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Some general Gompertz and Gompertz-Makeham life expectancy models

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Abstract

Life expectancy models are highly important, as they indicate the populations health. The present models consider several types of factors, and by analyzing them we extend the formulae to new, mixed types in order to create particular representations that are beneficial for creating future results and estimations.

Keywords: Gompertz distribution, Makeham term, life expectancy, mortality, Gamma function, gamma-heterogeneous populations, frailty.

1 INTRODUCTION

The techniques of constructing Life tables were initially formulated for populations with members that have diverse characteristics to which concerns their endowment for longevity. These new methods own a specific approach for calculating the cohort construct, period, and individual life tables. For a population of individuals, with mortality governed by a GompertzMakeham hazard, we can derive closed-form solutions to the life-expectancy integral, which correspond to homogeneous and gamma-heterogeneous populations, regardless the presence/absence of the Makeham term. The obtained expressions hold special functions that help constructing approximations with more precision. These can be used to further study the elasticity of life expectancy with respect to model parameters.
The gamma-Gompertz multiplicative frailty model is the main parametric

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model applied nowadays to human mortality data at adult and old ages. Is necessary to find a characterization for the life expectancy emerged from a gamma-Gompertz force of mortality in. The paper [10] offered an introduction which describes the Gompertz-Makeham context and model, its connection to frailty, survival function and hypergeometrical gaussian function, and its representation with regards to the new classes of distributions based on the single-parameter Lindleys distribution. In the preliminaries, we described the several types of life expectancy and approximations: Gompertz, Gompertz-Makeham, Lindley-Gompertz, Lindley-Gompertz-Makeham, Shanker, Shanker-Makeham. It provides an exact formula for gamma-Gompertz life expectancy at birth, which may also add value in practice for computational convenience. We compare actual (lifetable) to model-based (gamma-Gompertz) life expectancy, in order to assess (on aggregate) how many years of life expectancy are not captured (or overestimated) by the gamma-Gompertz mortality mechanism.

Recent studies show the diversity of paths and models that can be studied and applied. We can mention amongst them for example a study of Hokov-Mayerov, Flaut and Maturo [16], which analyzez the socio-economic determinants of life expectancy at birth in the European countries starting with year 2000 and ending with 2017. Instead of using only the so-called traditional variables, and here we can mention the per-capita income and education, they are adding in this analysis sociocultural differences and public expenditure on social protection. They used the Random Forest algorithm to measure the importance of certain variables related to life expectancy at birth. Their results show that the variables with greatest relative importance into explaining life expectancy at birth are: the "area" variable, the population education level, public expedicture on social protection. They also show that the least important factors are inflation, public environmental expenditure, unemployment.

Parametric models of human mortality were introduced by Gompertz [15] by offering his vision that death rates at adult ages will increase exponentially with age. Makeham [19] added to the initial model an age-independent constant which performs for mortality that is not related to aging and so, adds a third parameter that improves the model fit.

Vaupel et al. [24] introduced a positive random variable Z, called frailty, that acts upon individual hazards. The framework of the individual frailty Z begins with the following a p.d.f. representation:

$$\pi\left(Z\right)=\frac{\lambda^{k}}{\Gamma\left(k\right)}Z^{k-1}e^{-\lambda z},k,\lambda>0 \tag{1}$$

where frailty is considered to be fixed, i.e. ones frailty is configured at the starting point of the study by a value that remains the same throughout ones life. The given context is based on the force of mortality and the survival function of an individual with frailty Z = z at given age x, namely by (2) and (3):

$$\mu\left(x|z\right) = zae^{bx} + c \tag{2}$$

$$s\left(x|z\right) = exp\{-z\frac{a}{b}\left(e^{bx}-1\right) - cx\} \tag{3}$$

where $a, b > 0$ are the Gompertz parameters and $c \geq 0$ stands for the level of age-independent extrinsic mortality (Kirkwood [17]). Is shown that $\mu(x|z)$ follows a Gompertz curve when c=0. Otherwise $\mu(x|z)$ has a GM shape. The distribution of lifetimes in a ΓG mixture model is described by a survival function:

$$s\left(x\right) = e^{-cx}\left(1+\frac{a}{b\lambda}\left(e^{bx}-1\right)\right)^{-k} \tag{4}$$

$$\mu\left(x|Z\right) = Z\mu\left(x\right), \tag{5}$$

where Z is a random variable, the frailty more specifically (Vaupel, Manton, and Stallard [24], which accounts for the unobserved heterogeneity across individuals. In this framework, $\mu(x)$ will be the base line force of mortality. Model (4) is called a multiplicative (frailty) model.

The gamma-Gompertz multiplicative frailty model (1) has been introduced in demography by Vaupel, Manton, and Stallard [24]. By capturing the observed bending of human mortality rates at older ages (Beard [5]), it also takes into consideration, as previously mentioned, the unobserved heterogeneity.

In [24] and [7], formula (6) describes the remaining life expectance at birth for this specific model,

$$e_0 = \frac{1}{bk} \cdot {}_2F_1\left(k, 1; k+1; 1-\frac{a}{b\lambda}\right), \tag{6}$$

where ${}_2F_1\left(\alpha, \beta; \gamma; z\right)$is the Gaussian hypergeometric function.

Relationship (5) describes the first moment of the mixture gamma-Gompertz distribution. From a demographic perspective, it will mark the expected lifetime duration under the gamma-Gompertz assumption.

There is a large variety of mixtures where the Gompertz distribution is present. To give an example, Bakouch and Abd El-Bar [3] introduced a new weighted version of the Gompertz distribution. Their model consists in a mixture of the classical Gompertz and second upper record value of Gompertz densities. By using a certain tranformation, it gives another version of the two-parameter Lindley distribution. It can be interpreted as a dual member of the log-Lindley-X family. Several properties are obtained. The method of maximum likelihood has been used for the estimation of the model parameters. The results are: the variance-covariance matrix, the confidence interval of the parameters. Bantan et al. [2] assembled their research on unit distributions (quantities with values between 0 and 1 that describe proportions, probabilities, and percentages). They introduced the unit gamma/Gompertz distribution, based on the gamma/Gompertz distribution and the inverse-exponential scheme. Their aim was to transpose the flexibility of the gamma/Gompertz distribution (three-parameter lifetime distribution) to the unit interval. First, they probed it with the analytical behavior of the primary functions. It is indicated that the pdf can be increasing, decreasing, or even increasing-decreasing, decreasing-increasing.

Contrastingly, the hazard rate function has monotonically either increasing, decreasing, or constant shapes. Theyve concluded the theoretical part with propositions concerning stochastic ordering, moments, quantiles, the reliability coefficient. The maximum likelihood method is used also on this model to estimate the parameters from unit data. Simulation results are furnished to evaluate this method: applications using real data sets, one on trade shares and one on flood levels, reveal the importance of this model when contrasted with other unit models.

