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Finite-dimensional Zinbiel algebras and combinatorial structures

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Abstract

In this paper, we study the link between finite-dimensional Zinbiel algebras and combinatorial structures or (pseudo)digraphs determining which configurations are associated with those algebras. Some properties of Zinbiel algebras that can be read from their associated combinatorial structures are studied. We also analyze the isomorphism classes for each configuration associated with these algebras providing a new method to classify them and we compare our results with the current classifications of 2- and 3-dimensional Zinbiel algebras. We also obtain the 3-vertices combinatorial structures associated with such algebras. In order to complement the theoretical study, we have designed and performed the implementation of an algorithm which constructs and draws the (pseudo)digraph associated with a given Zinbiel algebra and, conversely, another procedure to test if a given combinatorial structure is associated with some Zinbiel algebra.

1 Introduction

In these days, one of the most important and stimulating research in Science and, particularly, in Mathematics is finding and studying new links between different fields. Alternative techniques and procedures allow researchers to solve many open problems, improve known theories and achieve new results.

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This paper deals with the relation between Zinbiel algebras and Graph Theory. More concretely, our main goal is to continue with the research line started in [1, 3, 4], where a link between combinatorial structures and Lie or Leibniz algebras was established. Due to this link, several properties on those non-associative algebras can be translated into the field of Graph Theory and vice versa. Now, we want to extend these studies to the case of Zinbiel algebras.

Non-associative algebras have been deeply studied due to its own theoretical importance and its many applications to different fields like Physics, Engineering or Applied Mathematics. A particular type of these algebras is formed by Zinbiel algebras. They were introduced by J.-L. Loday [7] in 1995. They are the Koszul dual of Leibniz algebras and, in fact, J.M. Lemaire (see [8]) proposed the name of Zinbiel as a mirror of Leibniz algebras. As happens with any class of non-associative algebras, there exist many general questions to be solved and these questions (as, for example, the classification of Zinbiel algebras) require alternative techniques since the traditional ones are not sufficient.

Analogously, Graph Theory is also running at a very high level nowadays. Graphs have been used to deal with a wide range of problems in many different fields including non-associative algebras. For example, they have been really helpful in order to compute degenerations of Zinbiel algebras in [6]. In this way, we believe that graphs and simplicial complexes (its generalization to higher dimensions) might be an useful tool in the study of non-associative algebras, providing new ways to solve many open problems, like the above-mentioned classification problem of Zinbiel algebras.

Hence, our main goal is to study the link between combinatoral structures and Zinbiel algebras, giving a generalization for the techniques introduced in [1] and then developed in [3, 4] to the case of Zinbiel algebras. We will also provide a new method to classify this type of algebras.

The structure of this paper is the following: Section 2 focuses on reviewing some well-known results on Zinbiel algebras and Graph Theory. Then, Section 3 is devoted to the association between combinatorial structures and Zinbiel algebras. In Section 4, we study some properties of Zinbiel algebras that can be read from their associated combinatorial structure. Next, Section 5 analyzes the structure of (pseudo)digraphs associated with Zinbiel algebras and some of their properties. For each configuration, we study both the solvability and the isomorphism classes for its associated Zinbiel algebra. In Section 6 we study the 3-vertex combinatorial structures including full triangles that are associated with Zinbiel algebras. In addition, we also show how our algorithm may be useful for the classification of Zinbiel algebras. Finally, Section 7 shows the implementation of the two algorithmic procedures used in the previous sections. The first one is devoted to check if a given combinatorial structure (not necessarily a (pseudo)digraph) is associated or not with a Zinbiel algebra and, conversely, the second one computes the (pseudo)digraph, if possible, associated with a given finite-dimensional Zinbiel algebra starting from its law. In addition, we give a brief computational study, showing the complexity order and computing time of the procedures here presented.

We believe that the tools and results achieved in this paper might be useful to advance in the research line connecting Zinbiel algebras and Graph Theory. In addition, combinatorial structures may provide us with a new method to classify Zinbiel algebras.

2 Preliminaries

For a general overview on Zinbiel algebras and Graph Theory, the reader can consult [7]. We only consider finite-dimensional Zinbiel algebras over the complex number field $\mathbb{C}$.

Definition 1. *A* Zinbiel algebra $\mathcal{Z}$ *is a vector space with a second bilinear inner composition law* $([\cdot,\cdot])$*, called the* bracket product *or* commutator*, which satisfies*

$$[[A,B],C] = [A,[B,C]] + [A,[C,B]], \forall A,B,C \in \mathcal{Z}.$$

The latter is called the Zinbiel identity*. From here on, we use the notation* $Z(A,B,C) = [[A,B],C] - [A,[B,C]] - [A,[C,B]]$.

Given a basis $\{e_i\}_{i=1}^n$ *of* $\mathcal{Z}$*, its* structure (or Maurer-Cartan) constants *are defined by* $[e_i,e_j] = \sum_{h=1}^n c_{i,j}^h e_h$*, for* $1 \le i < j \le n$.

Definition 2. *Given a Zinbiel algebra* $\mathcal{Z}$*, its* center *is defined as* $Z(\mathcal{Z}) = \{X \in \mathcal{Z} \mid [X,Y] = 0, \forall Y \in \mathcal{Z}\}$.

Definition 3. *Given a finite-dimensional Zinbiel algebra* $\mathcal{Z}$*, its* derived series *is*

$$\mathcal{Z}_1 = \mathcal{Z},\ \mathcal{Z}_2 = [\mathcal{Z},\mathcal{Z}],\ \ldots,\ \mathcal{Z}_k = [\mathcal{Z}_{k-1},\mathcal{Z}_{k-1}],\ \ldots$$

Thus, $\mathcal{Z}$ *is called* solvable *if there exists* $m \in \mathbb{N}$ *such that* $\mathcal{Z}_m = \{0\}$*. In addition, if* $\mathcal{Z}_{m-1} \neq \{0\}$ *also holds, then* $\mathcal{Z}$ *is* $(m-1)$*-step solvable.*

Definition 4. *Given a finite-dimensional Zinbiel algebra* $\mathcal{Z}$*, its* central series *is*

$$\mathcal{Z}^1 = \mathcal{Z},\ \mathcal{Z}^2 = [\mathcal{Z},\mathcal{Z}],\ \ldots,\ \mathcal{Z}^k = [\mathcal{Z}^{k-1},\mathcal{Z}],\ \ldots$$

Thus, $\mathcal{Z}$ *is called* nilpotent *if there exists* $m \in \mathbb{N}$ *such that* $\mathcal{Z}^m = \{0\}$*. In addition, if* $\mathcal{Z}^{m-1} \neq \{0\}$ *also holds, then* $\mathcal{Z}$ *is* $(m-1)$*-step nilpotent.*

Remark 1. *Every nilpotent algebra is trivially solvable because* $\mathcal{Z}_i \subseteq \mathcal{Z}^i$*, for all* $i \in \mathbb{N}$.

Remark 2. *The derived algebra of a Zinbiel algebra $\mathcal{Z}$ will be denoted by $D\mathcal{Z} = \mathcal{Z}_2 = \mathcal{Z}^2$.*

Although the reader can consult [5] as an introductory reference to Graph Theory, some notions are recalled next.

Definition 5. *A* digraph *consists in an ordered pair $G = (V, E)$, where V is a non-empty set called vertex-set and E is a set of ordered pairs (edges) of two vertices, called edge-set.*

Definition 6. *A* loop *in the digraph G is an edge that connects a vertex with itself. If the digraph G contains loops, then G is called a* pseudodigraph.

Throughout the paper, we consider (pseudo)digraphs admitting double edges.

Definition 7. *Given a digraph $G = (V, E)$, a vertex $v \in V$ is a* sink *(resp. a* source*) if each edge incident with v is oriented towards v (resp. from v). See Figure 1.*

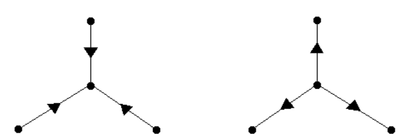

Figure 1: Example of sink and source, respectively.

Definition 8. *A vertex v in G is said to be* simple *if v has no loop. Otherwise, it will be* non-simple.

3 Associating combinatorial structures with Zinbiel algebras

Let $\mathcal{Z}$ be a n-dimensional Zinbiel algebra with basis $\mathcal{B} = \{e_i\}_{i=1}^n$ and law $[e_i, e_j] = \sum_{h=1}^n c_{i,j}^h e_h$. The pair $(\mathcal{Z}, \mathcal{B})$ can be associated with a combinatorial structure by following the procedure introduced in [4, Section 3] for Leibniz algebras.

Therefore, every Zinbiel algebra with a fixed basis can be associated with a combinatorial structure. This method for Zinbiel algebras provides a generalization of the one described in [1] for Lie algebras, as we prove in the following.

Example 1. *The 3-dimensional Zinbiel algebra with law* $[e_1, e_1] = e_2 - e_3$, $[e_1, e_2] = [e_1, e_3] = -e_2 + e_3$ *is associated with the combinatorial structure shown in Figure 2.*

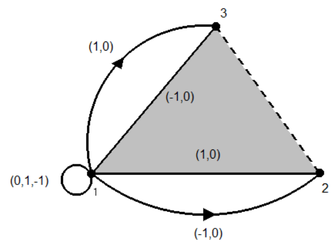

Figure 2: Combinatorial structure associated with a 3-dimensional Zinbiel algebra.

4 Reading properties from the combinatorial structure

In this section, given a Zinbiel algebra $\mathcal{Z}$, we analyze the properties of $\mathcal{Z}$ algebras that can be read from its associated combinatorial structure, G. More concretely, we characterize the case when $\mathcal{Z}$ is a Lie algebra and then we study the center, $Z(\mathcal{Z})$, and the derived algebra $D\mathcal{Z}$.

Proposition 1. *Let G be the combinatorial structure associated with a Zinbiel algebra $\mathcal{Z}$. Then $\mathcal{Z}$ is a Lie algebra, if G satisfies the following conditions*

1. *There are no loops.*
2. *The weight for the edge from vertex i to vertex j is given by $(c_{i,j}^j, -c_{i,j}^j)$.*
3. *The weight for the edges in a full triangle ijk is given by $(c_{i,j}^k, -c_{i,j}^k)$, $(c_{j,k}^i, -c_{j,k}^i)$ and $(c_{i,k}^j, -c_{i,k}^j)$.*

Proof. Trivial from the self-annihilation and the skew-symmetry of the commutator. □

Remark 3. *Given an edge in a combinatorial structure associated with a Zinbiel algebra, both coordinates in Conditons 2 and 3 from Proposition 1 are opposite each other and, then, only one coordinate is required for saving the information of the structure constants as happened in [1].*

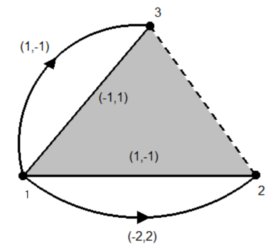

Figure 3: Combinatorial structure associated with a 3-dimensional Lie algebra.

Example 2. *The combinatorial structure of Figure 3 is associated with the 3-dimensional Lie algebra with law* $[e_1, e_2] = -[e_2, e_1] = -2e_2 + e_3, [e_1, e_3] = -[e_3, e_1] = -e_2 + e_3$.

