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Operational matrix method to solve nonlinear reaction-advection-diffusion equation in fractional order system

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Dedicated to the memory of Professor Constantin Popa

Abstract

In the present paper, a numerical scheme is discussed to solve one-dimensional nonlinear diffusion equation of fractional order in which collocation is performed using the Lucas operational matrix. Since the spectral collocation method is used in the proposed method, therefore the residual, initial and boundary conditions of the presented problem are collocated at fixed collocation points. The result is a system of nonlinear equations that can be solved by using Newton's method. Through error analysis and application to some existing problems, the accuracy of the method is confirmed. The obtained results are presented in tabular forms, which clearly show the higher accuracy of the proposed method. The variations of the solute profile of the proposed model are shown graphically due to the presence or absence of advection and reaction terms for different particular cases.

1 Introduction

The fractional order diffusion equation (FDE) is used to deal with the problems related to chemical reactions, environmental pollution, physics, biological

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system, and hydrology, etc. In most of the cases exact solutions are not available for the FDE especially for nonlinear cases, so it is important to obtain the numerical solutions for such types of problems, and therefore it has received more attention from the researchers to solve diffusion equations. In this article, an approximate solution of one-dimensional FDE is obtained with the operational matrix method. Over the past few years, the development of many different methods has taken place. [1] have numerically solved diffusion equations using the quadrature method, [2] have established a method to find exact solution of nonlinear diffusion equation by using variational iteration method, [3] have developed a method to find the unknown boundary conditions of diffusion equation of fractional order, [4] have developed a scheme to find the approximate solution of diffusion equation of fractional order, [5] have discussed operational matrix (OM) of Chebyshev polynomial to solve FDE, [6]-[7] have established a method to find approximate solution of nonlinear diffusion equation of fractional order, [8]-[10] have found approximate analytical solution of coupled FDE.

The ground water is a huge source for drinking water as well in agriculture and industrial sectors. Nowadays we are facing a threat to groundwater resources from expanding demand, waste use and contamination. Understanding the behavior of contaminants as they travel through different media, many researchers are working on numerical as well as analytical studies to simulate the movement of contaminants in groundwater. Mass conservation of the solutes transported through porous media is depicted by a partial differential equation called reaction-advection-diffusion equation. Here the authors have considered the transport of the solute through an aquifer with homogeneous porous medium. The problem can be illustrated physically by assuming the length of the aquifer as l, and pollutants enter into the groundwater through the porous medium. Since pollutants concentration is greater than that of the concentration of the groundwater, so there will be diffusion. The governing equation of the solute can be expressed by the following nonlinear nonhomogeneous fractional-order reaction-advection-diffusion equation (FRADE) as

$$\frac{\partial^\alpha C(x,t)}{\partial t^\alpha} = \frac{\partial}{\partial x}\left(C(x,t)\frac{\partial^\beta C(x,t)}{\partial x^\beta}\right) - v(x)\frac{\partial C(x,t)}{\partial x} + kC(x,t) + f(x,t), \tag{1}$$

where $0 < \beta, \alpha \leq 1$,
with the initial condition as

$$C(x,0) = \rho_0(x) \tag{2}$$

and the boundary conditions as

$$C(0,t) = \rho_1(t), \;\; C(l,t) = \rho_2(t). \tag{3}$$

Here $C(x,t)$ is solute concentration, k is reaction term, v is the advection coefficient and $f(x,t)$ is the forced term. For $\alpha = 1$ and $\beta = 2$, the mathematical model (1) becomes the integer order classical RADE model. Here one dimensional FDE is approximated using Lucas operational matrix together with collocation method. In the discussed method the solution is expressed as a series of Lucas polynomial $\sum c_{i,j}\phi_i\phi_j$, where ϕ is the set of polynomials and coefficients $c_{i,j}$ are calculated by taking help of collocation method. In which boundary, initial conditions and residual are to be collocated at the certain collocation points. In the present article, the solution is approximated by using Lucas polynomials as basis functions. Many authors have used Lucas OM to solve the problem. The authors of [11] have proposed a method with the help of the said matrix to solve the differential equation. In the next year the same authors [12] had generalized the matrix to solve fractional order differential equation. The novelty of the present article is the drive taken by the authors to extend the method to solve the nonlinear PDE in fractional order systems. OM method is better than the other existing methods because as it involves sparse matrix which reduces the computational time. The use of OM method can be seen in many articles, [13] have used Legendre wavelet method for integration, [14] used direct method to solve integro-differential equations and nonlinear Volterra-Fredholm integral equations. An OM to solve differential equations of fractional order was developed in [15]. Thus our effort to apply the method in nonlinear fractional order PDEs is first of its kind. The salient part of this study is the effect of advection and convection terms on the solution profile and damping of concentrations due to presence of sink term for different fractional order parameters.