Given the COVID-19 pandemic and the demand of forecasting the progress of the contagious disease while estimating the procedures of reducing infections, Berihuete et al. [6] performed a Bayesian inference for a non-homogeneous Poisson process, having intensity function that is based on the Gompertz curve. They discussed the prior distribution of the parameter and generated samples from the posterior distribution. For this, they used Markov Chain Monte Carlo (MCMC) methods. In the latter, they have exemplified the method by analyzing real data sets associated with COVID-19, targeted from a specific region that is located in the south of Spain.

In numerous applied sciences such as economics, engineering or finance, modeling and analyzing lifetime data are critical. Lifetime distributions are used to describe, statistically the length of the life of a system. In [11] Ekhouseuhi and Opone introduced a new class of lifetime distribution. They considered the mathematical properties of one of the sub models of the LD, a three parameter generalized Lindley distribution (TPGLD). Bncescu [4] proposed a new method of constructing statistical models. It can be interpreted as the lifetime distributions of series-parallel or parallel-series systems that are used in characterizing coherent systems. The studied problem concerns coherent systems by comparing the expected system lifetimes. Were discussed and established conditions for ordering of the expected system lifetimes of complex series-parallel or parallel-series systems. Also, were considered parameter es-

timation and then the analysis of two real data sets. The formula for the reliability was given, for the hazard rate and for mean hazard rate functions as well.
This paper is organized as follows: In Section 1 we offer an Introduction from the points of view concerning the environment and the mixtures derived over time. Section 2 shows the Preliminaries for the Gompertz model and augmented from it, the Gompertz-Makeham model and the gamma-Gompertz-Makeham model. In continuation, one of the most important classes of probability distributions is studied: Lindleys with its special occurences: one-parameter, two-parameter and extensions. For the last part we consider the PDF and CDF of the Proposed Distribution (NG2PLD). In Section 3 we will give new results for new models, derived from the representations of Damodaran, Zeghdoudi-Nedjar, Ghitany and Ekhosuehi by considering their properties and estimations.

2 Preliminaries

We offer the context for the Gompertz life expectancy particular models: gamma-Gompertz, Gompertz-Makeham and gamma-Gompertz-Makeham as general results to provide a better understanding on the past and recent studies.

2.1 Gompertz life expectancy and its approximation

In the Gompertz case [15], when the force of mortality is given as (7):

$$\mu_G(x) = ae^{bx} \tag{7}$$

Castellares et al. [7] showed that the corresponding remaining life expectancy at age x can be expressed by (8):

$$e_G(x) = \frac{1}{b} E_1\left(\frac{a}{b}e^{bx}\right) exp\left(\frac{a}{b}e^{bx}\right) \tag{8}$$

where $E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt$ denotes the exponential integral.
Relationship (9), as shown by Abramowitz and Stegun [1],

$$E_1(t) = -\gamma - ln\ t - \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n \cdot n!} \tag{9}$$

states that if $t = ae^{bx}/b$ is close to 0, then $e_G(x)$ of (10) can be approximated by:

$$e_G(x) \approx \frac{1}{b}\left(-\gamma - ln\frac{a}{b}\right) exp\left(\frac{a}{b}e^{bx}\right) \tag{10}$$

2.2 Gompertz-Makeham life expectancy and its approximation

In the GompertzMakeham case (through mixing of [15] and [19]), when the force of mortality is given as (11):

$$\mu_{GM}(x) = ae^{bx} + c \tag{11}$$

with its afferent remaining life expectancy at age x (12) from [7], equal to:

$$e_{GM}(x) = \frac{1}{b}\left(\frac{a}{b}e^{bx}\right)^{\frac{c}{b}} exp\left(\frac{ae^{bx}}{b}\right)\Gamma\left(-\frac{c}{b}, \frac{a}{b}e^{bx}\right) \tag{12}$$

Note that:

$$E_1(z) = \lim_{s\to 0}\int_0^\infty t^{s-1}e^{-t}dt = \lim_{s\to 0}\Gamma(s,z)$$

i.e. $e_G(x)$ is a degenerate form of $e_{GM}(x)$ when the Makeham term equals zero.
If a is close to 0, life expectancy at birth $e_{GM}(0)$ can be approximated by:

$$e_{GM}(0) = \frac{1}{c} - \frac{\left(\frac{a}{b}e^{\gamma-1}\right)^{\frac{c}{b}}}{c\left(1-\frac{c}{b}\right)}$$

where $\gamma \approx 0.57722$ is the EulerMascheroni constant.
For parameter values corresponding to mortality patterns in modern societies ($0 < \frac{a}{b}e^{bx} \leq 1$ $\quad and \quad$ $0 < \frac{c}{b} \leq 0.1$),the incomplete gamma function $\Gamma\left(-\frac{c}{b}, \frac{a}{b}e^{bx}\right)$ can be approximated by:

$$\Gamma(s,z) = \frac{s}{s+s^2}exp\{(1-\gamma)s + 0.3225s^2\} - \sum_{k=0}^{\infty}(-1)^k\frac{z^{s+k}}{k!(s+k)}$$

where $\zeta(n) = \sum_{k=1}^{\infty}k^{-n}$ the Riemann zeta function and $0.3225 \approx \frac{\zeta(2)-1}{2}$.
The closer the z-argument of the upper incomplete gamma function to 0, i.e. at younger ages, the fewer terms of $\sum_{k=0}^{\infty}\frac{(-1)^{k+1}z^{s+k}}{k!(s+k)}$ we need to use. To achieve a high accuracy ε, the number of terms m in the latter series to be taken into account can be determined by:

$$\frac{z^{s+m+1}}{(m+1)!(s+m+1)} \leq \varepsilon$$

2.3 Some results in Gamma-Gompertz and Gamma-Gompertz-Makeham life expectancy

Castellares et al. presented in [7] a closed-form expression for the remaining life expectancy based on the gamma-Gompertz model, at age x.

The following Theorem, (used in [7]) presents a complete representation for the hypergeometrical function (13), denoted ${}_2F_1$:

Theorem 1.

If $|z| < 1 \quad and \quad Re(p) > Re(n) > 0$ then:

$$ {}_2F_1(m,n,p;z) = \frac{\Gamma(p)}{\Gamma(n)\Gamma(p-n)} \int_0^1 u^{n-1}(1-u)^{p-n-1}(1-zu)^{-m}du \qquad (13) $$

Where $\Gamma(\cdot)$denotes the complete gamma function.
The following result represent formulas for the gamma-Gompertz and the gamma-Gompertz-Makeham models presented recently by Castellares in [7].

Theorem 2.