Proposition 2. *Let G be the combinatorial structure associated with a Zinbiel algebra $\mathcal{Z}$. Then,*

$$Z(\mathcal{Z}) \supset \operatorname{span}\{e_i \mid i \in \Gamma\} \cup \operatorname{span}\{e_j \mid j \in \Lambda\}, \quad \text{where}$$

Γ is the set of simple and isolated vertices of G and Λ is the set of simple vertices in full triangles that are adjacent only to ghost edges.

Proof. Let us assume that $G = (V, E)$ with $V = \{1, \ldots, n\}$, therefore $B = \{e_k\}_{k=1}^n$ is a basis of $\mathcal{Z}$. First, if the vertex $i \in V$ is a source then $\exists j \in V$ such that $(c_{i,j}^j, c_{j,i}^j) \neq (0, 0)$. Consequently, $e_i \notin Z(\mathcal{Z})$. In case that $i \in V$ is a simple and isolated vertex then $e_i \in Z(\mathcal{Z})$ since $[e_i, e_j] = [e_j, e_i] = 0$, $\forall 1 \leq j \leq n$. Now, we suppose that $i \in V$ is a simple vertex in a full triangle which is adjacent only to ghost edges. In that case, $[e_i, e_j] = [e_j, e_i] = 0$, $\forall 1 \leq j \leq n$. Therefore, $e_i \in Z(\mathcal{Z})$.

□

Remark 4. *Notice that the center of a Zinbiel algebra can be given by a linear combination of the basis vectors associated to the type of vertices indicated in Proposition 2. For example, the 3-dimensional Zinbiel algebra $\mathcal{Z}$ of Example 1 verifies $Z(\mathcal{Z}) = \operatorname{span}\{e_2 - e_3\}$*

Proposition 3. *Let G be the combinatorial structure associated with a Zinbiel algebra $\mathcal{Z}$. Then, the derived algebra of $\mathcal{Z}$ is given by*

$$D\mathcal{Z} = \operatorname{span}\left\{\sum_{i \in \Upsilon} c_{i,i}^h e_h\right\} \cup \operatorname{span}\left\{c_{i,j}^j e_j + c_{i,j}^k e_k, c_{j,i}^j e_j + c_{j,i}^k e_k \mid j \in \Pi\right\}, \quad \text{where}$$

Υ is the set of non-simple vertices (vertices with a loop) and Π is the set of vertices that are not sources.

Proof. Let us assume that $G = (V, E)$ with $V = \{1, \ldots, n\}$, therefore $B = \{e_k\}_{k=1}^n$ is a basis of $\mathcal{Z}$. First, if i is a non-simple vertex, then $[e_i, e_i] = \sum_{h=1}^{n} c_{i,i}^h e_h$. Consequently, $\sum_{h=1}^{n} c_{i,i}^h e_h \in D\mathcal{Z}$. From now on, we will only consider simple vertices. In case that i is a source, then $c_{j,i}^i = c_{i,j}^i = 0$, $\forall 1 \leq j \leq n$. Therefore, $e_i \notin D\mathcal{Z}$. If i is not a source, then $\exists j$ with $1 \leq j \leq n$ such that $(c_{i,j}^i, c_{j,i}^i) \neq (0,0)$. Consequently, $c_{i,j}^i\, e_i$, $c_{j,i}^i\, e_i \in D\mathcal{Z}$. Moreover, if i is a vertex in a full triangle with vertices $\{i, j, k\}$, then it may happen that $(c_{i,j}^k, c_{j,i}^k) \neq (0,0)$, so we would have to consider the terms $c_{i,j}^i\, e_i + c_{i,j}^k\, e_k$ and $c_{j,i}^i\, e_i + c_{j,i}^k\, e_k$.

□

5 Zinbiel algebras and (pseudo)digraphs

In this section, we study the structure of (pseudo)digraphs associated with Zinbiel algebras. For each case, we analyze the type of Zinbiel algebra according to its solvability and isomorphism class. Let $\mathcal{Z}$ be a Zinbiel algebra with basis $\mathcal{B}$ such that the combinatorial structure G associated with $(\mathcal{Z}, \mathcal{B})$ consists of a (pseudo)digraph; that is, there are no triangles in G. This assertion is equivalent to affirm that the law of $\mathcal{Z}$ with respect to the basis $\mathcal{B} = \{e_i\}_{i=1}^n$ is given by

$$[e_i, e_j] = c_{i,j}^i e_i + c_{i,j}^j e_j,\ 1 \leq i \neq j \leq n; \quad [e_k, e_k] = \sum_{h=1}^{n} c_{k,k}^h e_h \tag{1}$$

and the rest of products are null.

Proposition 4. *The unique (pseudo)digraph associated with a* 1*-dimensional Zinbiel algebra is that formed by an isolated vertex.*

Remark 5. *Clearly, there is only one isomorphism class,* $\mathcal{Z}^1$*, which corresponds to the* 1*-dimensional abelian Zinbiel algebra.*

Proposition 5. *If G is a (pseudo)digraph of 2 vertices, then G is associated with a* 2*-dimensional Zinbiel algebra $\mathcal{Z}$ if and only if G is isomorphic to configurations a), b) or j) in Figure 4. The rest of configurations cannot be associated with Zinbiel algebras.*

Proof. Figure 4 includes all the possible (pseudo)digraphs of two vertices. Configuration $c)$ cannot be associated with a Zinbiel algebra since $Z(e_1, e_1, e_1) = Z(e_1, e_1, e_2) = Z(e_2, e_2, e_2) = 0$ implies that there is no loop on vertex 1. Configurations $d), e)$ and $f)$ cannot be associated with a Zinbiel algebra since

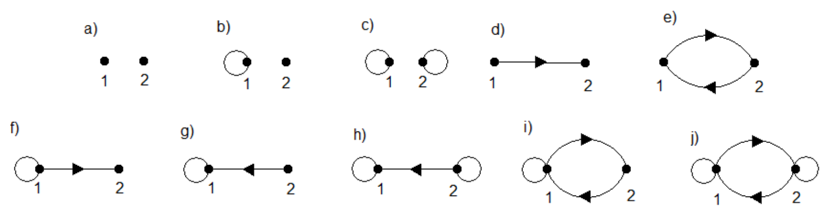

Figure 4: Pseudodigraphs with two vertices.

$Z(e_1, e_2, e_1)= Z(e_2, e_1, e_1) = 0$ implies the non-existence of edges from vertex 1 into vertex 2. Finally, configurations $g), h)$ and $i)$ cannot be associated with a Zinbiel algebra since $Z(e_2, e_1, e_2) = Z(e_1, e_2, e_2) = 0$ implies the non-existence of edges from vertex 2 into vertex 1.

Any other configuration in Figure 4 is associated with a Zinbiel algebra if and only if the following restrictions hold for each of them

i) Configuration a): No constraints.

ii) Configuration $b)$: $c^1_{1,1} = 0 \wedge c^2_{1,1} \neq 0$.

iii) Configuration $j)$: $c^1_{1,1} = \frac{(c^1_{2,1})^2}{c^1_{2,2}} \wedge c^2_{1,1} = -\frac{(c^1_{2,1})^3}{(c^1_{2,2})^2} \wedge c^1_{1,2} = c^1_{2,1} \wedge c^2_{1,2} = -\frac{(c^1_{2,1})^2}{c^1_{2,2}} \wedge c^2_{2,1} = -\frac{(c^1_{2,1})^2}{c^1_{2,2}} \wedge c^2_{2,2} = -c^1_{2,1}$. □

Proposition 6. *Under the assumptions in Proposition 5,*

- *Configuration $a)$ is associated with the abelian 2-dimensional Zinbiel algebra.*
- *Configurations $b)$ and $j)$ are associated with 2-step nilpotent Zinbiel algebras.*

Proof. Let $\mathcal{Z}$ be the Zinbiel algebra associated with Configuration $b)$. From Proposition 5, $\mathcal{Z}_2 = \mathcal{Z}^2 = \text{span}(e_2)$ is an abelian ideal and, hence, $\mathcal{Z}_3 = \mathcal{Z}^3 = \{0\}$. Consequently, $\mathcal{Z}$ is 2-step nilpotent. On the other hand, if $\mathcal{Z}$ is the Zinbiel algebra associated with Configuration $j)$, Proposition 5 implies that

$$\mathcal{Z}_2 = \mathcal{Z}^2 = \text{span}\left(\frac{(c^1_{2,1})^2}{c^1_{2,2}}e_1 - \frac{(c^1_{2,1})^3}{(c^1_{2,2})^2}e_2, c^1_{2,1}e_1 - \frac{(c^1_{2,1})^2}{c^1_{2,2}}e_2, c^1_{2,2}e_1 - c^1_{2,1}e_2\right) =$$

$$span\left(e_1 - \frac{c^1_{2,1}}{c^1_{2,2}}e_2\right)$$

Moreover,

$$\left[e_1, e_1 - \frac{c_{2,1}^1}{c_{2,2}^1}e_2\right] = \frac{(c_{2,1}^1)^2}{c_{2,2}^1}e_1 - \frac{(c_{2,1}^1)^3}{(c_{2,2}^1)^2}e_2 - \frac{c_{2,1}^1}{c_{2,2}^1}\left(c_{2,1}^1 e_1 - \frac{(c_{2,1}^1)^2}{c_{2,2}^1}e_2\right) = 0 \quad \text{and}$$

$$\left[e_2, e_1 - \frac{c_{2,1}^1}{c_{2,2}^1}e_2\right] = [e_2, e_1] - \frac{c_{2,1}^1}{c_{2,2}^1}[e_2, e_2] = 0$$

Therefore, $\mathcal{Z}_3 = \mathcal{Z}^3 = \{0\}$ and $\mathcal{Z}$ is a 2-step nilpotent Zinbiel algebra. □

Now, we study the isomorphism class for Zinbiel algebras associated with Configurations b) and j) from Figure 4. Note that Zinbiel algebras associated with Configuration a) correspond to the 2-dimensional abelian Zinbiel algebra and its isomorphism class is denoted by $\mathcal{Z}_a^2$.