The article is arranged as follows. Section 2 consists of some basic definitions and formulae of fractional calculus are introduced. Section 3 contains OM for derivative using Lucas polynomial and in Section 4 the details of the proposed numerical tool is provided. In Section 5 the authors have applied the proposed method on three existing problems and compare their solutions with the existing analytical results. The solutions of the concerned scientific model by applying the proposed method is given in Section 6, which is followed by the section Conclusion of the overall work.

2 Preliminaries

2.1 Some useful definitions and formulae

In this section, the following basic properties and definitions of fractional calculus have been used.
The Caputo derivative of fractional order ϑ, where $i-1<\vartheta\leq i$ of function $f(x,t)$ w.r. to the time variable t is given as

$$D_t^{\vartheta}f(x,t)=\begin{cases}\frac{1}{\Gamma(i-\vartheta)}\int_0^t(t-s)^{i-\vartheta-1}\frac{\partial^i f(x,s)}{\partial s^i}ds, & i-1<\vartheta<i,\\ \\ \frac{\partial^i f(x,t)}{\partial t^i}, & \vartheta=i.\end{cases}\tag{4}$$

Similarly the fractional order derivative of function $f(x,t)$ w.r. to variable x is given by

$$D_x^{\vartheta}f(x,t)=\begin{cases}\frac{1}{\Gamma(i-\vartheta)}\int_0^x(x-s)^{i-\vartheta-1}\frac{\partial^i f(s,t)}{\partial s^i}ds, & i-1<\vartheta<i,\\ \\ \frac{\partial^i f(x,t)}{\partial x^i}, & \vartheta=i.\end{cases}\tag{5}$$

In addition, Caputo differential operator satisfies the linearity property

$$D^{\vartheta}(\mu f(x,t)+\lambda g(x,t)=\mu D^{\vartheta}f(x,t)+\lambda D^{\vartheta}g(x,t),\tag{6}$$

where μ and λ are constants. According to the definition of the Caputo differential operator we have

$$D_x^{\vartheta}x^m=\begin{cases}\frac{\Gamma(m+1)}{\Gamma(m+1-\vartheta)}x^{m-\vartheta}, & m=1,2\cdots,\\ \\ 0, & m=0.\end{cases}\tag{7}$$

2.2 Lucas polynomial

Lucas polynomials can be obtained from the following relation

$$L_{n+2}(x)=L_n(x)+xL_{n+1}(x),\ \ n\geq 0,\tag{8}$$

with the conditions

$$L_0(x)=2,\ \ L_1(x)=x.$$

From the above relation the Lucas polynomial is obtained as

$$L_n(x)=n\sum_{m=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(n-2m+1)_{m-1}}{m!}x^{n-2m},\tag{9}$$

where $\lfloor . \rfloor$ denotes the floor function and Pochhammer symbol $(k)_m$ is defined as

$$(k)_m = \frac{\Gamma(m+k)}{\Gamma(k)}.$$

The polynomial x^m for $m \geq 1$ can be expressed as linear combination of Lucas polynomial as

$$x^m = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^k \delta_{m-2k}(m-k+1)_k}{k!} L_{m-2k}(x), \tag{10}$$

where δ_s is defined by

$$\delta_s = \begin{cases} 1 & s > 0, \\ \frac{1}{2}, & s = 0. \end{cases} \tag{11}$$

Suppose $f(x,t)$ be a function in $[0,1] \times [0,1]$. Then $f(x,t)$ can be written in series form of Lucas polynomial $L_n(x)$ as

$$f(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{ij} L_i(x) L_j(t),$$

where coefficients u_{ij}'s are not known . We can approximate the above function in first $(n+1)$ terms in combination of Lucas polynomial as

$$f(x,t) = \sum_{i=0}^{n} \sum_{j=0}^{n} u_{ij} L_i(x) L_j(t) = \Psi^T(x) U \Psi(t),$$

where U is $(1+n) \times (1+n)$ unknown matrix and $\Psi(x)$ is $(1+n) \times 1$ column vector given by

$$U = \begin{pmatrix} u_{0,0} & u_{0,1} & \cdots & u_{0,n+1} \\ u_{1,0} & u_{1,1} & \cdots & u_{1,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n+1,1} & u_{n+1,2} & \cdots & u_{n+1,n+1} \end{pmatrix}_{(n+1)\times(n+1)} \tag{12}$$

and

$$\Psi(x) = [L_0(x), L_1(x), ..., L_n(x)]^T. \tag{13}$$

3 Formation of Lucas OM for fractional order derivative

The differentiation of the vector $\Psi(x)$ w.r. to x is given by

$$\frac{d\Psi(x)}{dx} = P^{(1)} \Psi(x), \tag{14}$$

where $P^{(1)} = (m_{ij}^{(1)})$ *is* an $(1+n)$-dimensional square matrix, whose components are calculated as

$$m_{ij}^{(1)} = \begin{cases} i(-1)^{\frac{i-j-1}{2}}\delta_j, & if \;\; j < i, and \;\; (j+i) \;\; odd, \\ 0, & else. \end{cases}$$