The remaining life expectancy at age x (14) for the gamma-Gompertz model is:

$$ e(x) = \frac{1}{bk} \cdot {}_2F_1\left(k, 1, k+1; \frac{\left(1-\frac{a}{b\lambda}\right)}{1-\frac{1}{b\lambda}+\frac{a}{b\lambda}e^{bx}}\right) \qquad (14) $$

Remark 3.

For $x = 0$, (15) occurs,

$$ e(0) = \frac{1}{bk} \cdot {}_2F_1\left(k, 1, k+1; 1-\frac{a}{b\lambda}\right) \qquad (15) $$

which coincides with the life expectancy at birth derived by Missov [21], Eq. [30].

Proposition 1.

Relationship (16) describes the remaining life expectancy at age x for the Gompertz model:

$$ e(x) = exp\left[-\frac{a}{b}\left(e^{bx}-1\right)\right], \quad x \geq 0. \qquad (16) $$

Proposition 2.

Relationship (17) describes the remaining life expectancy at age x for the Gompertz-Makeham model:

$$e(x) = \frac{1}{b} exp\left(\frac{ae^{bx}}{b}\right)\left(\frac{ae^{bx}}{b}\right)^{\frac{c}{b}} \Gamma\left(-\frac{c}{b}, \frac{ae^{bx}}{b}\right). \tag{17}$$

Proposition 3.

Relationship (18) describes the remaining life expectancy at age x for the gamma-Gompertz-Makeham model:

$$e(x) = \frac{1}{bk+c} \cdot {}_2F_1\left(k, 1, k+1+\frac{c}{b}; \frac{\left(1-\frac{a}{b\lambda}\right)}{1-\frac{a}{b\lambda}+\frac{a}{b\lambda}e^{bx}}\right) \tag{18}$$

2.4 Some Lindley Probability Distribution types.

a) A one parameter Lindley distribution 1LD

Lindley [18] introduced a one-parameter lifetime distribution, known as Lindley distribution (1LD). Its probability density function (19), abbreviated as pdf is obtained by combining the densities exponential (θ) and gamma($2;\theta$) with mixing probabilities $\theta/(1+\theta)$ and $1/(1+\theta)$ respectively. Regarding the above description, the pdf of one parameter Lindley distribution is given by:

$$f_1(x) = \frac{\theta^2}{1+\theta}(1+x)\,e^{-\theta x}; \quad x > \theta, \theta > 0. \tag{19}$$

The corresponding cumulative distribution function (cdf) $F_1(x)$ has been obtained by using:

$$F_1(x) = 1 - \frac{\theta+1+\theta x}{1+\theta} e^{-\theta x}; \quad x > 0, \theta > 0,$$

where θ is the scale parameter. Recently researchers have proposed new classes of distributions based this time on the modification of that singular parameter present in Lindleys distribution.
Ghitany et al. [13], [14] studied various properties of this distribution and showed that (1) provides a better model for some applications than the exponential distribution. Mazucheli and Achcar [20] presented the applications of Lindley distribution to competing risks lifetime data. Deniz and Ojeda [9] discussed a discrete version of this distribution and outlined its applications in count data related to insurance.

b) A two-parameter Lindley distribution 2LD

Shanker et al. [23] introduced the two-parameter Lindley distribution (2LD) for modeling waiting and survival times data. Its pdf (20) is given by:

$$f_2(x) = \frac{\theta^2}{\theta+\alpha}\left(1+\alpha x\right)e^{-\theta x}; \quad x>0, \theta>0, \alpha>-\theta. \tag{20}$$

The moments of the 2LD, its failure rate function, its mean residual life function, and stochastic orderings have been studied. Were established the expressions for: the failure rate function, for mean residual life function, stochastic orderings of the two-parameter LD which have shown the flexibility over the one-parameter LD and over the exponential distribution. The methods discussed for estimating its parameters were the maximum likelihood and the method of moments. This distribution has been found to fit data-sets that are related to waiting and survival times, as well as to test its goodness of fit to which the one parameter LD has been suitable for others. The outcome revealed that almost all these data-sets from the two parameter LD distribution provided closer fits than the ones resulted from the one parameter LD.

c) Generalized Lindley Distribution - GLD

Zakerzadeh and Dolati [25] obtained a generalized Lindley distribution (GLD) and discussed various properties having the following pdf (21):

$$f_3(x) = \frac{\theta^2\left(\theta x\right)^{\alpha-1}\left(\alpha+\gamma x\right)}{\left(\gamma+\theta\right)\Gamma\left(\alpha+1\right)}e^{-\theta x}; \quad x>0, \alpha, \theta, \gamma>0, \tag{21}$$

where

$$\Gamma(c) = \int_0^\infty y^{c-1}e^{-y}dy, \quad c>0,$$

is the complete gamma function.
Bakouch et al. [3] obtained an extended Lindley distribution and discussed some of its properties and applications.

d) New Generalized Lindley Distribution NGLD

Elbatal et al. [12] introduced a new generalized Lindley distribution (NGLD) with pdf (22):

$$f_4(x) = \frac{1}{1+\theta}\left[\frac{\theta^{\alpha+1}\theta^{\alpha-1}}{\Gamma(\alpha)} + \frac{\theta^\beta x^{\beta-1}}{\Gamma(\beta)}\right]e^{-\theta x}; \quad x>0, \theta, \alpha>0, \beta>0. \tag{22}$$

In this section, we present definition and some important properties of the extended new generalized Lindley distribution. Hereafter we use the short form ENGLD for extended new generalized Lindley distribution based on Damodarans formulation [8].

e) Extended New Generalized Lindley Distribution - ENGLD

A continuous random variable X is said to follow ENGLD if its pdf f(x) has the following form:

$$f(x) = \sum_{i=1}^{n} p_i g_i(x)$$

where

$$g_i(x) = \frac{\theta^{\alpha_i}}{\Gamma(\alpha_i)} x^{\alpha_i - 1} e^{-\theta x} \quad for \quad \theta > 0, \alpha_i > 0 \quad for \quad i = 1, 2, 3, \ldots, n.$$

