae, **32** (2018), 215–225.

[17] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[18] D. H. Hyers, G. Isac and T. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhauser, 1998.

[19] D. H. Hyers and M. T. Rassias, *Approximate homomorphisms*, Aequ. Math. **44** (1992), 125–153.

[20] Y. F. Jin, C. Park and M. T. Rassias, *Hom-derivations in C^*-ternary algebras*, Acta Mathematica Sinica, English Series, **36** (9) (2020), 1025–1038.

[21] S. M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis. In: Springer Optimization and Its Applications*, **48**, Springer, New York, 2011.

[22] S. M. Jung, D. Popa and M. T. Rassias, *On the stability of the linear functional equation in a single variable on complete metric groups*, J. Global Optimization, **59** (2014), 165–171.

[23] S. M. Jung and M. T. Rassias, *A linear functional equation of third order associated to the Fibonacci numbers*, Abst. Appl. Anal. **2014** (2014), Article ID 137468, 1–7.

[24] S. M. Jung, M. T. Rassias and C. Mortici, *On a functional equation of trigonometric type*, Appl. Math. Comp. **252** (2015), 294–303.

[25] S. M. Jung, M. T. Rassias and S. M. Yang, *Approximation properties of solutions of a mean value-type functional inequality, II*, Mathematics, **1299** (2020), 1–8.

[26] D. Kang and H. B. Kim, *Generalized cubic functions on a quasi-fuzzy normed space*, J. Chungcheong Math. Soc. **32** (1) (2019), 29–46.

[27] P. Kannappan, *Functional Equations and Inequalities with Applications. In: Springer Monographs in Mathematics*, Springer, New York, 2009.

[28] J. R. Lee, S. Y. Jang, C. Park and D. Y. Shin, *Fuzzy stability of quadratic functional equations*, Adv. Difference Equ. **2010**, Article ID 412160 (2010), 1–16.

[29] Y. H. Lee, S. M. Jung and M. T. Rassias, *On an n-dimensional mixed type additive and quadratic functional equation*, Appl. Math. Comp. **228** (2014), 13–16.

[30] Y. H. Lee, S. M. Jung and M. T. Rassias, *Uniqueness theorems on functional inequalities concerning cubic-quadratic-additive equation*, J. Math. Inequ. **12** (1) (2018), 43–61.

[31] T. Miura, S. Miyajima and S. E. Takahasi, *A characterization of Hyers-Ulam stability of first order linear differential operators*, J. Math. Anal. Appl. **286** (2003) 136–146.

[32] S. A. Mohiuddine and H. Şevli, *Stability of Pexiderized quadratic functional equation in intuitionistic fuzzy normed space*, J. Comp. and Appl. Math. **235** (2011), 2137–2146.

[33] C. Mortici, M. T. Rassias and S. M. Jung, *On the stability of a functional equation associated with the Fibonacci numbers*, Abst. Appl. Anal. **2014** (2014), Article ID 546046, 1–6.

[34] M. Mursaleen and K. J. Ansari, *Stability results in intuitionistic fuzzy normed spaces for a cubic functional equation*, Appl. Math. Inf. Sci. **7** (5) (2013), 1677–1684.

[35] P. Narasimman, H. Dutta and I. H. Jebril, *Stability of mixed type functional equation in normed spaces using fuzzy concept*, Int. J. General Syst. **48** (5) (2019), 507–522.

[36] M. T. Rassias, *Solution of a functional equation problem of Steven Butler*, Octogon Math. Mag. **12** (2004), 152–153.

[37] T. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[38] T. M. Rassias, *Functional Equations and Inequalities*, Kluwer Academic Publishers, 2000.

[39] K. Ravi and B. V. Senthil Kumar, *Ulam-Gavruta-Rassias stability of Rassias reciprocal functional equation*, Global J. Appl. Math. Sci. **3** (1-2) (2010), 57–79.

[40] R. Saadati and C. Park, *Non-Archimedean L-fuzzy normed spaces and stability of functional equations*, Computers and Math. Appl. **60** (2010), 2488–2496.

[41] P. K. Sahoo and P. Kannappan, *Introduction to Functional Equations*, CRC Press, 2011.

[42] B. V. Senthil Kumar and H. Dutta, *Non-Archimedean stability of a generalized reciprocal-quadratic functional equation in several variables by direct and fixed point methods*, Filomat, **32** (9) (2018), 3199–3209.

[43] B. V. Senthil Kumar and H. Dutta, *Fuzzy stability of a rational functional equation and its relevance to system design*, Int. J. General Syst. **48**(2) (2019), 157–169.

[44] B. V. Senthil Kumar and H. Dutta, *Approximation of multiplicative inverse undecic and duodecic functional equations*, Math. Meth. Appl. Sci. **42** (2019), 1073–1081.

[45] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, Inc. New York, 1960.

[46] S.M. Ulam, *Problems in Modern Mathematics*, Wiley, 1964.

[47] J. Xiao and X. Zhu, *Topological degree theory and fixed point theorems in fuzzy normed space*, Fuzzy Sets Syst. **147** (2004), 437–452.

[48] L. A. Zadeh, *Fuzzy sets*, Inform Control, **8** (1965), 338–353.

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