In general, for any positive integer r,

$$\frac{d^r\Psi(x)}{dx^r} = P^{(r)}\Psi(x) = (P^{(1)})^r\Psi(x). \tag{15}$$

From [11], it can be seen for the fractional derivative of order $\alpha > 0$,

$$D^\alpha\Psi(x) = x^{-\alpha}P^{(\alpha)}\Psi(x) \;\; , \tag{16}$$

where $\Psi(x)$ be the Lucas polynomial vector already defined in equation (13), and $P^{(\alpha)}$=$(m_{i,j}^{(\alpha)})$ is the Lucas OM of derivative of order α is given by

$$P^{(\alpha)} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \eta_\alpha(\lceil\alpha\rceil,0) & \eta_\alpha(\lceil\alpha\rceil,\lceil\alpha\rceil) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \eta_\alpha(i,0) & \cdots & \eta_\alpha(i,i) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \eta_\alpha(n,0) & \eta_\alpha(n,1) & \eta_\alpha(n,2) & \cdots & \eta_\alpha(n,n) \end{pmatrix}_{(n+1)\times(n+1)} \tag{17}$$

the elements $m_{i,j}^{\alpha}$'s are given explicitly as

$$m_{ij}^{(\alpha)} = \begin{cases} \eta_\alpha(i,j), & if \;\; j \le i, i \ge \lceil\alpha\rceil, \\ 0, & elsewhere \;, \end{cases}$$

where

$$\eta_\alpha(i,j) = \sum_{\substack{k=\lceil\alpha\rceil \\ (k+i)even \\ (k+j)even}}^{i} \frac{(-1)^{\frac{k-j}{2}}\delta_j\left(\frac{i+k-2}{2}\right)!\left(\frac{2+j+k}{2}\right)_{\frac{k-j}{2}}}{\left(\frac{i-k}{2}\right)!\left(\frac{k-j}{2}\right)!\Gamma(1-\alpha+k)}. \tag{18}$$

4 Numerical method to solve one dimensional Diffusion equation

In this section, we will discuss the method in brief for the approximate solution of the FDE .

To approximate the problem numerically, we assume

$$C(x,t) \approx \tilde{C}(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} u_{i,j} L_i(x) L_j(t) = \Psi^T(x) U \Psi(t), \tag{19}$$

where

$$\Psi(x) = [L_0(x), L_1(x), ..., L_n(x)]^T, \quad \Psi(t) = [L_0(t), L_1(t), ..., L_n(t)]^T, \tag{20}$$

and U is defined in equation (12) whose unknown coefficients have to be determined.
According to the problem, we have

$$\begin{aligned} t^{-\alpha}\Psi^T(x) U P^{(\alpha)} \Psi(t) &= x^{-\beta}\Psi^T(x) U \Psi(t) (P^{(\beta)}\Psi(x))^T U \Psi(t) \\ &+ x^{-(\beta-1)}\Psi^T(x)(P^1)^T U \Psi(t) \Psi^T(x)(P^{(\beta-1)})^T U \Psi(t) \\ &- v(x)(P^{(1)}\Psi(x))^T U \Psi(t) + k\Psi^T(x) U \Psi(t) + f(x,t), \end{aligned} \tag{21}$$

where $\Psi_t(t) = [\frac{\partial}{\partial t} L_0(t), \cdots, \frac{\partial}{\partial t} L_n(t)]^T, \Psi_x(x) = [\frac{\partial}{\partial x} L_0(x), \cdots, \frac{\partial}{\partial x} L_n(x)]^T$,
$\Psi_{xx}(x) = [\frac{\partial^2}{\partial x^2} L_0(x), \cdots, \frac{\partial^2}{\partial x^2} L_n(x)]^T$.
If we approximate $u(x,t)$, then the residual $R(x,t)$ is

$$\begin{aligned} R(x,t) &= t^{-\alpha}\Psi^T(x) U P^{(\alpha)} \Psi(t) - x^{-\beta}\Psi^T(x) U \Psi(t) (P^{(\beta)}\Psi(x))^T U \Psi(t) \\ &- x^{-(\beta-1)}\Psi^T(x)(P^1)^T U \Psi(t) \Psi^T(x)(P^{(\beta-1)})^T U \Psi(t) \\ &+ v(x)(P^{(1)}\Psi(x))^T U \Psi(t) - k\Psi^T(x) U \Psi(t) - f(x,t). \end{aligned} \tag{22}$$