Proposition 7. *Zinbiel algebras associated with Configurations b) and j) from Figure 4 belong to the isomorphism class $\mathcal{Z}_b^2 = \operatorname{span}(e_1, e_2)$ with law $[e_1, e_1] = e_2$.*

Proof. First, we prove that Zinbiel algebras associated with Configurations b) and j) are isomorphic each other. Let $\mathcal{Z}_j^2$ be a Zinbiel algebra associated with Configuration j). According to Proposition 5, its law can be expressed as

$$[w_1, w_1] = \frac{(c_{2,1}^1)^2}{c_{2,2}^1}w_1 - \frac{(c_{2,1}^1)^3}{(c_{2,2}^1)^2}w_2, \quad [w_1, w_2] = [w_2, w_1] = c_{2,1}^1 w_1 - \frac{(c_{2,1}^1)^2}{c_{2,2}^1}w_2$$

$$[w_2, w_2] = c_{2,2}^1 w_1 - c_{2,1}^1 w_2$$

By considering the basis change $\phi : \mathcal{Z}_j^2 \to \mathcal{Z}_j^2$ given by $v_1 = \phi(w_1) = w_2$; $v_2 = \phi(w_2) = w_1 - \frac{c_{2,1}^1}{c_{2,2}^1}w_2$, we obtain the following expression for the law

$$[v_1, v_1] = c_{2,2}^1 v_2, \quad [v_1, v_2] = [v_2, v_1] = [v_2, v_2] = 0.$$

Finally, to obtain the law of $\mathcal{Z}_b^2$, we only have to apply the basis change $\phi' : \mathcal{Z}_j^2 \to \mathcal{Z}_j^2$ given by $e_1 = \phi'(v_1) = \frac{1}{c_{2,2}^1}v_1$; $e_2 = \phi'(v_2) = \frac{1}{c_{2,2}^1}v_2$. □

Proposition 8. *If G is a non-connected (pseudo)digraph of 3 vertices, then G is associated with a 3-dimensional Zinbiel algebra $\mathcal{Z}$ if and only if G is isomorphic to Configurations i), ii), iii), $xvii$) or $xviii$) from Figure 5. Any other configuration in Figure 5 cannot be associated with Zinbiel algebras.*

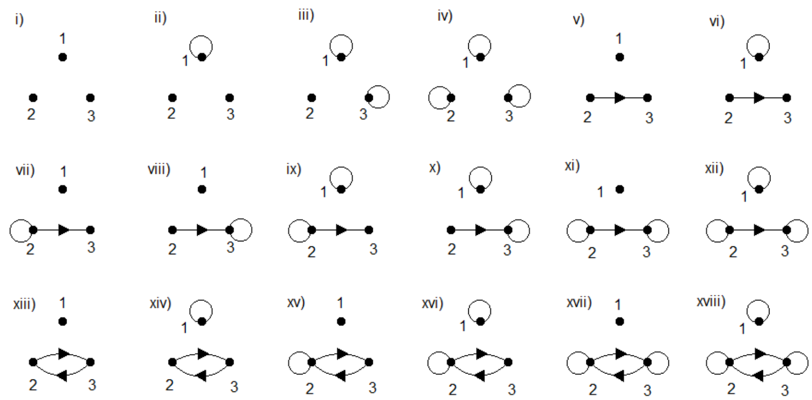

Figure 5: Disconnected (pseudo)digraphs with 3 vertices.

Proof. Figure 5 includes all the possible disconnected (pseudo)digraphs of three vertices. Using an analogous reasoning to that given for Configuration c) in the proof of Proposition 5, it is easy to prove that Configuration iv) cannot be associated with a Zinbiel algebra. Moreover, Configurations v) to xii) are not associated with Zinbiel algebras since $Z(e_2,e_2,e_3)=Z(e_2,e_3,e_2)=Z(e_3,e_2,e_2)=0$ implies $c^3_{2,3}=c^3_{3,2}=0$ and, hence, there should be no edge from vertex 2 to vertex 3. Finally, Configurations $xiii$) to xvi) cannot be associated with Zinbiel algebras since $Z(e_2,e_3,e_3)=Z(e_3,e_2,e_3)=Z(e_3,e_3,e_2)=Z(e_2,e_2,e_3)=0$ implies that there is no double edge between vertices 2 and 3.

Any other configuration from Figure 5 is associated with a Zinbiel algebra if and only if the following restrictions hold for each of them

- Configuration i): No constraints.
- Configuration ii): $c^1_{1,1}=0\wedge\ (c^2_{1,1}\neq 0 \vee c^3_{1,1}\neq 0)$.
- Configuration iii): $c^1_{1,1}=c^3_{1,1}=c^1_{3,3}=c^3_{3,3}=0\wedge c^2_{1,1}\neq 0\wedge c^2_{3,3}\neq 0$.
- Configuration $xvii$): $c^1_{2,2}=0\wedge c^3_{3,2}=-c^2_{2,2}=c^3_{2,3}\wedge c^3_{2,2}=-\frac{(c^3_{2,3})^2}{c^2_{3,2}}\wedge c^2_{2,3}=-c^3_{3,3}=c^2_{3,2}\wedge c^1_{3,3}=0\wedge c^2_{3,3}=-\frac{(c^2_{3,2})^2}{c^3_{2,3}}\wedge c^2_{3,2}\neq 0\neq c^3_{2,3}$.
- Configuration $xviii$): $c^1_{1,1}=c^1_{2,2}=c^1_{3,3}=0\wedge c^3_{3,2}=-c^2_{2,2}=c^3_{2,3}\wedge c^2_{2,3}=-c^3_{3,3}=c^2_{3,2}\wedge c^2_{1,1}=\frac{c^3_{1,1}c^2_{3,2}}{c^3_{2,3}}\wedge c^3_{2,2}=-\frac{(c^3_{2,3})^2}{c^2_{3,2}}\wedge c^2_{3,3}=-\frac{(c^2_{3,2})^2}{c^3_{2,3}}\wedge c^3_{1,1}\neq 0\neq c^2_{3,2}\neq 0\neq c^3_{2,3}$.

□

Proposition 9. *Under the assumptions in Proposition 8,*

- *Configuration* $i)$ *is associated with the abelian* 3*-dimensional Zinbiel algebra.*
- *Configurations* $ii), iii), xvii)$ *and* $xviii)$ *are associated with* 2*-step nilpotent Zinbiel algebras.*

Proof. Let $\mathcal{Z}$ be a Zinbiel algebra associated with Configuration $ii)$. According to Proposition 8, $\mathcal{Z}_2 = \mathcal{Z}^2 = \mathrm{span}(c_{1,1}^2 e_2 + c_{1,1}^3 e_3)$ is an abelian ideal and, hence, $\mathcal{Z}_3 = \mathcal{Z}^3 = \{0\}$. Consequently, $\mathcal{Z}$ is 2-step nilpotent.

The reasoning for Configuration $iii)$ is analogous to that for Configuration $b)$ in Proposition 6.

Next, if $\mathcal{Z}$ denotes the Zinbiel algebra associated with Configuration $xviii)$, then, according to Proposition 8, we obtain

$$[e_1, e_1] = -\frac{c_{1,1}^3}{c_{3,2}^2}[e_3, e_3],\ [e_2, e_2] = \frac{(c_{2,3}^3)^2}{(c_{3,2}^2)^2}[e_3, e_3],\ [e_2, e_3] = [e_3, e_2] = -\frac{c_{2,3}^3}{c_{3,2}^2}[e_3, e_3]$$

Therefore, $\mathcal{Z}_2 = \mathcal{Z}^2 = \mathrm{span}([e_3, e_3])$. In addition,

$$[e_1, [e_3, e_3]] = 0;$$

$$[e_2, [e_3, e_3]] = -\frac{(c_{3,2}^2)^2}{c_{2,3}^3}[e_2, e_2] + c_{3,2}^2[e_2, e_3] = -c_{2,3}^3[e_3, e_3] + c_{2,3}^3[e_3, e_3] = 0;$$

$$[e_3, [e_3, e_3]] = -\frac{(c_{3,2}^2)^2}{c_{2,3}^3}[e_3, e_2] - c_{3,2}^2[e_3, e_3] = +c_{3,2}^2[e_3, e_3] + c_{3,2}^2[e_3, e_3] = 0.$$

Therefore, $\mathcal{Z}_3 = \mathcal{Z}^3 = \{0\}$ and $\mathcal{Z}$ is 2-step nilpotent. An analogous reasoning can be used for the proof of Configuration $xvii)$. □

Proposition 10. *Let G be a connected (pseudo)digraph of* 3 *vertices. Then, G cannot be associated with any Zinbiel algebra.*

Proof. First, we prove the non-existence of Zinbiel algebras associated with connected digraphs of three vertices (see Figure 6 for all the possible digraphs). It suffices to indicate the Zinbiel identities which involve that the weight of some edge in the digraph is zero and, hence, there does not exist that edge. Table 1 includes the list of those Zinbiel identities for each configuration and the non-existing edge (i.e. its weight is zero).

Next, we prove that no pseudodigraph of 3 vertices is associated with a Zinbiel algebra. To do so, we should consider all the possibilities by taking

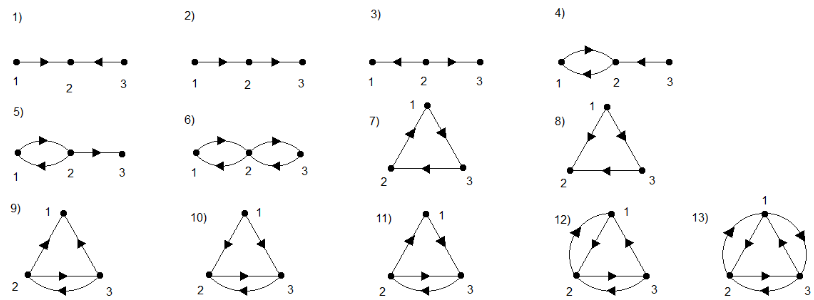

Figure 6: Connected digraphs with 3 vertices.

Conf.	Zinbiel identities	Non-existing Edge
1	$Z(e_1, e_2, e_1) = Z(e_2, e_1, e_3) = Z(e_3, e_2, e_3) = 0$	$1 \to 2$ or $3 \to 2$
2	$Z(e_1, e_2, e_1) = Z(e_2, e_3, e_2) = Z(e_3, e_1, e_2) = 0$	$1 \to 2$ or $2 \to 3$
3	$Z(e_1, e_2, e_2) = Z(e_2, e_1, e_2) = 0$	$2 \to 1$
4	$Z(e_3, e_2, e_3) = Z(e_2, e_1, e_3) = Z(e_1, e_3, e_2) = 0$	$3 \to 2$
5	$Z(e_3, e_2, e_2) = Z(e_2, e_3, e_2) = 0$	$2 \to 3$
6	$Z(e_1, e_2, e_3) = Z(e_2, e_1, e_3) = Z(e_2, e_3, e_1) = Z(e_3, e_2, e_1) = 0$	$3 \to 2$
7	$Z(e_2, e_1, e_2) = Z(e_1, e_3, e_1) = Z(e_3, e_2, e_1) = Z(e_3, e_2, e_3) = 0$	$2 \to 1$ or $1 \to 3$
8	$Z(e_1, e_2, e_1) = Z(e_2, e_1, e_1) = 0$	$2 \to 1$
9	$Z(e_1, e_2, e_2) = Z(e_2, e_1, e_2) = 0$	$1 \to 2$
10	$Z(e_1, e_2, e_1) = Z(e_2, e_1, e_1) = 0$	$1 \to 2$
11	$Z(e_2, e_1, e_2) = Z(e_1, e_3, e_1) = Z(e_1, e_2, e_3) = Z(e_2, e_3, e_1) = 0$	$2 \to 1$ or $3 \to 2$ or $1 \to 3$
12	$Z(e_3, e_1, e_3) = Z(e_1, e_2, e_3) = Z(e_2, e_1, e_3) = Z(e_3, e_2, e_1) = 0$	$2 \to 3$ or $3 \to 1$
13	$Z(e_1, e_2, e_1) = Z(e_2, e_1, e_1) = 0$	$1 \to 2$

Table 1: Zinbiel identities involving the non-existence of some edge in the configuration.

a configuration from Figure 6 and adding loops. Due to reasons of length, we only give an explicit proof for the case of Configuration 3); an analogous reasoning can be given for the rest of cases. Consider Configuration 3) but inserting a loop on each vertex. From $Z(e_2, e_1, e_2) = Z(e_2, e_3, e_2) = 0$, we can conclude that $c_{2,1}^1 = c_{2,3}^3 = 0$. By using $Z(e_2, e_2, e_2) = Z(e_1, e_1, e_2) = 0$, we obtain $c_{2,2}^2 = c_{1,1}^2 = 0$. Finally, from $Z(e_1, e_1, e_1) = 0$, $c_{1,1}^1 = 0$ holds and therefore $c_{1,2}^1 = 0$ according to $Z(e_1, e_2, e_2) = 0$; this comes into contradiction with the fact that there exists an edge from vertex 2 into vertex 1. □

Now, we study the isomorphism classes of Zinbiel algebras associated with Configurations *ii*), *iii*), *xvii*) and *xviii*) from Figure 5. Notice that Zinbiel algebras associated with Configuration *i*) from Figure 5 correspond to the 3-dimensional abelian Zinbiel algebra, and its isomorphism class is denoted by $\mathfrak{Z}^3_{i)}$.