The mixing weights p_i are given by $p_i = \frac{1}{n-1} \frac{\beta\gamma}{\theta+\beta\gamma}$ for $i = 2, 3, \ldots, n$ and

$$p_1 = 1 - \sum_{i=2}^{n} p_i, \beta > 0, \gamma > 0.$$

f) The three component mixture ENGLD

For the three component mixture ENGLD (n=3),

$$p_1 = \frac{\theta}{\theta + \beta\gamma}, p_2 = \frac{\beta\gamma}{2(\theta + \beta\gamma)} \quad and \quad p_3 = \frac{\beta\gamma}{2(\theta + \beta\gamma)}.$$

g) A pseudo Lindley distribution PLD

In this case, the density function of X is given by:

$$f_{PsLD}(x; \theta, \beta) = \begin{cases} \dfrac{\theta(\beta - 1 + \theta x)}{\beta} \cdot exp(-\lambda x), & \text{x}, \theta, \beta \geq 1 \\ 0, & otherwise \end{cases}$$

h) The (two-parameter) weighted Lindley distribution NG2PLD

A random variable X is said to follow the new generalized two-parameter Lindley distribution, if its density function is defined by:

$$f(x, \theta) = \frac{\theta^2}{1 + \theta} \left(1 + \frac{\theta^{\alpha-2} x^{\alpha-1}}{\Gamma(\alpha)}\right) e^{-\theta x}; \quad x > 0, \alpha, \theta > 0$$

The pdf of the NG2PLD is a two-component mixture of Exponential (θ) and Gamma (α, θ) distributions which can be expressed as,

$$f(x, \theta) = pf_1(x) + (1-p)f_2(x)$$

where $f_1(x) = \theta e^{-\theta x}$, $f_2(x) = \frac{\theta^2 x^{\alpha-1}}{\Gamma(\alpha)} e^{-\theta x}$ are the pdf of the Exponential (θ) and Gamma (α, θ) distributions respectively and $p = \frac{\theta}{\theta+1}$ is the mixing proportion.
A two-parameter weighted Lindley distribution and its applications to survival data.
The weighted Lindley distribution considered in [13] has p.d.f.

$$f(x) = \frac{\theta^{c+1}}{(\theta+c)\Gamma(c)} x^{c-1}(1+x)e^{-\theta x}, \quad x > 0, \theta > 0 \tag{23}$$

The pdf (23) can be expressed as a two-component mixture:

$$f(x) = pf_1(x) + (1-p)f_2(x) \tag{24}$$

where $p = \frac{\theta}{\theta+c}$ and

$$f_j(x) = \frac{\theta^{c+j-1}}{\Gamma(c+j-1)} x^{c+j-2} e^{-\theta x}, \quad x > 0, \quad c, \theta > 0, \quad j = 1, 2$$

is the pdf of the gamma distribution with shape parameters $c+j-1$ and scale parameter θ, denoted by Gamma $(c+j-1, \theta)$, $\quad j = 1, 2$.

3 Main Results: New statistical models.

Given some typical life expectancy models, we update them to provide a higher accuracy. In the aftermath, through mixing we found the following results for the mixed gamma-Gompertz and Gompertz-Makeham models, for the GLD, GLD-Makeham, the NGLD, and NGLD-Makeham models, for the PLD-Gompertz, PLD-Gompertz Makeham, WLD-Gompertz and WLD-Gompertz-Makeham models, for the the Ghitany-Gompertz-Makeham, the TPGLD, and the Ekhousehi-Gompertz-Makeham models, and for the Damodaran-Gompertz model, and Damodaran-Gompertz-Makeham models.

3.1 A mixed gamma-Gompertz and Gompertz-Makeham models

Damodaran and Irshard [8] extended the new generalized Lindley distribution (ENGLD) and discussed some of its basic properties, among which we can mention reliability and inequality measures, the expression for Rnyi entropy. For this portrayal were shown properties as well for the: moments and recurrence relation, characteristic function, conditional moments, hazard rate function, vitality function and mean residual life function, geometric vitality function, mean inactivity time and reversed hazard rate function, entropy measures (Sharma-Mittal entropy, Mathai-Haubold entropy, residual entropy, past entropy). Their estimation of parameters for the model was done using the method of moments and the maximum likelihood method. To compare the performance of ENGLD with other Lindley distribution forms, they considered a real life data set.

A mixed gamma-Gompertz model

$$\mu(x) = ae^{bx}$$

Force of mortality for the Gompertz case

$$\pi(z) = \sum_{i=1}^{n} p_i \pi_i(z)$$

where

$$\pi_i(z) = \frac{\theta^{\alpha_i}}{\Gamma(\alpha_i)} x^{\alpha_i - 1} e^{-\theta x}, \quad \alpha_i > 0, i = \overline{1, n}.$$

$$\mu(x|Z = z) = Z\mu(x)$$

Force of mortality for the Gompertz case, Z random variable called frailty, where $Z = z$:

$$S(x|z) = exp\left(-\int_0^x \mu(t/z)dt\right) = exp\{-\frac{az}{b}\left(e^{bx} - 1\right)\}$$

Survival function of an individual with frailty $Z = z$ at age x:

$$S_i(x) = \int_0^\infty S(x/z)\pi_i(z)dz = \left(1 + \frac{a}{b\theta}\left(e^{bx} - 1\right)\right)^{-\alpha_i}$$

Survival function when the force of mortality of an individual has frailty $Z = z$ at age x and $Z \sim Lindley(\theta)$.

Theorem

a) The remaining life expectance at age x for the gamma-Gompertz model is:

$$e(x) = \frac{1}{S(x)} \sum_{i=1}^{n} S_i(x) e^i(x)$$

where

$$e^i(x) = \frac{1}{b\alpha_i} \cdot {}_2F_1 \left(\alpha_i, 1, 1 + \alpha_i, \frac{1 - \frac{a}{b\theta}}{1 - \frac{a}{b\theta} + \frac{a}{b\theta} e^{bx}} \right)$$

b) The remaining life expectance at birth for the gamma-Gompertz model is:

$$e(0) = \sum_{i=1}^{n} \frac{1}{b\alpha_i} \cdot {}_2F_1 \left(\alpha_i, 1, 1 + \alpha_i, 1 - \frac{a}{b\theta} \right)$$

where

$$e^i(0) = \frac{1}{b\alpha_i} \cdot {}_2F_1 \left(\alpha_i, 1, 1 + \alpha_i, 1 - \frac{a}{b\theta} \right)$$

The mixed gamma-Gompertz-Makeham model

$$\mu(x) = ae^{bx} + c$$

Force of mortality for the Gompertz-Makeham case:

$$\mu(x|z) = zae^{bx} + c$$

$$S(x|z) = exp\{ -\frac{za}{b} \left(e^{bx} - 1 \right) - cx \}$$

Survival function of an individual with frailty $Z = z$ at age x:

$$S_i(x) = e^{-cx} \left(1 + \frac{a}{b\theta} \left(e^{bx} - 1 \right) \right)^{\alpha_i}$$

$$S(x) = \sum_{i=1}^{n} p_i S_i(x) = e^{-cx} \sum_{i=1}^{n} p_i \left(1 + \frac{a}{b\theta} \left(e^{bx} - 1 \right) \right)^{-\alpha_i}$$

Survival function when the force of mortality of an individual has frailty $Z = z$ at age x and $Z \sim Lindley(\theta)$.