Now, from the initial condition and boundary conditions, we have

$$\Psi^T(x) U \Psi(0) = \rho_0(x), \tag{23}$$

$$\Psi^T(0) U \Psi(t) = \rho_1(t), \tag{24}$$

$$\Psi^T(l) U \Psi(t) = \rho_2(t). \tag{25}$$

While using the collocation method, $R(x,t)$ vanishes at certain collocation points. The collocation points are chosen as $\left(\frac{i}{n+1}, \frac{j}{n+1}\right)$.
Now, making residual $R(x,t)$ zero at certain collocation points, we have

$$R(x_i, t_j) = 0, \;\; i = 1, 2, \cdots, n-1, \;\; j = 1, 2, \cdots, n, \tag{26}$$

and collocating the equation (23) at $(n+1)$ points $\{x_i : i = 0, 1, \cdots, n\}$, we get

$$\Psi^T(x_i)U\Psi(0) = \rho_0(x_i). \tag{27}$$

Also collocating equations (24) and (25) at n points $\{t_j : j = 1, 2, \cdots, n\}$, we get

$$\Psi^T(0)U\Psi(t_j) = \rho_1(t_j), \tag{28}$$

$$\Psi^T(1)U\Psi(t_j) = \rho_2(t_j). \tag{29}$$

Equation (26) with the aid of equations (27), (28) and (29) generates a $(n+1) \times (n+1)$ nonlinear system. whose solution can be obtained by using well known Newton's iteration method.

5 Error bound

In this section, authors aim is to find an upper bound for the error which may be expected in the presented method. Consider the space

$$\prod_n = \text{span}\left\{L_i(x)L_j(t),\ \ i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, n\right\}.$$

If we assume that $\tilde{C}(x,t)$ is the best approximation of $C(x,t) \in \prod_n$, then using the definition of best approximation, we have

$$||C(x,t) - \tilde{C}(x,t)||_\infty \leq ||C(x,t) - S(x,t)||_\infty, \quad \forall S(x,t) \in \prod_n. \tag{30}$$

Inequality (31) is also true if $S(x,t)$ is the interpolating polynomial of $C(x,t)$ at points (x_i, t_j), where x_i, t_j $0 < i, j \leq n$ are roots of the polynomial $L_n(x)$ and $L_n(t)$ respectively. Then using [16] we get

$$\begin{aligned} C(x,t) - S(x,t) = {} & \frac{1}{(1+n)!}\frac{\partial^{1+n}C(\gamma,t)}{\partial x^{1+n}}\prod_{i=0}^{n}(x - x_i) + \frac{1}{(1+n)!}\frac{\partial^{1+n}C(x,\tau)}{\partial t^{1+n}} \\ & \times \prod_{i=0}^{n}(t - t_i) - \frac{1}{(1+n)!}\frac{1}{(1+n)!}\frac{\partial^{2+2n}C(\gamma',\tau')}{\partial x^{1+n}\partial t^{1+n}} \\ & \times \prod_{i=0}^{n}(x - x_i)\prod_{j=0}^{n}(t - t_j) \end{aligned}$$

where $\gamma, \gamma', \tau, \tau' \in [0,1]$, now taking norm on both the sides, we have

$$\begin{aligned}\|C(x,t)-S(x,t)\|_\infty \leq{}& \frac{1}{(1+n)!}\max_{(x,t)\in[0,1]}\left|\frac{\partial^{1+n}C(\gamma,t)}{\partial x^{1+n}}\right| \times \left\|\prod_{i=0}^{n}(x-x_i)\right\|_\infty \\ &+ \frac{1}{(1+n)!}\max_{(x,t)\in[0,1]}\left|\frac{\partial^{1+n}C(x,\tau)}{\partial t^{1+n}}\right| \\ &\times \left\|\prod_{i=0}^{n}(t-t_j)\right\|_\infty + \frac{1}{(1+n)!^2}\max_{(x,t)\in[0,1]}\left|\frac{\partial^{2+2n}C(\gamma',\tau')}{\partial x^{1+n}\partial t^{1+n}}\right| \\ &\times \left\|\prod_{i=0}^{n}(x-x_i)\right\|_\infty \times \left\|\prod_{j=0}^{n}(t-t_j)\right\|_\infty .\end{aligned}$$

Since $C(x,t)$ has continuous derivative over $[0,1]$ therefore there exist constants K_1, K_2 and K_3 such that

$$\max_{(x,t)\in[0,1]}\left|\frac{\partial^{n+1}C(\gamma,t)}{\partial x^{n+1}}\right| \leq K_1, \qquad \max_{(x,t)\in[0,1]}\left|\frac{\partial^{n+1}C(x,\tau)}{\partial t^{n+1}}\right| \leq \quad K_2,$$

and

$$\max_{(x,t)\in[0,1]}\left|\frac{\partial^{2n+2}C(\gamma',\tau')}{\partial x^{n+1}\partial t^{n+1}}\right| \leq K_3.$$

Now to minimize the factor $\prod_{i=0}^{n}(x-x_i)$, we proceed in the following way

$$\min_{x_i\in[0,1]}\max_{(x,t)\in[0,1]}\left|\prod_{i=0}^{n}(x-x_i)\right| = \min_{x_i\in[0,1]}\max_{(x,t)\in[0,1]}\left|L_{n+1}(x)\right|,$$

where $L_{n+1}(x)$ is the Lucas polynomial of degree $(n+1)$. Now from the inequality of Lucas polynomial

$$|L_n(x)| \leq 2\sigma^n,$$

where $\sigma = \frac{1+\sqrt{5}}{2}$ is known as golden ratio. Now from the above inequalities, we get

$$||C(x,t)-\tilde{C}(x,t)|| \leq \frac{K_1\times 2\sigma^{n+1}}{(n+1)!} + \frac{K_2\times 2\sigma^{n+1}}{(n+1)!} + \frac{K_3\times 4\sigma^{2n+2}}{(n+1)!^2},$$

which is the required upper bound for the absolute error between analytical and numerical results. Therefore numerical approximation leads to the exact solution.