Proposition 11. *Zinbiel algebras associated with Configurations ii) and xvii) from Figure 5 belong to the isomorphism class* $\mathfrak{Z}^3_{ii)} = \operatorname{span}(e_1, e_2, e_3)$ *defined by the law* $[e_1, e_1] = e_3$.

Proof. First, we have to prove that Zinbiel algebras associated with Configurations *ii*) and *xvii*) are isomorphic each other. Let $\mathfrak{Z}^3_{ii)}$ and $\mathfrak{Z}^3_{xvii)}$ be the Zinbiel algebras associated with Configurations *ii*) and *xvii*) , respectively. According to Proposition 8, the law of $\mathfrak{Z}^3_{xvii)}$ can be expressed as

$$[w_2, w_2] = -\frac{c_{2,3}^3}{c_{3,2}^2}(c_{3,2}^2 w_2 + c_{2,3}^3 w_3), \quad [w_2, w_3] = [w_3, w_2] = c_{3,2}^2 w_2 + c_{2,3}^3 w_3$$

$$[w_3, w_3] = -\frac{c_{3,2}^2}{c_{2,3}^3}(c_{3,2}^2 w_2 + c_{2,3}^3 w_3).$$

If we consider the basis change $\phi : \mathfrak{Z}^3_{xvii)} \to \mathfrak{Z}^3_{xvii)}$ given by $v_1 = \phi(w_1) = w_3$; $v_2 = \phi(w_2) = w_1$; $v_3 = \phi(w_3) = -\frac{c_{3,2}^2}{c_{2,3}^3}(c_{3,2}^2 w_2 + c_{2,3}^3 w_3)$, we obtain the unique non-zero bracket $[v_1, v_1] = v_3$. Analogously, if we consider the law of $\mathfrak{Z}^3_{ii)}$ given by $[e_1, e_1] = c_{1,1}^2 e_2 + c_{1,1}^3 e_3$ with $(c_{1,1}^2, c_{1,1}^3) \neq (0, 0)$, we only need to apply the basis change $\phi' : \mathfrak{Z}^3_{ii)} \to \mathfrak{Z}^3_{ii)}$ given by $v_1 = \phi'(e_1) = e_1$; $v_2 = \phi'(e_2) = v_2$; $v_3 = \phi'(e_3) = c_{1,1}^2 v_2 + c_{1,1}^3 v_3$. □

Proposition 12. *Zinbiel algebras associated with Configuration iii) from Figure 5 correspond to the isomorphism class* $\mathfrak{Z}^3_{iii)} = \operatorname{span}(e_1, e_2, e_3)$ *defined by the law* $[e_1, e_1] = [e_3, e_3] = e_2$.

Proof. For Configuration iii), it suffices to consider the basis change $\phi : \mathcal{Z} \to \mathcal{Z}$ given by $e_1 = \phi(v_1) = \sqrt{c_{3,3}^2} v_1$; $e_2 = \phi(v_2) = c_{3,3}^2 c_{1,1}^2 v_2$; $e_3 = \phi(v_3) = \sqrt{c_{1,1}^2} v_3$, where $\mathcal{Z}$ is the Zinbiel algebra associated with Configuration *iii*), whose law is $[v_1, v_1] = c_{1,1}^2 v_2$, $[v_3, v_3] = c_{3,3}^2 v_2$.

□

Theorem 1. *Let G be a (pseudo)digraph with n vertices ($n > 3$). Then, G is associated with a Zinbiel algebra if and only if G satisfies one of the following conditions*

(i) G is given by n isolated vertices with, at most, $(n-1)$ loops.

(ii) G contains a unique connected component corresponding to Configuration j) from Figure 4 and the rest of connected components are isolated vertices (with or without loops).

Moreover, configurations (i) and (ii) correspond to abelian or 2-step nilpotent Zinbiel algebras.

Proof. Let us suppose that G is a (pseudo)digraph with n vertices ($n > 3$) verifying condition (i). We prove that G is associated with a Zinbiel algebra. We denote by $\{i_1, \ldots, i_n\}$ the vertex set of G and consider an n-dimensional vector space V , with basis $\{e_{i_1}, \ldots, e_{i_n}\}$. We define in V the inner products given in (1).

First, if we consider vertices without loop, then the corresponding Zinbiel identities are trivially satisfied. Assume that we have a loop on vertex i_j, with $1 \leq j \leq n$ and that i_ℓ is a vertex without loop. From $Z(e_{i_j}, e_{i_j}, e_{i_j}) = -c_{i_j,i_j}^{i_j} (\sum_{k=1}^n c_{i_j,i_j}^{i_k} e_k)$, we obtain that $c_{i_j,i_j}^{i_j} = 0$. Next, $Z(e_{i_j}, e_{i_j}, e_{i_m}) = c_{i_j,i_j}^{i_m} (\sum_{k=1}^n c_{i_m,i_m}^{i_k} e_k) = 0$ implies that $c_{i_j,i_j}^{i_m} = 0$, for every $1 \leq m \neq \ell \leq n$. Therefore, we obtain a Zinbiel algebra with these restrictions: $c_{i_j,i_j}^{i_j} = c_{i_j,i_j}^{i_m} = 0$, $\forall 1 \leq m \neq \ell \leq n$ and $c_{i_j,i_j}^{i_\ell} \neq 0$, for each vertex i_j containing a loop and every vertex i_ℓ without loop.

Next, we assume that G is a (pseudo)digraph with n vertices ($n > 3$) verifying condition (ii). Let us prove that G is associated with a Zinbiel algebra. We denote the vertices of G by $\{i, j, k_3, \ldots, k_n\}$, where ij is a double edge with loops and $\{k_3, \ldots, k_n\}$ is a set of isolated vertices. It is easy to prove that, from the Zinbiel identities, we obtain

$$c_{i,i}^i = \frac{(c_{j,i}^i)^2}{c_{j,j}^i}, c_{i,i}^j = -\frac{(c_{j,i}^i)^3}{(c_{j,j}^i)^2}, c_{i,j}^i = c_{j,i}^i, c_{i,j}^j = -\frac{(c_{j,i}^i)^2}{c_{j,j}^i}, c_{j,i}^j = -\frac{(c_{j,i}^i)^2}{c_{j,j}^i},$$

$$c_{j,j}^j = -c_{j,i}^i, c_{i,i}^{k_h} = c_{j,j}^{k_h} = 0, \quad \text{for } 3 \leq h \leq n.$$

Therefore, we obtain a Zinbiel algebra with the previous restrictions and additionally the following ones: $c^j_{i,i} \neq 0 \neq c^i_{j,j}$. Note that the vertices $\{k_h\}_{h=3}^n$ may have loops or not.

Conversely, we suppose that G is a (pseudo)digraph with n vertices ($n > 3$) and associated with a Zinbiel algebra. We prove that G must be as described in (i) or (ii). Let $\{i, j, k\}$ be three arbitrary vertices of G. According to Proposition 8, we only have 4 possible allowed configurations for these three vertices. Moreover, we cannot have two double edges from three vertices since Configuration 6) from Figure 6 is forbidden. Consequently, and from these considerations, it follows that G must be a set of n isolated vertices with a maximum of $(n-1)$ loops or a (pseudo)digraph formed by a double edge with loops plus isolated vertices.

Finally, if G is formed by isolated vertices with no loops, then we obtain an abelian Zinbiel algebra and otherwise, we can use an analogous reasoning to that considered in the proof of Proposition 9 to prove that Zinbiel algebras are 2-step nilpotent. $\square$

6 Combinatorial structures of three vertices associated with Zinbiel algebras

In this section, we study the combinatorial structures of 3 vertices including full triangles and being associated with Zinbiel algebras. We also analyze the isomorphism classes for those configurations. To do so, we consider a set of three vertices, $\{i, j, k\}$, and define a vector space V endowed with basis $\{e_i, e_j, e_k\}$ and law given by the following brackets

$$[e_h, e_h] = c^i_{h,h} e_i + c^j_{h,h} e_j + c^k_{h,h} e_k, \quad \text{for } h = i, j, k;$$

$$\begin{aligned}
[e_i, e_j] &= c^i_{i,j} e_i + c^j_{i,j} e_j + c^k_{i,j} e_k, & [e_j, e_i] &= c^i_{j,i} e_i + c^j_{j,i} e_j + c^k_{j,i} e_k,\\
[e_i, e_k] &= c^i_{i,k} e_i + c^j_{i,k} e_j + c^k_{i,k} e_k, & [e_k, e_i] &= c^i_{k,i} e_i + c^j_{k,i} e_j + c^k_{k,i} e_k,\\
[e_j, e_k] &= c^i_{j,k} e_i + c^j_{j,k} e_j + c^k_{j,k} e_k, & [e_k, e_j] &= c^i_{k,j} e_i + c^j_{k,j} e_j + c^k_{k,j} e_k.
\end{aligned}$$

where the structure constants may be zero or not. The main difficulty in this study consists in determining under what conditions the previous vector space is a Zinbiel algebra. By imposing the Zinbiel identities, we obtain an equation system which has to be solved. Therefore, we have obtained the following

Proposition 13. *Let G be a combinatorial structures of three vertices containing full triangles. Then, G is associated with a 3-dimensional Zinbiel algebra $\mathcal{Z}$ if and only if G is isomorphic to configurations shown in Figure 7. Moreover, the restrictions for each configuration are*

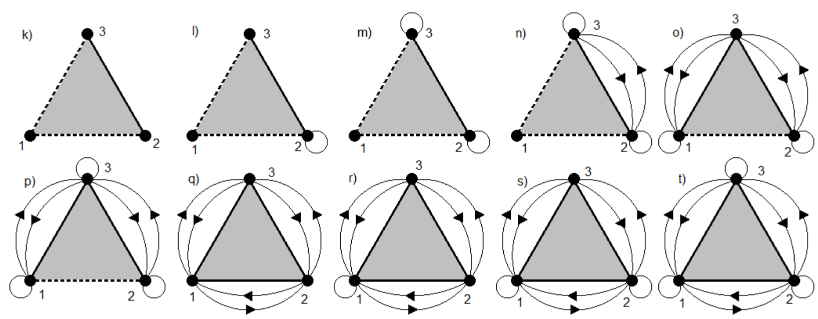

Figure 7: Combinatorial structures of 3 vertices associated with Zinbiel algebras.