Theorem

a) The remaining life expectance at age x for the gamma-Gompertz-Makeham model is:

$$e(x) = \frac{1}{S(x)} \sum_{i=1}^{n} S_i(x) e^i(x)$$

where

$$e^i(x) = \frac{1}{b\alpha_i + c} \cdot {}_2F_1\left(\alpha_i, 1, 1 + \alpha_i + \frac{c}{b}, \frac{1 - \frac{a}{b\theta}}{1 - \frac{a}{b\theta} + \frac{a}{b\theta} e^{bx}}\right)$$

b) The remaining life expectance at birth for the gamma-Gompertz-Makeham model is:

$$e(0) = \sum_{i=1}^{n} \frac{1}{b\alpha_i + c} \cdot {}_2F_1\left(\alpha_i, 1, 1 + \alpha_i + \frac{c}{b}, 1 - \frac{a}{b\theta}\right)$$

where

$$e^i(0) = \frac{1}{b\alpha_i + c} \cdot {}_2F_1\left(\alpha_i, 1, 1 + \alpha_i + \frac{c}{b}, 1 - \frac{a}{b\theta}\right)$$

The following results represent special models deducted from the above models

3.2 The GLD and GLD-Makeham models

Theorem

a) The remaining life expectancy at age x for the GLD model is:

$$e(x) = \frac{1}{S(x)} \left(S_1(x) e^1(x) + S_2(x) e^2(x)\right)$$

$$S_1(x) = \left(1 + \frac{1}{b\theta}(e^{bx} - 1)\right)^{\alpha}, \quad S_2(x) = \left(1 + \frac{a}{b\theta}(e^{bx} - 1)\right)^{\alpha+1}$$

$$S(x) = \frac{\theta}{\gamma + \theta}\left(1 + \frac{1}{b\theta}(e^{bx} - 1)\right)^{\alpha} + \frac{\gamma}{\gamma + \theta}\left(1 + \frac{a}{b\theta}(e^{bx} - 1)\right)^{\alpha+1}$$

$$e_1(x) = \frac{1}{b\alpha} \cdot {}_2F_1\left(\alpha, 1, 1 + \alpha, \frac{1 - \frac{a}{b\theta}}{1 - \frac{a}{b\theta} + \frac{a}{b\theta} e^{bx}}\right)$$

$$e_2(x) = \frac{1}{b(\alpha + 1)} \cdot {}_2F_1\left(\alpha + 1, 1, 2 + \alpha, \frac{1 - \frac{a}{b\theta}}{1 - \frac{a}{b\theta} + \frac{a}{b\theta} e^{bx}}\right)$$

b) The remaining life expectancy at birth for the GLD model is:

$$e(0) = \frac{1}{b\alpha} \cdot {}_2F_1\left(\alpha, 1, 1+\alpha, 1-\frac{a}{b\theta}\right) + \frac{1}{b(\alpha+1)} \cdot {}_2F_1\left(\alpha, 1, 1+\alpha, 1-\frac{a}{b\theta}\right)$$

Theorem

a) The remaining life expectancy at age x for the GLD-Makeham model is:

$$e(x) = \frac{1}{S(x)}\left(S_1(x)e^1(x) + S_2(x)e^2(x)\right)$$

$$S_1(x) = e^{-cx}\left(1+\frac{1}{b\theta}(e^{bx}-1)\right)^{\alpha}, \quad S_2(x) = e^{-cx}\left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{\alpha+1}$$

$$S(x) = e^{-cx}\left[\frac{\theta}{\gamma+\theta}e^{-cx}\left(1+\frac{1}{b\theta}\left(e^{bx}-1\right)\right) + \frac{\gamma}{\gamma+\theta}e^{-cx}\left(1+\frac{a}{b\theta}\left(e^{bx}-1\right)\right)^{\alpha+1}\right]$$

$$e_1(x) = \frac{1}{b\alpha+c} \cdot {}_2F_1\left(\alpha, 1, 1+\alpha+\frac{c}{b}, 1-\frac{a}{b\theta}\right)$$

$$e_2(x) = \frac{1}{b(\alpha+1)+c} \cdot {}_2F_1\left(\alpha+1, 1, 2+\alpha+\frac{c}{b}, 1-\frac{a}{b\theta}\right)$$

b) The remaining life expectancy at birth for the GLD-Makeham model is:

$$e(0) = \frac{1}{b\alpha+c} \cdot {}_2F_1\left(\alpha+1, 1, 2+\alpha+\frac{c}{b}, 1-\frac{a}{b\theta}\right)$$
$$+\frac{1}{b(\alpha+1)+c} \cdot {}_2F_1\left(\alpha+1, 1, 2+\alpha+\frac{c}{b}, 1-\frac{a}{b\theta}\right)$$

3.3 The NGLD and NGLD-Makeham models

Theorem

a) The remaining life expectancy at age x for the NGLD model is:

$$e(x) = \frac{1}{S(x)}\left(S_1(x)e_1(x) + S_2(x)e_2(x)\right)$$

$$S_1(x) = \left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{\alpha}, \quad S_2(x) = \left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{\beta}$$

$$S(x) = \frac{\theta}{1+\theta}\left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{\alpha} + \frac{1}{1+\theta}\left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{\beta}$$

$$e_1(x) = \frac{1}{b\alpha} \cdot {}_2F_1\left(\alpha, 1, 1+\alpha, \frac{1-\frac{a}{b\theta}}{1-\frac{a}{b\theta}+\frac{a}{b\theta}e^{bx}}\right)$$

$$e_2(x) = \frac{1}{b\beta} \cdot {}_2F_1\left(\beta, 1, 1+\beta, \frac{1-\frac{a}{b\theta}}{1-\frac{a}{b\theta}+\frac{a}{b\theta}e^{bx}}\right)$$

b) The remaining life expectancy at birth for the NGLD model is:

$$e(0) = \frac{1}{b\alpha} \cdot {}_2F_1\left(\alpha, 1, 1+\alpha, 1-\frac{a}{b\theta}\right) + \frac{1}{b\beta} \cdot {}_2F_1\left(\beta, 1, 1+\beta, 1-\frac{a}{b\theta}\right)$$