6 Numerical Examples

To validate the effectiveness of the approach, we have applied our proposed approach to three existing problems and have compared the obtained approximate solution with their analytical results. To evaluate the accuracy of the method, the Root-Mean-Square error (RMSE) are used through the formula $\sqrt{\frac{1}{N}\sum_{i=1}^{N}|C(x_i)-C_{exact}(x_i)|^2}$.

Example 1. Consider FDE equation with variable coefficient as

$$\frac{\partial^\alpha C(x,t)}{\partial t^\alpha} = (x^2+t+1)\frac{\partial^2 C(x,t)}{\partial x^2} - t^2 e^x \frac{\partial C(x,t)}{\partial x} + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - 2(x^2+t) + 2t^2 x e^x - 2,$$

where $0 \le x \le 1, \quad t > 0,$
having initial condition

$$C(x,0) = x^2,$$

and boundary conditions

$$C(0,t) = t^2,$$
$$C(1,t) = t^2 + 1,$$

whose exact solution is $C(x,t) = t^2 + x^2$ given in [17].

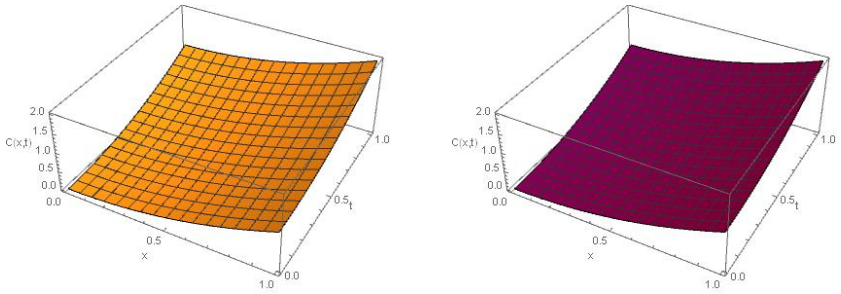
Table 1: The Root mean square (RMS) error for n=7 and $\alpha = 0.2,\ 0.6$, for Example 1

	$\alpha = 0.2$		$\alpha = 0.6$	
$t\downarrow$	Our method at n=7	[17] at n=11	Our method at n=7	[17] at n=11
0.2	5.1819×10^{-16}	2.5257×10^{-6}	6.2789×10^{-16}	4.7226×10^{-5}
0.4	2.8210×10^{-16}	2.3959×10^{-6}	3.0519×10^{-16}	4.3875×10^{-5}
0.6	2.5134×10^{-16}	2.2295×10^{-6}	4.2020×10^{-16}	4.0071×10^{-5}
0.8	6.7896×10^{-16}	2.0723×10^{-6}	5.6392×10^{-16}	3.6685×10^{-5}
1.0	1.5651×10^{-14}	1.9307×10^{-6}	1.5599×10^{-14}	3.3755×10^{-5}

Table 1 shows the RMSE between the exact and approximate solutions. It is shown that RMS error obtained by using our proposed method for the order of the operational matrix $n = 7$ at $\alpha = 0.2$ and $\alpha = 0.8$ are much higher compared to meshless method given in [17]. The error can be decreased by increment in the value of n.

Example 2. Consider the problem

$$\frac{\partial^\alpha C(x,t)}{\partial t^\alpha} = \frac{\partial}{\partial x}\left(C(x,t)\frac{\partial^\beta C(x,t)}{\partial x^\beta}\right) - \frac{\partial C(x,t)}{\partial x} + C(x,t) + f(x,t),$$



(a) Exact solution (b) Numerical solution

Figure 1: Behaviour of exact and numerical solution at $n = 7$ for $\alpha = 0.6$ for Example 1 .

with the initial condition

$$C(x, 0) = 0,$$

and boundary conditions

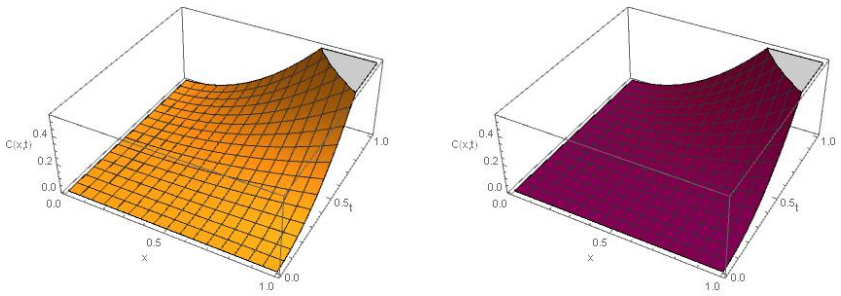
$$C(0, t) = 0, \quad C(1, t) = t^2.$$

The analytical solution of the present time-space fractional nonlinear problem for $\alpha = 0.6$ and $\beta = 0.8$ for the appropriate choice of $f(x, t)$, is given by $C(x, t) = x^2t^2$. In Table 2, the maximum absolute error (MAE) $L_\infty = \max_{0<x,t\leq 1} |C(x, t) - \tilde{C}(x, t)|$ for various n and x values is presented at $t = 0.5$.