k) $(c_{2,3}^1, c_{3,2}^1) \neq (0,0)$

l) $c_{2,2}^2 = 0,\ c_{2,2}^3(2c_{2,3}^1 - c_{3,2}^1) = 0,\ (c_{2,3}^1, c_{3,2}^1) \neq (0,0),\ (c_{2,2}^1, c_{2,2}^3) \neq (0,0)$

m) $c_{2,2}^2 = c_{2,2}^3 = c_{3,3}^2 = c_{3,3}^3 = 0,\ c_{2,2}^1 \neq 0,\ c_{3,3}^1 \neq 0,\ (c_{2,3}^1, c_{3,2}^1) \neq (0,0)$

n) $c_{2,2}^1 = \dfrac{-c_{2,3}^3(2c_{2,3}^3 c_{3,3}^1 + 3c_{3,2}^1 c_{3,2}^2)}{(c_{3,2}^2)^2},\ c_{2,2}^2 = -c_{2,3}^3,\ c_{2,2}^3 = -\dfrac{(c_{2,3}^3)^2}{c_{3,2}^2},\ c_{2,3}^1 = \dfrac{c_{2,3}^3 c_{3,3}^1 + 2c_{3,2}^1 c_{3,2}^2}{c_{3,2}^2},\ c_{2,3}^2 = c_{3,2}^2,\ c_{3,2}^3 = c_{2,3}^3,\ c_{3,3}^2 = -\dfrac{(c_{3,2}^2)^2}{c_{2,3}^3},\ c_{3,3}^3 = -c_{3,2}^2,$ $(c_{2,3}^1, c_{3,2}^1) \neq (0,0),\ c_{2,3}^2 \neq 0,\ c_{2,3}^3 \neq 0,$

o) $c_{1,1}^1 = -\dfrac{(c_{3,1}^1)^2 c_{2,2}^1}{(c_{3,2}^1)^2},\ c_{1,1}^2 = \dfrac{(c_{3,1}^1)^3 c_{2,2}^1}{(c_{3,2}^1)^3},\ c_{1,1}^3 = -\dfrac{(c_{3,1}^1)^3 (c_{2,2}^1)^2}{(c_{3,2}^1)^4},\ c_{1,3}^1 = c_{3,1}^1,$ $c_{1,3}^2 = \dfrac{-(c_{3,1}^1)^2}{c_{3,2}^1},\ c_{1,3}^3 = \dfrac{(c_{3,1}^1)^2 c_{2,2}^1}{(c_{3,2}^1)^2},\ c_{2,2}^2 = \dfrac{-c_{2,2}^1 c_{3,1}^1}{c_{3,2}^1},\ c_{2,2}^3 = \dfrac{(c_{2,2}^1)^2 c_{3,1}^1}{(c_{3,2}^1)^2},$ $c_{2,3}^1 = c_{3,2}^1,\ c_{2,3}^2 = -c_{3,1}^1,\ c_{2,3}^3 = \dfrac{c_{2,2}^1 c_{3,1}^1}{c_{3,2}^1},\ c_{3,1}^2 = \dfrac{-(c_{3,1}^1)^2}{c_{3,2}^1},\ c_{3,1}^3 = \dfrac{(c_{3,1}^1)^2 c_{2,2}^1}{(c_{3,2}^1)^2},\ c_{3,2}^2 = -c_{3,1}^1,\ c_{3,2}^3 = \dfrac{c_{2,2}^1 c_{3,1}^1}{c_{3,2}^1},\ c_{3,2}^1 \neq 0,\ c_{3,1}^1 \neq 0,\ c_{2,2}^1 \neq 0;$

p) $c_{1,1}^1 = -c_{1,3}^3,\ c_{1,1}^2 = \dfrac{-c_{1,3}^3 c_{3,1}^2}{c_{3,1}^1},\ c_{1,1}^3 = -\dfrac{(c_{1,3}^3)^2}{c_{3,1}^1},\ c_{1,3}^1 = c_{3,1}^1,\ c_{1,3}^2 = c_{3,1}^2,\ c_{2,2}^1 = \dfrac{-c_{1,3}^3 c_{3,2}^2 c_{3,1}^1}{(c_{3,1}^2)^2},\ c_{2,2}^2 = -\dfrac{c_{1,3}^3 c_{3,2}^2}{c_{3,1}^2},\ c_{2,2}^3 = -\dfrac{(c_{1,3}^3)^2 c_{3,2}^2}{(c_{3,1}^2)^2},\ c_{2,3}^1 =$

$\frac{c_{3,2}^2c_{3,1}^1}{c_{3,1}^2}$, $c_{2,3}^2 = c_{3,2}^2$, $c_{2,3}^3 = \frac{c_{1,3}^3c_{3,2}^2}{c_{3,1}^2}$, $c_{3,1}^3 = c_{1,3}^3$, $c_{3,2}^1 = \frac{c_{3,2}^2c_{3,1}^1}{c_{3,1}^2}$, $c_{3,2}^3 = \frac{c_{1,3}^3c_{3,2}^2}{c_{3,1}^2}$, $c_{3,3}^1 = -\frac{c_{3,1}^1(c_{3,1}^1+c_{3,2}^2)}{c_{1,3}^3}$, $c_{3,3}^2 = -\frac{c_{3,1}^2(c_{3,1}^1+c_{3,2}^2)}{c_{1,3}^3}$, $c_{3,3}^3 = -c_{3,1}^1 - c_{3,2}^2$, $c_{3,1}^1 \neq 0$, $c_{3,1}^2 \neq 0$, $c_{1,3}^3 \neq 0$, $c_{3,2}^2 \neq 0$, $c_{3,1}^1 \neq -c_{3,2}^2$;

q) $c_{1,2}^1 = -c_{3,2}^3$, $c_{1,2}^2 = \frac{c_{2,3}^2c_{3,2}^3}{c_{3,2}^1}$, $c_{1,2}^3 = -\frac{(c_{3,2}^3)^2}{c_{3,2}^1}$, $c_{1,3}^1 = -c_{2,3}^2$, $c_{1,3}^2 = \frac{(c_{2,3}^2)^2}{c_{3,2}^1}$, $c_{1,3}^3 = -\frac{c_{2,3}^2c_{3,2}^3}{c_{3,2}^1}$, $c_{2,1}^1 = c_{3,2}^3$, $c_{2,1}^2 = -\frac{c_{2,3}^2c_{3,2}^3}{c_{3,2}^1}$, $c_{2,1}^3 = \frac{(c_{3,2}^3)^2}{c_{3,2}^1}$, $c_{2,3}^1 = -c_{3,2}^1$, $c_{2,3}^3 = -c_{3,2}^3$, $c_{3,1}^1 = c_{2,3}^2$, $c_{3,1}^2 = -\frac{(c_{2,3}^2)^2}{c_{3,2}^1}$, $c_{3,1}^3 = \frac{c_{2,3}^2c_{3,2}^3}{c_{3,2}^1}$, $c_{3,2}^2 = -c_{2,3}^2$, $c_{3,2}^1 \neq 0$, $c_{2,3}^2 \neq 0$, $c_{3,2}^3 \neq 0$

r) $c_{1,1}^1 = -\frac{c_{3,1}^3(c_{2,3}^1+c_{3,2}^1)}{c_{3,2}^1}$, $c_{1,1}^2 = \frac{c_{3,1}^1c_{3,1}^3(c_{2,3}^1+c_{3,2}^1)}{(c_{3,2}^1)^2}$, $c_{1,2}^1 = -\frac{c_{3,1}^3c_{3,2}^1}{c_{3,1}^1}$, $c_{1,2}^2 = c_{3,1}^3$, $c_{1,2}^3 = -\frac{(c_{3,1}^3)^2c_{3,2}^1}{(c_{3,1}^1)^2}$, $c_{1,3}^1 = \frac{c_{2,3}^1c_{3,1}^1}{c_{3,2}^1}$, $c_{1,3}^2 = -\frac{c_{2,3}^1(c_{3,1}^1)^2}{(c_{3,2}^1)^2}$, $c_{1,3}^3 = \frac{c_{2,3}^1c_{3,1}^3}{c_{3,2}^1}$, $c_{2,1}^1 = -\frac{c_{2,3}^1c_{3,1}^3}{c_{3,1}^1}$, $c_{2,1}^2 = \frac{c_{2,3}^1c_{3,1}^3}{c_{3,2}^1}$, $c_{2,1}^3 = -\frac{c_{2,3}^1(c_{3,1}^3)^2}{(c_{3,1}^1)^2}$, $c_{2,3}^2 = -\frac{c_{2,3}^1c_{3,1}^1}{c_{3,2}^1}$, $c_{2,3}^3 = \frac{c_{2,3}^1c_{3,1}^3}{c_{3,1}^1}$, $c_{3,1}^2 = -\frac{(c_{3,1}^1)^2}{c_{3,2}^1}$, $c_{3,2}^2 = -c_{3,1}^1$, $c_{3,2}^3 = \frac{c_{3,1}^3c_{3,2}^1}{c_{3,1}^1}$, $c_{1,1}^3 = -\frac{(c_{3,1}^3)^2(c_{2,3}^1+c_{3,2}^1)}{c_{3,2}^1c_{3,1}^1}$, $c_{3,2}^1 \neq 0$, $c_{3,1}^1 \neq 0$

s) $c_{1,1}^1 = \frac{(c_{3,1}^1+c_{3,2}^2)^2}{c_{3,3}^1}$, $c_{1,1}^2 = -\frac{(c_{3,1}^1+c_{3,2}^2)^2c_{3,2}^2}{c_{2,3}^1c_{3,3}^1}$, $c_{1,1}^3 = -\frac{(c_{3,1}^1+c_{3,2}^2)^3}{(c_{3,3}^1)^2}$, $c_{1,2}^1 = -\frac{(c_{3,1}^1+c_{3,2}^2)c_{2,3}^1}{c_{3,3}^1}$, $c_{1,2}^2 = \frac{c_{3,2}^2(c_{3,1}^1+c_{3,2}^2)}{c_{3,3}^1}$, $c_{1,2}^3 = \frac{(c_{3,1}^1+c_{3,2}^2)^2c_{2,3}^1}{(c_{3,3}^1)^2}$, $c_{1,3}^1 = c_{3,1}^1 + 2c_{3,2}^2$, $c_{1,3}^2 = -\frac{c_{3,2}^2(c_{3,1}^1+2c_{3,2}^2)}{c_{2,3}^1}$, $c_{2,1}^1 = \frac{(c_{3,1}^1+c_{3,2}^2)c_{2,3}^1}{c_{3,3}^1}$, $c_{2,1}^2 = -\frac{c_{3,2}^2(c_{3,1}^1+c_{3,2}^2)}{c_{3,3}^1}$, $c_{2,1}^3 = -\frac{(c_{3,1}^1+c_{3,2}^2)^2c_{2,3}^1}{(c_{3,3}^1)^2}$, $c_{2,3}^2 = -c_{3,2}^2$, $c_{2,3}^3 = -\frac{(c_{3,1}^1+c_{3,2}^2)c_{2,3}^1}{c_{3,3}^1}$, $c_{3,1}^2 = -\frac{c_{3,1}^1c_{3,2}^2}{c_{2,3}^1}$, $c_{3,1}^3 = -\frac{c_{3,1}^1(c_{3,1}^1+c_{3,2}^2)}{c_{3,3}^1}$, $c_{3,2}^1 = -c_{2,3}^1$, $c_{3,2}^3 = \frac{(c_{3,1}^1+c_{3,2}^2)c_{2,3}^1}{c_{3,3}^1}$, $c_{3,3}^2 = -\frac{c_{3,2}^2c_{3,3}^1}{c_{2,3}^1}$, $c_{3,3}^3 = -c_{3,1}^1 - c_{3,2}^2$, $c_{1,3}^3 =$