Theorem

a) The remaining life expectancy at age x for the NGLD-Makeham model is:

$$e(x) = \frac{1}{S(x)}\left(S_1(x)e_1(x) + S_2(x)e_2(x)\right)$$

$$S_1(x) = e^{-cx}\left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{\alpha}, \quad S_2(x) = e^{-cx}\left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{\beta}$$

$$S(x) = e^{-cx}\left[\frac{\theta}{1+\theta}\left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{\alpha} + \frac{1}{1+\theta}\left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{\beta}\right]$$

$$e_1(x) = \frac{1}{b\alpha+c} \cdot {}_2F_1\left(\alpha, 1, 1+\alpha+\frac{c}{b}, \frac{1-\frac{a}{b\theta}}{1-\frac{a}{b\theta}+\frac{a}{b\theta}e^{bx}}\right)$$

$$e_2(x) = \frac{1}{b\beta+c} \cdot {}_2F_1\left(\beta, 1, 1+\beta+\frac{c}{b}, \frac{1-\frac{a}{b\theta}}{1-\frac{a}{b\theta}+\frac{a}{b\theta}e^{bx}}\right)$$

b) The remaining life expectancy at birth for the NGLD-Makeham model is:

$$e(0) = \frac{1}{b\alpha+c} \cdot {}_2F_1\left(\alpha, 1, 1+\alpha+\frac{c}{b}, 1-\frac{a}{b\theta}\right) + \frac{1}{b\beta+c} \cdot {}_2F_1\left(\beta, 1, 1+\beta+\frac{c}{b}, 1-\frac{a}{b\theta}\right)$$

3.4 PLD Gompertz and PLD-Gompertz-Makeham models

Zeghdoudi and Nedjar continued in [22] the study on the Gamma Lindley proposed more properties and simulations to provide a more flexible model for lifetime data. Statistical properties such as the quantile function, the Lorenz curve, the moment method, maximum likelihood estimation, entropy, or limiting distribution of extreme order statistics were established. Their simulation study examined the bias and the mean square error of the maximum likelihood estimators of the parameters.

They built an application of this model for a real data set and compared

with the fit grasped by other two (three)-parameter distributions. Further, They gave the new distribution named Gamma Lindley distribution (GaLD), of which the Lindley distribution (LD) is a particular case. In [27] theyve discussed and added even more properties. An application of this distribution was given.
In [26] the Poisson pseudo Lindley distribution (PPsLD) has been obtained by compounding Poisson and pseudo Lindley distributions. Moments, the Lorenz curve, the quantile function, the maximum likelihood estimation were all described by giving their properties. Simulations studies, as well as data driven applications were reported.
The real interested thing started happening here, in [27] where they introduced a new distribution called the Pseudo Lindley Distribution (PsLD) which is a generalization of the Lindley distribution (LD). Many properties were studied, amongst which we can list some: moments, characteristic function, failure, rate function, distributions of sums, parameters estimation.

$$\pi(z) = \frac{\theta(\beta - 1 + \theta z)e^{-\theta z}}{\beta}, \quad \beta \geq 1, \theta > 0, z > 0$$

$$\pi(z) = p\pi_1 + (1-p)\pi_2, \quad p = \frac{\beta - 1}{\beta}$$

$$\pi_1 = \theta e^{-\theta z}, \quad \alpha_1 = 1$$

$$\pi_2 = \theta^2 z e^{-\theta z}, \quad \alpha_2 = 2$$

Theorem

a) The remaining life expectance at age x for the PLD-Gompertz model is:

$$e(x) = \frac{1}{S(x)}(S_1(x)e^1(x) + S_2(x)e^2(x))$$

where

$$S(x) = \frac{\beta - 1}{\beta} \cdot \left(1 + \frac{a}{b\theta}\left(e^{bx} - 1\right)\right)^{-1} + \frac{1}{\beta} \cdot \left(1 + \frac{a}{b\theta}\left(e^{bx} - 1\right)\right)^{-2}$$

$$S_1(x) = \left(1 + \frac{a}{b\theta}\left(e^{bx} - 1\right)\right)^{-1}$$

$$S_2(x) = \left(1 + \frac{a}{b\theta}\left(e^{bx} - 1\right)\right)^{-2}$$

$$e^1(x) = \frac{1}{b} \cdot {}_2F_1\left(1, 1, 2; \frac{1 - \frac{a}{b\theta}}{1 - \frac{a}{b\theta} + \frac{a}{b\theta}e^{bx}}\right)$$

$$e^2(x) = \frac{1}{2b} \cdot {}_2F_1\left(2, 1, 3; \frac{1-\frac{a}{b\theta}}{1-\frac{a}{b\theta}+\frac{a}{b\theta}e^{bx}}\right)$$

b) The remaining life expectance at birth for the PLD-Gompertz model is:

$$e(0) = \frac{1}{b} \cdot {}_2F_1\left(1, 1, 2; 1-\frac{a}{b\theta}\right) + \frac{1}{2b} \cdot {}_2F_1\left(2, 1, 3; 1-\frac{a}{b\theta}\right)$$

The adaptation of the above result for the mixture with the Makeham model is:

Theorem

a) The remaining life expectance at age x for the PLD-Gompertz-Makeham model is:

$$e(x) = \frac{1}{S(x)}(S_1(x)e^1(x) + S_2(x)e^2(x))$$

where:

$$S(x) = e^{-cx}\left[\frac{\beta-1}{\beta} \cdot \left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{-1} + \frac{1}{\beta} \cdot \left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{-2}\right]$$

$$S_1(x) = e^{-cx}\left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{-1}$$

$$S_2(x) = e^{-cx}\left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{-2}$$

$$e^1(x) = \frac{1}{b+c} \cdot {}_2F_1\left(1, 1, 2+\frac{c}{b}; \frac{1-\frac{a}{b\theta}}{1-\frac{a}{b\theta}+\frac{a}{b\theta}e^{bx}}\right)$$

$$e_2(x) = \frac{1}{2b+c} \cdot {}_2F_1\left(2, 1, 3+\frac{c}{b}; \frac{1-\frac{a}{b\theta}}{1-\frac{a}{b\theta}+\frac{a}{b\theta}e^{bx}}\right)$$

b) The remaining life expectance at birth for the PLD-Gompertz-Makeham model is:

$$e(0) = \frac{1}{b+c} \cdot {}_2F_1\left(1, 1, 2+\frac{c}{b}; 1-\frac{a}{b\theta}\right) + \frac{1}{2b+c} \cdot {}_2F_1\left(2, 1, 3+\frac{c}{b}; 1-\frac{a}{b\theta}\right)$$

3.5 The WLD Gompertz and WLD-GM models

In [13] a two-parameter weighted Lindley distribution was proposed for modeling survival data. This distribution has the property that the hazard rate (or the mean residual life) function exhibits bathtub (upside-down bathtub) or increasing (decreasing) shapes. The application consists in simulation studies conducted to investigate the performance of the MLE and the asymptotic confidence intervals of the parameters. Then the proposed model has been used for the representation of real survival data.