Table 2: The maximum absolute error for $n = 4$ and $n = 5$ at $t = 0.5$ for Example 2

	L_∞	L_∞
$x \downarrow$	$n = 4$	$n = 5$
0.2	2.7062×10^{-16}	4.8752×10^{-17}
0.4	1.0825×10^{-15}	2.7756×10^{-17}
0.6	1.4710×10^{-15}	5.8287×10^{-16}
0.8	1.2212×10^{-15}	9.9920×10^{-16}
1.0	6.3726×10^{-14}	7.8381×10^{-14}

It is found that the error reduces due to increase in n. It is also observed that even for $n = 5$, the error is of order 10^{-17} for our proposed method.



(a) Exact solution (b) Numerical solution

Figure 2: Behaviour of Exact and Numerical solutions at $n = 4$ for Example 2.

Example 3. Consider the nonlinear PDE

$$\frac{\partial C(x,t)}{\partial t} = \frac{\partial^2 C(x,t)}{\partial x^2} + \frac{\partial C(x,t)}{\partial x} C(x,t) + (1 - C(x,t))\, C(x,t),$$

with the conditions

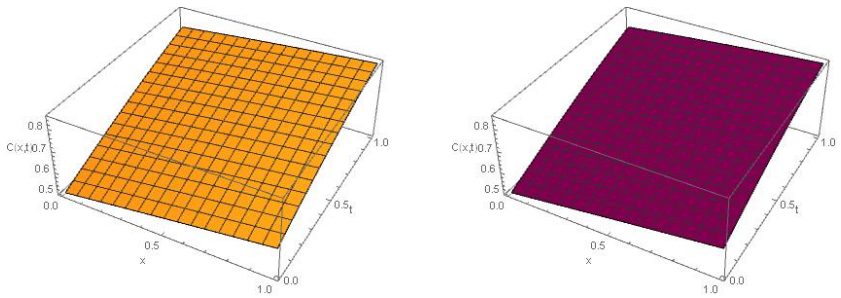
$$C(x,0) = \frac{1}{2}\tanh\left(\frac{x}{4}\right) + \frac{1}{2},$$
$$C(0,t) = \frac{1}{2}\tanh\left(\frac{5t}{8}\right) + \frac{1}{2},$$
$$C(1,t) = \frac{1}{2}\tanh\left[\frac{1}{4}\left(1 + \frac{5t}{2}\right)\right] + \frac{1}{2},$$

which has the analytical solution as $C(x,t) = \frac{1}{2}\tanh\left[\frac{1}{4}\left(x + \frac{5t}{2}\right)\right] + \frac{1}{2}$.

Table 3: The maximum absolute error L_∞ for $n = 3$ and $n = 4$ for Example 3

	Ł$_\infty$	L_∞
$x \downarrow$	$n = 3$	$n = 4$
0.2	7.5843×10^{-4}	2.6815×10^{-5}
0.4	6.3607×10^{-4}	4.4925×10^{-6}
0.6	6.6367×10^{-4}	2.5863×10^{-5}
0.8	8.1003×10^{-4}	4.6403×10^{-5}
1.0	1.0508×10^{-3}	7.0425×10^{-5}

For the above nonlinear problem, it is seen from Table 3 that the absolute error defined in Example 3 is decreased with the increase in n. Thus for a



(a) Exact solution (b) Numerical solution

Figure 3: Behaviour of Exact and Numerical solution at $n = 4$ for Example 3.

nonlinear problem, for a less value of n, the numerical results obtained by our proposed approach are similar to the analytical solution, which clearly show that the proposed numerical scheme is very accurate and effective even for solving nonlinear PDEs.

7 Results and discussion

The present section attempts to solve the considered model using our proposed method for distinct values of α, β, v and k as initial and boundary conditions

$$C(x,0) = (1-x)x,$$

and

$$C(0,t) = 0, \qquad C(1,t) = 0.$$

The residual with prescribed initial and boundary conditions are given in equations (22)-(25). Now varying i from 1 to $n-1$ and j from 1 to n the residual (26) along with equations (27)-(29) already discussed in Section 4 gives a system of algebraic equations of order $(n+1)\times(n+1)$, which has been solved by using Newton method with the help of Wolfram Mathematica software version 11.3 for $n = 3$.