$-\dfrac{(c_{3,1}^1 + c_{3,2}^2)(c_{3,1}^1 + 2c_{3,2}^2)}{c_{3,3}^1}$, $c_{3,3}^1 \neq 0$, $c_{2,3}^1 \neq 0$

t) $c_{1,1}^1 = -\dfrac{c_{3,1}^3(c_{2,2}^2 + c_{3,2}^3)}{c_{3,2}^3}$, $c_{1,1}^2 = \dfrac{c_{2,2}^2(c_{3,1}^3)^2}{(c_{3,2}^3)^2}$, $c_{1,2}^1 = -c_{2,2}^2 - c_{3,2}^3$, $c_{1,2}^2 = \dfrac{c_{2,2}^2 c_{3,1}^3}{c_{3,2}^3}$, $c_{1,2}^3 = \dfrac{c_{1,1}^3 c_{3,2}^3}{c_{3,1}^3}$, $c_{1,3}^1 = -\dfrac{(c_{3,1}^3)^2(c_{2,2}^2 + c_{3,2}^3)}{c_{1,1}^3 c_{3,2}^3}$, $c_{1,3}^2 = \dfrac{c_{2,2}^2(c_{3,1}^3)^3}{(c_{3,2}^3)^2 c_{1,1}^3}$, $c_{1,3}^3 = c_{3,1}^3$, $c_{2,1}^1 = -c_{2,2}^2 - c_{3,2}^3$, $c_{2,1}^2 = \dfrac{c_{2,2}^2 c_{3,1}^3}{c_{3,2}^3}$, $c_{2,1}^3 = \dfrac{c_{1,1}^3 c_{3,2}^3}{c_{3,1}^3}$, $c_{2,2}^1 = -\dfrac{(c_{2,2}^2 + c_{3,2}^3)c_{3,2}^3}{c_{3,1}^3}$, $c_{2,2}^3 = \dfrac{c_{1,1}^3(c_{3,2}^3)^2}{(c_{3,1}^3)^2}$, $c_{2,3}^1 = -\dfrac{c_{3,1}^3(c_{2,2}^2 + c_{3,2}^3)}{c_{1,1}^3}$, $c_{2,3}^2 = \dfrac{c_{2,2}^2(c_{3,1}^3)^2}{c_{3,2}^3 c_{1,1}^3}$, $c_{2,3}^3 = c_{3,2}^3$, $c_{3,1}^1 = -\dfrac{(c_{3,1}^3)^2(c_{2,2}^2 + c_{3,2}^3)}{c_{1,1}^3 c_{3,2}^3}$, $c_{3,1}^2 = \dfrac{c_{2,2}^2(c_{3,1}^3)^3}{(c_{3,2}^3)^2 c_{1,1}^3}$, $c_{3,2}^1 = -\dfrac{c_{3,1}^3(c_{2,2}^2 + c_{3,2}^3)}{c_{1,1}^3}$, $c_{3,2}^2 = \dfrac{c_{2,2}^2(c_{3,1}^3)^2}{c_{3,2}^3 c_{1,1}^3}$, $c_{3,3}^1 = -\dfrac{(c_{3,1}^3)^3(c_{2,2}^2 + c_{3,2}^3)}{c_{3,2}^3(c_{1,1}^3)^2}$, $c_{3,3}^2 = \dfrac{(c_{3,1}^3)^4 c_{2,2}^2}{(c_{3,2}^3)^2(c_{1,1}^3)^2}$, $c_{3,3}^3 = \dfrac{(c_{3,1}^3)^2}{c_{1,1}^3}$, $c_{3,2}^3 \neq 0$, $c_{3,1}^3 \neq 0$, $c_{1,1}^3 \neq 0$

Proof. Similar to the proof of Proposition 10. □

Now, we study the isomorphism class for Zinbiel algebras associated with configurations shown in Figure 7.

Theorem 2. *Let $\mathcal{Z}_\alpha^3$ be the Zinbiel algebra associated with Configuration α, where $\alpha \in \{k, l, m, n, o, p, q, r, s, t\}$ from Figure 7 and let $\mathcal{Z}_{xviii)}^3$ be the Zinbiel algebra associated to Configuration xviii) from Figure 5. Then, we have the following isomorphisms ($\cong$)*

1) $\mathcal{Z}_m^3 \cong \mathcal{Z}_p^3 \cong \mathcal{Z}_t^3 \cong \mathcal{Z}_{xviii)}^3$

2) $\mathcal{Z}_l^3 \cong \mathcal{Z}_n^3 \cong \mathcal{Z}_o^3 \cong \mathcal{Z}_r^3 \cong \mathcal{Z}_s^3$

Proof. We start proving 1) that $\mathcal{Z}_m^3 \cong \mathcal{Z}_{xviii)}^3$. According to Proposition 8, the law of $\mathcal{Z}_{xviii)}^3$ is given by

$$[v_1, v_1] = \frac{c_{1,1}^3 c_{3,2}^2}{c_{2,3}^3} v_2 + c_{1,1}^3 v_3, \quad [v_2, v_2] = -c_{2,3}^3 v_2 - \frac{(c_{2,3}^3)^2}{c_{3,2}^2} v_3,$$

$$[v_3, v_3] = -\frac{(c_{3,2}^2)^2}{c_{2,3}^3} v_2 - c_{3,2}^2 v_3, \quad [v_2, v_3] = [v_3, v_2] = c_{3,2}^2 v_2 + c_{2,3}^3 v_3$$

By considering the basis change $\phi : \mathcal{Z}^3_{xviii)} \to \mathcal{Z}^3_{xviii)}$ given by $e_1 = \phi(v_1) = c^2_{3,2}v_2 + c^3_{2,3}v_3$; $e_2 = \phi(v_2) = v_2 + 2c^3_{2,3}v_1$; $e_3 = \phi(v_3) = -v_3 + \frac{1}{c^3_{1,1}}v_1$,

$$[e_2, e_2] = c^3_{2,3}(4c^3_{1,1} - \frac{1}{c^2_{3,2}})e_1,\ [e_3, e_3] = \frac{1}{c^3_{2,3}}(\frac{1}{c^3_{1,1}} - c^2_{3,2})e_1,\ [e_2, e_3] = [e_3, e_2] = e_1$$

With a trivial scale on the coefficients, we obtain the law of $\mathcal{Z}^3_m$. In order to prove 2), we bear in mind that, from Proposition 13, the law of $\mathcal{Z}^3_n$ is

$$[v_2, v_2] = -\frac{c^3_{2,3}(2c^3_{2,3}c^1_{3,3} + 3c^1_{3,2}c^2_{3,2})}{(c^2_{3,2})^2}v_1 - c^3_{2,3}v_2 - \frac{(c^3_{2,3})^2}{c^2_{3,2}}v_3,$$

$$[v_3, v_3] = c^1_{3,3}v_1 - \frac{(c^2_{3,2})^2}{c^3_{2,3}}v_2 - c^2_{3,2}v_3, \quad [v_3, v_2] = c^1_{3,2}v_1 + c^2_{3,2}v_2 + c^3_{2,3}v_3$$

$$[v_2, v_3] = \frac{c^3_{2,3}c^1_{3,3} + 2c^1_{3,2}c^2_{3,2}}{c^2_{3,2}}v_1 + c^2_{3,2}v_2 + c^3_{2,3}v_3$$

Next, we consider the basis change $\phi : \mathcal{Z}^3_n \to \mathcal{Z}^3_n$ given by $e_1 = \phi(v_1) = v_1$; $e_2 = \phi(v_2) = c^2_{3,2}v_2 + c^3_{2,3}v_3$; $e_3 = \phi(v_3) = v_3$, we obtain the law

$$[e_3, e_3] = c^1_{3,3}e_1 - \frac{c^2_{3,2}}{c^3_{2,3}}e_2, \quad [e_2, e_3] = 2[e_3, e_2] = 2(c^3_{2,3}c^1_{3,3} + c^1_{3,2}c^2_{3,2})e_1$$

Relabeling the vertices, we get the law of $\mathcal{Z}^3_l$. We continue proving that $\mathcal{Z}^3_o \cong \mathcal{Z}^3_l$. According to Proposition 13, the non-zero brackets of $\mathcal{Z}^3_o$ are

$$[v_1, v_1] = -\frac{(c^1_{3,1})^2c^1_{2,2}}{(c^1_{3,2})^2}v_1 + \frac{(c^1_{3,1})^3c^1_{2,2}}{(c^1_{3,2})^3}v_2 - \frac{(c^1_{3,1})^3(c^1_{2,2})^2}{(c^1_{3,2})^4}v_3,$$

$$[v_1, v_3] = [v_3, v_1] = c^1_{3,1}v_1 - \frac{(c^1_{3,1})^2}{c^1_{3,2}}v_2 + \frac{(c^1_{3,1})^2c^1_{2,2}}{(c^1_{3,2})^2}v_3,$$

$$[v_2, v_2] = c^1_{2,2}v_1 - \frac{c^1_{2,2}c^1_{3,1}}{c^1_{3,2}}v_2 + \frac{(c^1_{2,2})^2c^1_{3,1}}{(c^1_{3,2})^2}v_3,$$

$$[v_2, v_3] = [v_3, v_2] = c^1_{3,2}v_1 - c^1_{3,1}v_2 + \frac{c^1_{2,2}c^1_{3,1}}{c^1_{3,2}}v_3$$

Now, we consider the basis change $\phi : \mathcal{Z}^3_o \to \mathcal{Z}^3_o$ given by $e_1 = \phi(v_1) = v_3$; $e_2 = \phi(v_2) = c^1_{2,2}v_1 - \frac{c^1_{3,1}c^1_{2,2}}{c^1_{3,2}}v_2 + \frac{c^1_{3,1}(c^1_{2,2})^2}{(c^1_{3,2})^2}v_3$; $e_3 = \phi(v_3) = v_1$, we obtain the law

$$[e_3, e_3] = -\frac{(c^1_{3,1})^2}{(c^1_{3,2})^2}e_2, \quad [e_1, e_3] = [e_3, e_1] = \frac{c^1_{3,1}}{c^1_{3,2}}e_2$$