Theorem

a) The remaining life expectance at age x for the WLD-Gompertz model is:

$$e(x) = \frac{1}{S(x)}(S_1(x)e^1(x) + S_2(x)e^2(x))$$

where

$$S(x) = \frac{\theta}{\theta+\gamma} \cdot \left(1 + \frac{a}{b\theta}(e^{bx}-1)\right)^{-\gamma} + \frac{\gamma}{\theta+\gamma} \cdot \left(1 + \frac{a}{b\theta}(e^{bx}-1)\right)^{-\gamma-1}$$

$$S_1(x) = \left(1 + \frac{a}{b\theta}(e^{bx}-1)\right)^{-\gamma}$$

$$S_2(x) = \left(1 + \frac{a}{b\theta}(e^{bx}-1)\right)^{-\gamma-1}$$

$$e^1(x) = \frac{1}{\gamma b} \cdot {}_2F_1\left(\gamma, 1, \gamma+1; \frac{1-\frac{a}{b\theta}}{1-\frac{a}{b\theta}+\frac{a}{b\theta}e^{bx}}\right)$$

$$e^2(x) = \frac{1}{b(\gamma+1)} \cdot {}_2F_1\left(\gamma+1, 1, \gamma+2; \frac{1-\frac{a}{b\theta}}{1-\frac{a}{b\theta}+\frac{a}{b\theta}e^{bx}}\right)$$

b) The remaining life expectance at age x for the WLD-Gompertz model is:

$$e(0) = \frac{1}{\gamma b} \cdot {}_2F_1\left(1, 1, 2; 1 - \frac{a}{b\theta}\right) + \frac{1}{b(\gamma+1)} \cdot {}_2F_1\left(\gamma+1, 1, \gamma+2; 1 - \frac{a}{b\theta}\right)$$

Adjusting our result with the Makeham model, we get:

Theorem

a) The remaining life expectance at age x for the WLD-Gompertz-Makeham model is:

$$e(x) = \frac{1}{S(x)}(S_1(x)e^1(x) + S_2(x)e^2(x))$$

where

$$S(x) = e^{-cx}\left[\frac{\theta}{\theta+\gamma} \cdot \left(1 + \frac{a}{b\theta}(e^{bx}-1)\right)^{-\gamma} + \frac{\gamma}{\theta+\gamma} \cdot \left(1 + \frac{a}{b\theta}(e^{bx}-1)\right)^{-\gamma-1}\right]$$

$$S_1(x) = e^{-cx}\left(1 + \frac{a}{b\theta}(e^{bx} - 1)\right)^{-\gamma}$$

$$S_2(x) = e^{-cx}\left(1 + \frac{a}{b\theta}(e^{bx} - 1)\right)^{-\gamma-1}$$

$$e^1(x) = \frac{1}{\gamma b + c} \cdot {}_2F_1\left(\gamma, 1, \gamma + 1 + \frac{c}{b}; \frac{1 - \frac{a}{b\theta}}{1 - \frac{a}{b\theta} + \frac{a}{b\theta}e^{bx}}\right)$$

$$e^2(x) = \frac{1}{b(\gamma + 1) + c} \cdot {}_2F_1\left(\gamma + 1, 1, \gamma + 2 + \frac{c}{b}; \frac{1 - \frac{a}{b\theta}}{1 - \frac{a}{b\theta} + \frac{a}{b\theta}e^{bx}}\right)$$

b) The remaining life expectance at age x for the WLD-Gompertz-Makeham model is:

$$e(0) = \frac{1}{\gamma b + c} \cdot {}_2F_1\left(1, 1, 2 + \frac{c}{b}; 1 - \frac{a}{b\theta}\right) + \frac{1}{b(\gamma + 1) + c} \cdot {}_2F_1\left(\gamma + 1, 1, \gamma + 2 + \frac{c}{b}; 1 - \frac{a}{b\theta}\right)$$

3.6 The Ghitany-Gompertz-Makeham model

In many applied sciences, such as economics, engineering and finance, amongst others, modeling and anaylizing lifetime data is crucial. Life time distributions are used to describe, statistically, the length of the life of a system. The adapted models, involving more parameters tend to drive more accurate results and can be applied for special cases, where an conformation of the model is needed.

Theorem

a) The remaining life expectance at age x for the Ghitany-Gompertz-Makeham model is:

$$e(x) = \frac{1}{S(x)}(S_1(x)e^1(x) + S_2(x)e^2(x))$$

where

$$S(x) = e^{-cx}\left[\frac{\beta - 1}{\beta} \cdot \left(1 + \frac{a}{b\theta}(e^{bx} - 1)\right)^{-1} + \frac{1}{\beta} \cdot \left(1 + \frac{a}{b\theta}(e^{bx} - 1)\right)^{-2}\right]$$

$$S_1(x) = e^{-cx}\left(1 + \frac{a}{b\theta}(e^{bx} - 1)\right)^{-1}$$

$$S_2(x) = e^{-cx}\left(1 + \frac{a}{b\theta}(e^{bx} - 1)\right)^{-2}$$

$$e^1(x) = \frac{1}{\gamma b + c} \cdot {}_2F_1\left(\gamma, 1, \gamma + 1 + \frac{c}{b}; \frac{1 - \frac{a}{b\theta}}{1 - \frac{a}{b\theta} + \frac{a}{b\theta}e^{bx}}\right)$$
$$e^2(x) = \frac{1}{b(\gamma + 1)} \cdot {}_2F_1\left(\gamma + 1, 1, \gamma + 2 + \frac{c}{b}; \frac{1 - \frac{a}{b\theta}}{1 - \frac{a}{b\theta} + \frac{a}{b\theta}e^{bx}}\right)$$

b) The remaining life expectance at birth for the Ghitany-Gompertz-Makeham model is:

$$e(0) = \frac{1}{\gamma b + c} \cdot {}_2F_1\left(1, 1, 2; 1 - \frac{a}{b\theta}\right) + \frac{1}{b(\gamma + 1)} \cdot {}_2F_1\left(\gamma + 1, 1, \gamma + 2 + \frac{c}{b}; 1 - \frac{a}{b\theta}\right)$$

3.7 The TPGLD model

TPGLD generalizes some of the Lindley family of distribution, and here we can mention: the power Lindley distribution, Sushila distribution, Lindley-Pareto distribution, Lindley-half logistic distribution, the classical Lindley distribution. An application for two real lifetime data sets of the TPGLD exposes its superiority over the exponentiated power LD, the exponentiated Lindley geometric distribution, the power LD, the Lindley-exponential distribution, and the traditional one parameter LD in modeling the lifetime data sets.