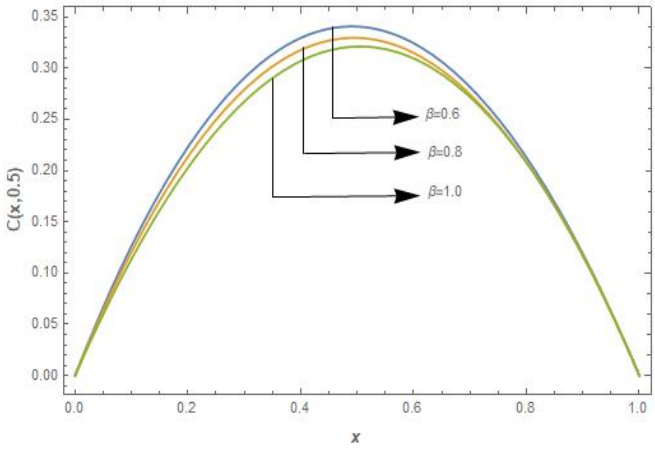


Figure 4: Variations of $C(x, 0.5)$ vs. x for various values of β at $\alpha = 1$,v=1.0, and t=0.5 for non-conservative system with $f(x, t) = xt + 1$.

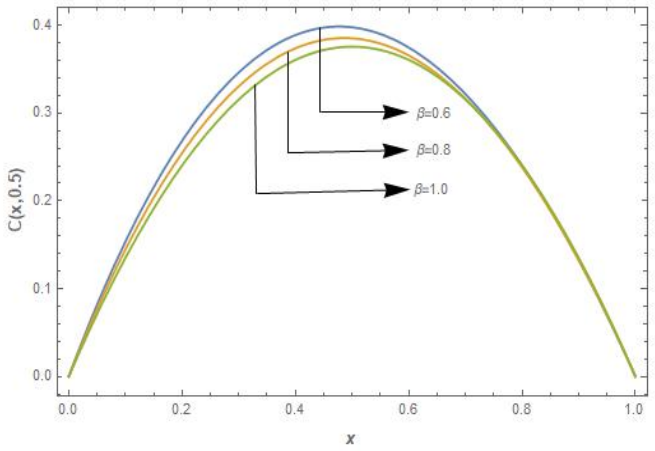


Figure 5: Variations of $C(x, 0.5)$ vs. x for various values of β at $\alpha = 1$,v=1.0, and t=0.5 for conservative system with $f(x, t) = xt + 1$.

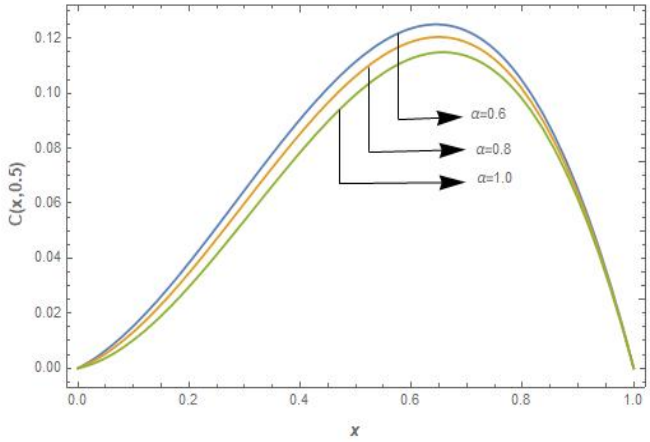


Figure 6: Variations of $C(x, 0.5)$ vs. x for various values of α at v=1.0, $\beta = 1$, and t=0.5 for non-conservative system with $f(x, t) = xt$.

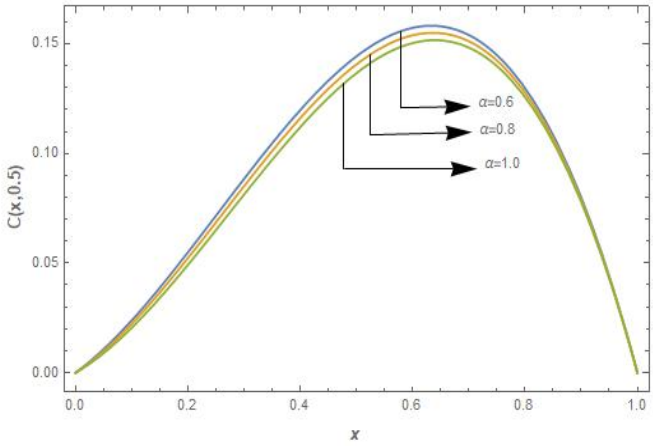


Figure 7: Variations of $C(x, 0.5)$ vs. x for various values of α at v=1.0, $\beta = 1$, and t=0.5 for conservative system with $f(x, t) = xt$.