After exchanging subindexes, we obtain the law of $\mathfrak{Z}_l^3$. By using the same reasoning, one can prove the isomorphism of statement 3). Next, we prove statement 4). In order to do so, we consider the law of $\mathfrak{Z}_r^3$ obtained from Proposition 13:

$$[v_1,v_1] = -\frac{c_{3,1}^3(c_{2,3}^1+c_{3,2}^1)}{c_{3,2}^1}v_1 + \frac{c_{3,1}^1c_{3,1}^3(c_{2,3}^1+c_{3,2}^1)}{(c_{3,2}^1)^2}v_2 - \frac{(c_{3,1}^3)^2(c_{2,3}^1+c_{3,2}^1)}{c_{3,2}^1c_{3,1}^1}v_3,$$

$$[v_1,v_2] = -\frac{c_{3,1}^3c_{3,2}^1}{c_{3,1}^1}v_1 + c_{3,1}^3v_2 - \frac{(c_{3,1}^3)^2c_{3,2}^1}{(c_{3,1}^1)^2}v_3,\ [v_1,v_3] = \frac{c_{2,3}^1c_{3,1}^1}{c_{3,2}^1}v_1 - \frac{c_{2,3}^1(c_{3,1}^1)^2}{(c_{3,2}^1)^2}v_2$$

$$+\frac{c_{2,3}^1c_{3,1}^3}{c_{3,2}^1}v_3,\ [v_2,v_1] = -\frac{c_{2,3}^1c_{3,1}^3}{c_{3,1}^1}v_1 + \frac{c_{2,3}^1c_{3,1}^3}{c_{3,2}^1}v_2 - \frac{c_{2,3}^1(c_{3,1}^3)^2}{(c_{3,1}^1)^2}v_3,$$

$$[v_2,v_3] = c_{2,3}^1v_1 - \frac{c_{2,3}^1c_{3,1}^1}{c_{3,2}^1}v_2 + \frac{c_{2,3}^1c_{3,1}^3}{c_{3,1}^1}v_3,$$

$$[v_3,v_1] = c_{3,1}^1v_1 - \frac{(c_{3,1}^1)^2}{c_{3,2}^1}v_2 + c_{3,1}^3v_3,\quad [v_3,v_2] = c_{3,2}^1v_1 - c_{3,1}^1v_2 + \frac{c_{3,1}^3c_{3,2}^1}{c_{3,1}^1}v_3$$

Let us consider $v = v_1 - \frac{c_{3,1}^1}{c_{3,2}^1}v_2 + \frac{c_{3,1}^3}{c_{3,1}^1}v_3$. Then,

$$[v_1,v_1] = -\frac{c_{3,1}^3(c_{2,3}^1+c_{3,2}^1)}{c_{3,2}^!}v,\quad [v_1,v_2] = -\frac{c_{3,1}^3c_{3,2}^1}{c_{3,1}^1}v,\quad [v_1,v_3] = \frac{c_{2,3}^1c_{3,1}^1}{c_{3,2}^1}v,$$

$$[v_2,v_1] = -\frac{c_{2,3}^1c_{3,1}^3}{c_{3,1}^1}v,\quad [v_2,v_3] = c_{2,3}^1v,\quad [v_3,v_1] = c_{3,1}^1v,\quad [v_3,v_2] = c_{3,2}^1v$$

Now, we consider the basis change $\phi : \mathfrak{Z}_r^3 \to \mathfrak{Z}_r^3$ given by $e_1 = \phi(v_1) = v_1$; $e_2 = \phi(v_2) = v$; $e_3 = \phi(v_3) = v_3$, we obtain the law

$$[e_1,e_1] = -\frac{c_{3,1}^3(c_{2,3}^1+c_{3,2}^1)}{c_{3,2}^1}e_2,\quad [e_1,e_3] = \frac{c_{2,3}^1}{c_{3,2}^1}[e_3,e_1] = \frac{c_{2,3}^1c_{3,1}^1}{c_{3,2}^1}e_2$$

Relabeling the vertices 1 and 2, the law of the algebra $\mathfrak{Z}_l^3$ is obtained. Finally, with a similar reasoning, it is possible to prove that $\mathfrak{Z}_s^3 \cong \mathfrak{Z}_l^3$ and $\mathfrak{Z}_t^3 \cong \mathfrak{Z}_m^3$. □

Finally, we clarify how all the isomorphism classes obtained in Sections 5 and 6 correspond to those in the well-known classification of 2- and 3-dimensional Zinbiel algebras. We will relate the notation given in [2, Theorem 1.7] with that used in this paper for the isomorphism classes.

$\mathfrak{Z}^1$	$\mathfrak{Z}^2_a$	$\mathfrak{Z}^2_b$	$\mathfrak{Z}^3_{i)}$	$\mathfrak{Z}^3_{ii)}$	$\mathfrak{Z}^3_{iii)}$	$\mathfrak{Z}^3_{xviii)}$	$\mathfrak{Z}^3_k$	$\mathfrak{Z}^3_l$
A	$Q(0)$	$Q(1)$	$R(0,0,0,0)$	$R(1,0,0,0)$	$R(1,0,0,1)$	$R(\alpha,1,1,\frac{-c^3_{2,3}}{c^2_{3,2}})$	$R(0,\beta,\gamma,0)$	W(3)

Table 2: Comparison with the classification of 2- and 3-dimensional Zinbiel algebras.

Remark 6. *We have not included configuration $q)$ from Figure 7 in Table 2, since it corresponds to a Lie algebra with the following law:*

$$[e_1,e_2]=-[e_2,e_1]=-c^3_{3,2}e_1+\frac{c^2_{2,3}c^3_{3,2}}{c^1_{3,2}}e_2-\frac{(c^3_{3,2})^2}{c^1_{3,2}}e_3,$$

$$[e_1,e_3]=-[e_3,e_1]=-c^2_{2,3}e_1+\frac{(c^2_{2,3})^2}{c^1_{3,2}}e_2-\frac{c^2_{2,3}c^3_{3,2}}{c^1_{3,2}}e_3,$$

$$[e_2,e_3]=-[e_3,e_2]=-c^1_{3,2}e_1+c^2_{2,3}e_2-c^3_{3,2}e_3$$

7 Algorithmic procedures

This section is devoted to introduce two algorithmic procedures: The first one checks if a given combinatorial structure is associated or not with a Zinbiel algebra; the second one, conversely, computes the (pseudo)digraph associated with a given finite-dimensional Zinbiel algebra starting from its law when this is not providing full triangles. Let us note that the procedure presented in Subsection 7.1 is developed starting from the techniques to prove the existence or non-existence of Zinbiel algebras associated with combinatorial (pseudo)digraphs in previous sections; moreover, this procedure has been run later to check the theoretical results for dimensions 2 and 3.

7.1 Checking if a given combinatorial structure is associated with a Zinbiel algebra

We have implemented this algorithmic procedure by using the symbolic computation package Maple, working the implementation in version 12 or higher. To do this, we have used the libraries `linalg` and `combinat` to activate commands related to Linear and Combinatorial Algebra. This algorithmic procedure consists of the following three steps:

a) Defining the values of the structure constants according to the combinatorial structure.

b) Generating the law which should be satisfied by the Zinbiel algebra, starting from the structure constants.

c) Checking if the Zinbiel identities are satisfied for this law.

In order to develop the implementation, we use three subprocedures for the two first steps and one main procedure for the last one. Before running the procedure, we must restart all the variables and delete all the computations saved in the kernel by using the command `restart`. The first step of this algorithm is executed by the subprocedure `assignment`, which allows us to define the dimension and the value of the structure constants of the vector space associated with the combinatorial structure and to determine the candidate for the bracket product. To do so, `assignment` receives the following two inputs: The list `V` with the vertices of the combinatorial structure as natural numbers, and the set `E` with its weighted, directed edges. The elements of the set `E` are inserted as $[[i,j,k],l]$, denoting $c_{i,j}^k = l$. As output, we obtain the value of the variable `dim` with the dimension of the combinatorial structure and also the value of all the non-zero structure constants.

```
> restart:
> assignment:=proc(V,E)
>  local B,L;
>  B:=[];L:=[];
>  for x from 1 to nops(V) do
>   B:=[op(B),e[x]]; od;
>  assign(dim,nops(V));
>  for i from 1 to nops(E) do
>   assign(c[E[i][1][1],E[i][1][2],E[i][1][3]],E[i][2]); od;
> end proc:
```

Now, we can run the second subprocedure, named `law`, which receives two natural numbers as inputs. These numbers represent the subindexes of two vectors in the endowed vector space or, equivalently, two vertices from the combinatorial structure. The subroutine computes the bracket of these two vectors. In the implementation, we use a local variable, `v`, to save the value of the bracket, which is computed by using the structure constants defined in the previous subprocedure.

```
> law:=proc(i,j)
>   local v; v:=0;
>   for k from 1 to dim do
>    if type(c[i,j,k],numeric)=true then
>       v:=v+c[i,j,k]*e[k]; fi; od;
>  v;
> end proc:
```

Next, we implement the subprocedure called `bracket` to compute the product between two arbitrary vectors expressed as linear combinations of the basis vectors used in the previous subprocedure.

```
> bracket:=proc(u,v,n)
>   local exp; exp:=0;
>   for i from 1 to n do
>      for j from 1 to n do
>         exp:=exp + coeff(u,e[i])*coeff(v,e[j])*law(i,j); od; od;
>   exp;
> end proc:
```

Finally, we show the implementation of the main procedure called `Zinbiel`, which checks if the vector space is or is not a Zinbiel algebra. This procedure receives as input the dimension n of the vector space $\mathcal{Z}$ and returns a message which will be "True" in case that the vector space $\mathcal{Z}$ is a Zinbiel algebra and "False" otherwise.

```
>Zinbiel:=proc(n)
> local L,M,N,P;
> L:=[];M:=[];N:=[];P:=[];
> for i from 1 to n do
>  L:=[op(L),i,i]; od;
> M:=permute(L,3);
> for j from 1 to nops(M) do
> eq[j]:=bracket(bracket(e[M[j][1]],e[M[j][2]],n),e[M[j][3]],n)
        -bracket(e[M[j][1]],bracket(e[M[j][2]],e[M[j][3]],n),n)
        -bracket(e[M[j][1]],bracket(e[M[j][3]],e[M[j][2]],n),n); od;
> N:=[seq(eq[k], k=1..nops(M))];
> for i from 1 to nops(N) do
>  if N[i]<>0 then
>    P:=[op(P),N[i]]; fi; od;
> if P=[] then return "True"
>  else return "False";  if;
> end proc:
```

Example 3. *To illustrate the algorithmic procedure, we show an example by considering the combinatorial structure of Figure 2.*

According to the notation that we are using in this algorithm, we have to consider

```
> V=[1,2,3];
> E={[[1,1,2],1],[[1,1,3],-1],[[1,2,2],-1],[[1,2,3],1],[[1,3,2],-1],[[1,3,3],1]};
```

Now, we run all the procedures obtaining

```
> assignment(V,E);
> Zinbiel(dim);
>     "True"
```

Therefore, the combinatorial structure is associated with a 3-dimensional Zinbiel algebra.

7.2 Obtaining (pseudo)digraphs associated with Zinbiel algebra

In this subsection, we show an algorithmic procedure that computes the (pseudo)digraph associated with a given Zinbiel algebra with finite dimension. According to the notation previously used, we consider a n-dimensional Zinbiel algebra $\mathcal{Z}$ with basis $\mathcal{B}$ and whose non-zero brackets are the ones given in (1). In this way, there will be no full triangles in the configuration.