Theorem

a) The remaining life expectance at age x for the TPGLD model is:

$$e(x) = \frac{1}{S(x)}(S_1(x)e^1(x) + S_2(x)e^2(x))$$

where

$$S(x) = \frac{\theta}{\theta + 1} \cdot \left(1 + \frac{a}{b\theta}(e^{bx} - 1)\right)^{-1} + \frac{1}{\theta + 1} \cdot \left(1 + \frac{a}{b\theta}(e^{bx} - 1)\right)^{-2}$$

$$S_1(x) = \left(1 + \frac{a}{b\theta}(e^{bx} - 1)\right)^{-1}$$
$$S_2(x) = \left(1 + \frac{a}{b\theta}(e^{bx} - 1)\right)^{-2}$$
$$e^1(x) = \frac{1}{b} \cdot {}_2F_1\left(1, 1, 2; \frac{1 - \frac{a}{b\theta}}{1 - \frac{a}{b\theta} + \frac{a}{b\theta}e^{bx}}\right)$$
$$e^2(x) = \frac{1}{2b} \cdot {}_2F_1\left(2, 1, 3; \frac{1 - \frac{a}{b\theta}}{1 - \frac{a}{b\theta} + \frac{a}{b\theta}e^{bx}}\right)$$

b) The remaining life expectance at birth for the TPGLD model is:

$$e(0) = \frac{1}{b} \cdot {}_2F_1\left(1, 1, 2; 1 - \frac{a}{b\theta}\right) + \frac{1}{2b} \cdot {}_2F_1\left(2, 1, 3; 1 - \frac{a}{b\theta}\right)$$

3.8 The Ekhosuehi-Gompertz-Makeham model

Theorem

a) The remaining life expectance at age x for the Ekhosuehi-Gompertz-Makeham model is:

$$e(x) = \frac{1}{S(x)}(S_1(x)e^1(x) + S_2(x)e^2(x))$$

where

$$S(x) = e^{-cx}\left[\frac{\theta}{\theta+1}\cdot\left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{-1} + \frac{1}{\theta+1}\cdot\left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{-2}\right]$$

$$S_1(x) = e^{-cx}\left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{-1}$$

$$S_2(x) = e^{-cx}\left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{-2}$$

$$e^1(x) = \frac{1}{b+c}\cdot {}_2F_1\left(1,1,2+\frac{c}{b};\frac{1-\frac{a}{b\theta}}{1-\frac{a}{b\theta}+\frac{a}{b\theta}e^{bx}}\right)$$

$$e^2(x) = \frac{1}{2b+c}\cdot {}_2F_1\left(2,1,3+\frac{c}{b};\frac{1-\frac{a}{b\theta}}{1-\frac{a}{b\theta}+\frac{a}{b\theta}e^{bx}}\right)$$

b) The remaining life expectance at birth for the Ekhosuehi-Gompertz-Makeham model is:

$$e(0) = \frac{1}{b+c}\cdot {}_2F_1\left(1,1,2;1-\frac{a}{b\theta}\right) + \frac{1}{2b+c}\cdot {}_2F_1\left(2,1,3+\frac{c}{b};1-\frac{a}{b\theta}\right)$$

3.9 The Damodaran-Gompertz model

Theorem

a) The remaining life expectance at age x for the Damodaran-Gompertz model is:

$$e(x) = \frac{1}{S(x)}(S_1(x)e^1(x) + S_2(x)e^2(x))$$

where

$$S(x) = \frac{\theta}{1+\theta}\cdot\left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{-\alpha} + \frac{1}{\theta+1}\cdot\left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{-\beta}$$

$$S_1(x) = \left(1+\frac{a}{b\theta}(e^{bx}-1)\right)^{-\alpha}$$

$$S_2(x) = \left(1 + \frac{a}{b\theta}(e^{bx} - 1)\right)^{-\beta}$$

$$e^1(x) = \frac{1}{\alpha b} \cdot {}_2F_1\left(\alpha, 1, \alpha + 1; \frac{1 - \frac{a}{b\theta}}{1 - \frac{a}{b\theta} + \frac{a}{b\theta}e^{bx}}\right)$$

$$e^2(x) = \frac{1}{\beta b} \cdot {}_2F_1\left(\beta, 1, \beta + 1; \frac{1 - \frac{a}{b\theta}}{1 - \frac{a}{b\theta} + \frac{a}{b\theta}e^{bx}}\right)$$

b) The remaining life expectance at birth for the Ekhosuehi-Gompertz-Makeham model is:

$$e(0) = \frac{1}{\alpha b} \cdot {}_2F_1\left(1, 1, 2; 1 - \frac{a}{b\theta}\right) + \frac{1}{\beta b} \cdot {}_2F_1\left(\beta, 1, \beta + 1; 1 - \frac{a}{b\theta}\right)$$

3.10 The Damodaran-Gompertz-Makeham model

Theorem

a)The remaining life expectance at age x for the Damodaran-Gompertz-Makeham model is:

$$e(x) = \frac{1}{S(x)}(S_1(x)e^1(x) + S_2(x)e^2(x))$$

where

$$S(x) = e^{-cx}\left[\frac{\theta}{\theta + 1} \cdot \left(1 + \frac{a}{b\theta}(e^{bx} - 1)\right)^{-1} + \frac{1}{\theta + 1} \cdot \left(1 + \frac{a}{b\theta}(e^{bx} - 1)\right)^{-2}\right]$$

$$S_1(x) = e^{-cx}\left(1 + \frac{a}{b\theta}(e^{bx} - 1)\right)^{-1}$$

$$S_2(x) = e^{-cx}\left(1 + \frac{a}{b\theta}(e^{bx} - 1)\right)^{-2}$$

$$e^1(x) = \frac{1}{\alpha b + c} \cdot {}_2F_1\left(\alpha, 1, \alpha + 1 + \frac{c}{b}; \frac{1 - \frac{a}{b\theta}}{1 - \frac{a}{b\theta} + \frac{a}{b\theta}e^{bx}}\right)$$

$$e^2(x) = \frac{1}{\beta b + c} \cdot {}_2F_1\left(\beta, 1, \beta + 1 + \frac{c}{b}; \frac{1 - \frac{a}{b\theta}}{1 - \frac{a}{b\theta} + \frac{a}{b\theta}e^{bx}}\right)$$

b) The remaining life expectance at birth for the Damodaran-Gompertz-Makeham model is:

$$e(0) = \frac{1}{\alpha b + c} \cdot {}_2F_1\left(1, 1, 2; 1 - \frac{a}{b\theta}\right) + \frac{1}{\beta b + c} \cdot {}_2F_1\left(\beta, 1, \beta + 1 + \frac{c}{b}; 1 - \frac{a}{b\theta}\right)$$

4 Conclusions

We studied the Gompertz, the Gompertz-Makeham and the gamma-Gompertz-Makeham models, as a beginning point.
We saw their extensions and understood that new classes of distributions and life time expectancies have risen and many are yet to be discovered and formulated.
Therefore, we gave new results for new models, derived from the representations of Damodaran, Zeghdoudi-Nedjar, Ghitany and Ekhosuehi by considering their properties and estimations.

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