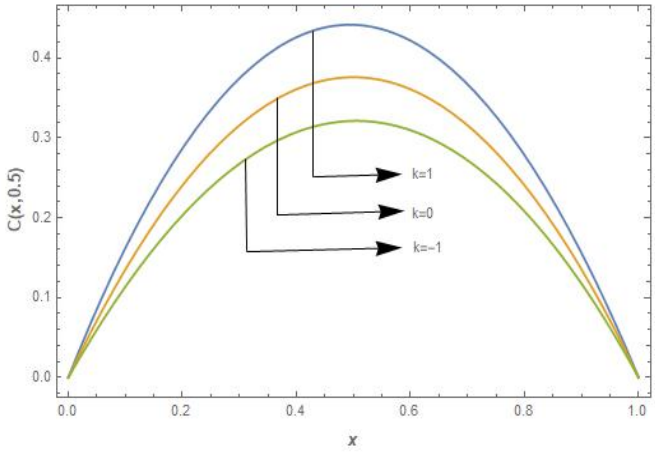


Figure 8: Variations of $C(x, 0.5)$ vs x at $\alpha = 1, \beta = 1$, $v = 1.0$, and $t = 0.5$ for conservative and non-conservative systems with $f(x, t) = xt + 1$.

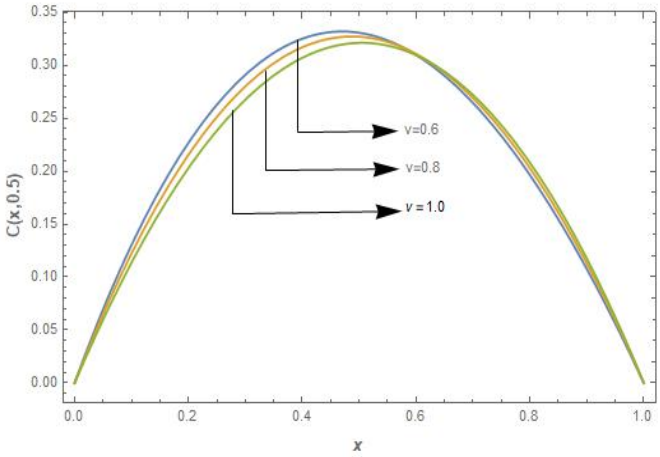


Figure 9: Variations of $C(x, 0.5)$ vs. x for various v at $\alpha = 1$, $\beta = 1$, and $t = 0.5$ for non-conservative system with $f(x, t) = xt + 1$.

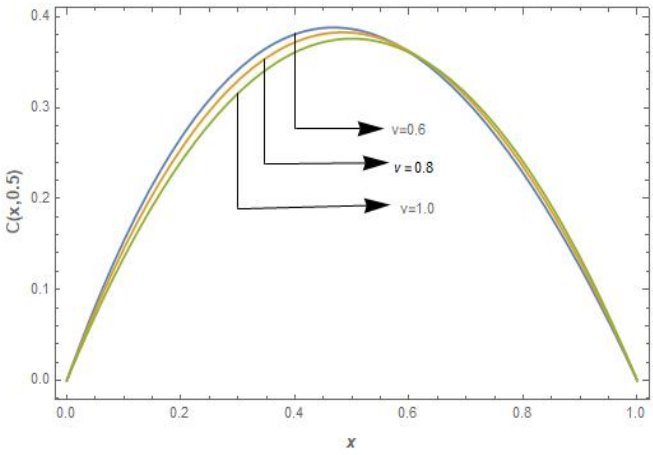


Figure 10: Variations of $C(x, 0.5)$ vs. x for various v at $\alpha = 1$, $\beta = 1$, and $t = 0.5$ for conservative system with $f(x, t) = xt + 1$.

The variations of the solute concentration due to the change in spatial order derivative β for temporal derivative $\alpha = 1$ and advection coefficient $v = 1.0$ for non-conservative system and conservative system are depicted through Figures 4-5. It is seen from the figures that due to increase in the order of the derivative β, the concentration is decreasing. Similarly, the variations in concentration due to increase in the order of the temporal derivative α for $\beta = 1$, $v = 1.0$ for non-conservative and conservative systems, respectively are shown in Figures 6-7. Here also the solute concentration profile decreases with the increase in α. The effects on the solute concentration due to the presence / absence of the reaction term are depicted through Figure 8. In contrast to the conservative system ($k = 0$), the solute profile in presence of the source ($k = 1$) and the sink ($k = -1$) terms is higher and lower, respectively. Figures 9-10 are drawn to observe the effects on solute profile due to the change in the advection coefficient for non-conservative and conservative systems, respectively. It is found that the concentration of the solute decreases rapidly with the increase in the velocity of the concentration.

8 Conclusion

In this article, the spectral collocation method with the aid of the Lucas OM is applied to solve the nonlinear time-space FRADE. The beauty of the method is that it reduces the problem to simultaneous algebraic equations which are solved by using Newton's method. The error analysis has been done to show the effectiveness and efficiency of the method while applying it to existing problems having analytical solutions. The important feature of the study is the error estimation of the numerical scheme which is applied to the proposed model. The most important point of the research is the graphical presentations of variations of the concentration due to the effect of temporal and spatial fractional order parameters and also due to the presence / absence of advection and reaction terms.

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