To implement the algorithm, we have used the symbolic computation package MAPLE 12, loading the libraries `linalg`, `combinat`, `GraphTheory` and `Maplets[Elements]`. The first three libraries allow us to apply commands of Linear Algebra, Combinatorics and Graph Theory, respectively; whereas the last is used to display a message so that the user introduces the required input in the first subprocedure, corresponding to the definition of the law of the algebra $\mathcal{Z}$. Our algorithm is based on the following four steps

1. Obtaining the bracket between basis vectors of $\mathcal{B}$.

   The implementation of this step is carried out by the subprocedure `law2`, which receives as input the subindexes of two basis vectors in $\mathcal{B}$. The output is the bracket between these vectors. Moreover, conditional sentences are necessary for the non-zero brackets. Since the subprocedure must be completed by the user, according to the law of $\mathcal{Z}$, we have added a sentence at the beginning recalling this fact. We also have to restart all the variables and delete the previous calculations before updating the value of `dim`, which is the variable saving the dimension of $\mathcal{Z}$.

```
> restart:
> maplet:=Maplet(AlertDialog("Don't forget to introduce non-zero brackets
of the algebra and its dimension in subprocedure law",
'onapprove'=Shutdown("Continue"),'oncancel'=Shutdown("Aborted"))):
> Maplets[Display](maplet):
> assign(dim,...):
> law2:=proc(i,j)
>  if (i,j)=... then ...; fi;
>  if ....
>  else 0; fi;
> end proc;
```

   The ellipsis in command `assign` are for $dim(\mathcal{Z})$. The other suspension points correspond to the introduction of the non-zero brackets of $\mathcal{Z}$.

2. Computing the bracket between two arbitrary vectors given as a linear combination of vectors from $\mathcal{B}$.

   We use the implementation of `bracket` shown in Subsection 7.1.

3. Solving the equation system obtained when imposing the Zinbiel identity.

   In this step we show the implementation of the routine `Zinbiel2`. This one is devoted to checking if the vector space $\mathcal{Z}$ is a Zinbiel algebra or not. The input is $dim(\mathcal{Z})$ and the output is the solution of the equation system obtained after imposing all the Zinbiel identities. In order to do that we use the permutations of each 3 basis vectors. In case that the system has no solution, then $\mathcal{Z}$ will not be a Zinbiel algebra. Otherwise, the conditions over the structure constants $c_{i,j}^k$ will be obtained so that $\mathcal{Z}$ is a Zinbiel algebra.

```
>Zinbiel2:=proc(n)
> local L,M,N,P;
> L:=[];M:=[];N:=[];P:=[];
> for i from 1 to n do
>   L:=[op(L),i,i,i]; od;
> M:=permute(L,3);
> for j from 1 to nops(M) do
> eq[j]:=bracket(bracket(e[M[j][1]],e[M[j][2]],n),e[M[j][3]],n)
        -bracket(e[M[j][1]],bracket(e[M[j][2]],e[M[j][3]],n),n)
        -bracket(e[M[j][1]],bracket(e[M[j][3]],e[M[j][2]],n),n);  od;
> N:=[seq(eq[k], k=1..nops(M))];
> for k from 1 to nops(N) do
>  for h from 1 to n do
>    P:=[op(P),coeff(N[k],e[h])=0]; od; od;
>solve(P);
>end proc:
```

4. Drawing the (pseudo)digraph associated with $\mathcal{Z}$.

   In this last step we implement the routine `drawing` to represent the (pseudo)digraph associated with the Zinbiel algebra obtained in the previous step. Firstly, for each solution generated by the main procedure `Zinbiel2` we execute the command `associate` in order to define the values of the structure constants. The input of this procedure `drawing` is $dim(\mathcal{Z})$ and the output is the drawing of the associated (pseudo)digraph. To implement the procedure, we have to consider five local variables: `E`, `G`, `L`, `S` and `V`. The list `E` saves all the edges of the (pseudo)digraph, `G` is the variable used to generate such a (pseudo)digraph, list `L` will save the vertices with loops, list `V` consists of the list of vertices in `G`, and `S` is used to save the permutations of vertices in `V` chosen two by two. The general idea of the implementation is to evaluate which edges appear in the (pseudo)digraph studying if their weight is zero or not.

   Let us note that Maple cannot draw loops within a pseudodigraph. In order to solve this drawback, we have used the command `HighlightVertex` to change colour to dark blue for those vertices with loops. We use yellow color for the remaining vertices.

```
>drawing:=proc(n)
>  local E,G,L,S,V;
>  L:=[];E:=[]; V:=[seq(i,i=1..n)]; S=permute(V,2);
>  for i from 1 to nops(S) do
>    if law(S[i][1],S[i][2])<>0 then
>      E:={op(E),S[i]}; fi; od;
>  for j from 1 to n do
>    if law(j,j)<>0 then
>      L:={op(L),[j,j]}; fi; od;
>  G:=Digraph(V,E);
>  for k from 1 to nops(L) do
>     HighlightVertex(G,L[k]); od;
>  DrawGraph(G);
>end proc:
```

Example 4. *Now, we show an example with Configuration xvii) from Figure 5 with the* 3*-dimensional Zinbiel algebra given by the law*

$$[e_2, e_3] = c_{2,3}^2 e_2 + c_{2,3}^3 e_3;\ [e_3, e_2] = c_{3,2}^2 e_2 + c_{3,2}^3 e_3;\ [e_2, e_2] = c_{2,2}^1 e_1 + c_{2,2}^2 e_2 + c_{2,2}^3 e_3,$$

$$[e_3, e_3] = c_{3,3}^1 e_1 + c_{3,3}^2 e_2 + c_{3,3}^3 e_3.$$

First, we have to complete the implementation of the subprocedure `law2` *as follows*

```
> if (i,j)=(2,2) then c221*e[1]+c222*e[2]+c223*e[3]; end if;
> if (i,j)=(2,3) then c231*e[1]+c232*e[2]+c233*e[3]; end if;
> if (i,j)=(3,2) then c321*e[1]+c322*e[2]+c323*e[3]; end if;
> if (i,j)=(3,3) then c331*e[1]+c332*e[2]+c333*e[3];
> else 0;
```

After that, we must run the subprocedure `bracket` *and procedure* `Zinbiel2`*. Now, if we evaluate the main procedure over the variable* `dim`*, we obtain the restrictions*

```
{c221=0,c222=-c233,c223=-c233^2/c322,c232=c322,c233=c233,c322=c322,c323=c233,
c331=0,c332=-c322^2/c233,c333=-c322}
```

From the previous output, we have obtained a family of Zinbiel algebras, where $c_{3,2}^2 \neq 0$ and $c_{2,3}^3 \neq 0$. In fact, it can be proved that every algebra in this family is isomorphic to the one with law

$$[e_2, e_2] = [e_3, e_3] = -e_2 - e_3;\ [e_2, e_3] = [e_3, e_2] = e_2 + e_3$$

Therefore we execute the order

```
> assign({c221=0,c222=-1,c223=-1,c232=1,c233=1,c322=1,c323=1,c331=0,c332=-1,
c333=-1});
```

Finally, we evaluate the procedure `drawing`*, obtaining Figure 8, which corresponds to Configuration xvii) from Figure 5.*

```
> drawing(dim);
```

Figure 8: Digraph corresponding to Configuration xvii).

7.3 Computational and complexity study

Here, we develop a computational study of the algorithmic procedure carried out in Subsection 7.2, which has been implemented with MAPLE 18, in an Intel(R) Core(TM) i7-4510U CPU with a 2.60 GHz processor and 12.00 GB of RAM. Table 3 shows computational data about the computing time and memory used to complete the whole procedure starting from the value of $dim(\mathcal{Z})$.

For this computational study, we have considered the family of 2-step nilpotent Zinbiel algebras associated with the generalization of configuration $xvii)$ from Figure 5.

Now, we show some brief statistics about the relation between the computing time and the memory used by the implementation of the previous procedure. Figures 9 and 10 show, respectively, the behavior of the computing time (C.T.) and used memory (U.M.) with respect to the dimension n. We can see how the computing time increases faster than the used memory and both of them fit a positive exponential model. Figure 11 represents a frequency diagram for the quotient between used memory and computing time. In this case, the behavior corresponds to a negative exponential model.

Finally, we compute the complexity of the algorithm taking into account the number of operations carried out in the worst case. We have used the

Input	Computing time	Used memory
$n = 3$	0.17 s	4.81 MB
$n = 4$	0.29 s	4.93 MB
$n = 5$	0.42 s	5.18 MB
$n = 6$	0.59 s	5.31 MB
$n = 7$	1.15 s	6.56 MB
$n = 8$	1.62 s	7.62 MB
$n = 9$	2.68 s	10.06 MB
$n = 10$	4.01 s	12.62 MB
$n = 11$	6.45 s	14.62 MB
$n = 12$	9.96 s	20.93 MB
$n = 13$	15.64 s	26.68 MB

Table 3: Computing time and used memory.

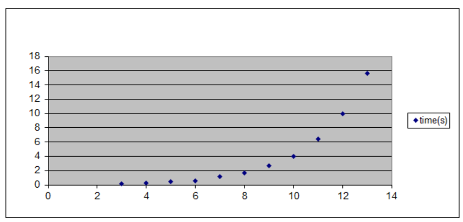

Figure 9: Graph for the C.T. with respect to dimension.

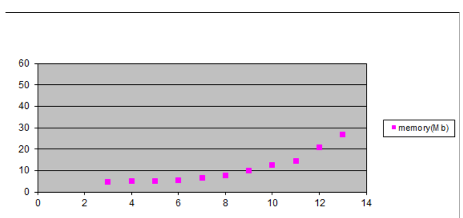

Figure 10: Graph for the U.M. with respect to dimension.

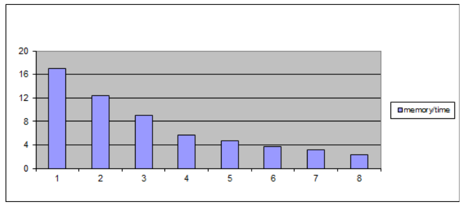

Figure 11: Graph for quotients U.M./C.T. with respect to dimension.

big O notation to express the complexity. To recall the big O notation, the reader can consult [9]: Given two functions $f, g : \mathbb{R} \to \mathbb{R}$, we could say that $f(x) = O(g(x))$ if and only if there exist $M \in \mathbb{R}^+$ and $x_0 \in \mathbb{R}$ such that $|f(x)| < M \cdot g(x)$, for all $x > x_0$.

We denote by $N_i(n)$ the number of operations when considering the Step i. This function depends on $dim(\mathcal{Z})$. Table 4 shows the number of computations and the complexity of each step.

Step	Routine	Complexity	Operations
1	`law2`	$O(n^2)$	$N_1(n) = 2 + \frac{n(n-1)}{2}$
2	`bracket`	$O(n^4)$	$N_2(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} N_1(n)$
3	`Zinbiel2`	$O(n^7)$	$N_3(n) = O(n) + O(n^3) + 2\sum_{i=1}^{n^3} N_2(n) + \sum_{j=1}^{n^3} \sum_{k=1}^{n} 1$
4	`drawing`	$O(n^4)$	$N_4(n) = O(n) + O(n^2) + 2\sum_{i=1}^{n^2} N_1(n)$

Table 4: Complexity and number of operations.

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